INTRODUCTION

In this thesis, we are working in Differential Geometric Control Theory. Two branches of mathematics, differential geometry and the Lie Theory are mainly involved. The thesis is divided into seven chapters. We start in the first chapter recalling some preliminaries on Lie groups and their Lie algebras and then in the next one giving a brief introduction to control systems as a first motivation.

Our principal interest throughout this thesis is to give a contribution on a new class of control systems, Linear Control Systems on Lie groups introduced in a paper published by *the American Mathematical Society Series : Symposia in Pure and Applied Mathematics, 1998.* In this connection, starting from the third chapter we especially deal with this class and give new results.

In particular, the contributions of the work are arranged into the last 4 chapters of the thesis. In fact, in Chapter 4 we establish a result about null controllable set containing some topological properties and extend well-known facts on null controllability property for linear control systems on R^n to linear control systems on Lie groups. We give a global sufficient condition to the null controllability of this class of control systems.

In Chapter 5, we introduce some associated systems to a given linear control system one which is to analize local controllability on a connected Lie group G and, one which is to study that system on a simply connected and nilpotent Lie group case.

In Chapter 6, we turn our interest to another fundamental problem in Control Theory, observability. The work in this section of the thesis is also related to [**3**] and indeed we generalize the notion of linear pair as introduced in *the paper published by Comput. Appl., Math., 1997.* As a matter of fact, we extend all the results appear in [**3**] and obtain more general results for general linear pairs where the dynamics of our system is given by a vector field in the normalizer.

We finish the thesis with the Chapter 7. In this part, we construct some original computational algorithms on the direction of our needs which will appear below in details.

CHAPTER I.

In this starting chapter, we first introduce some basic notions and facts from Differential Geometry that we shall need throughout the thesis , like tangent space and tangent maps, vector fields, Lie bracket, exponential map, etc.

CHAPTER II.

We deal with control systems in this section of the thesis as a brief introduction to control systems. By definition, a control system Σ on a differentiable manifold M is determined by the dynamics $\mathcal{D}$ which is a subfamily of $\mathcal{X}^{\infty}(M)$, the set of all vector fields of $C^{\infty}-$class. We first start with the class of linear control systems on $\mathbb{R}^n$ since this class is quite well-understood and is one of the most important class of control systems from both practical and theoretical points of view. Furthermore, it is the class that we are going to generalize. Then, respectively bilinear and invariant control systems follow.

We concentrate to the concepts of transitivity and controllability which is one of the fundamental and hard problems in Control Theory. Finally, the chapter ends with the Orbit Theorem, [**27**], which is due to H.Sussmann. In particular, this important theorem allows us to reduce controllability to system orbits.

CHAPTER III.

Starting from this chapter, we study with *Linear Control Systems on Lie Groups.* A linear control system on $\mathbb{R}^n$ is given by the family of differential equations on $\mathbb{R}^n$ of the form

$$\dot{x} = Ax + \sum_{j=1}^{m} u_j b_j$$

where b_j for each $j = 1, 2, ..., m$ defines an invariant vector field on $\mathbb{R}^n$. We note that the Lie bracket

$$[Ax, b_j] = -Ab_j \text{ for every } x \in \mathbb{R}^n.$$

That is, the resulting Lie bracket is by translation an invariant vector field on $\mathbb{R}^n$. The authors in [**6**] introduce the concept of normalizer of a Lie algebra $\mathbf{g}$ in the Lie algebra $\mathcal{X}^{\infty}(G)$ of all smooth vector fields defined on a Lie group G. In fact, this is the focul point of introducing this new and quite important notion. The set $norm_{\mathcal{X}^{\infty}(G)}(\mathbf{g})$ consists of such elements, *called linear vector fields.* More precisely, for any vector field X from the normalizer, the Lie bracket $[X, Y]$ is again a left-invariant vector field on G for every $Y \in \mathbf{g}$. After defining it, we mentione the identification between the normalizer and smooth functions defined on G into the Lie algebra $\mathbf{g}$. Later on, we give some properties of such functions and then motivate to the characterization of $norm_{\mathcal{X}^{\infty}(G)}$ $(\mathbf{g})$. In particular, as indicated in Theorem 3.10 the map

$$\Phi : norm_{\mathcal{X}^{\infty}(G)}(\mathbf{g}) \longrightarrow \mathbf{g} \otimes \partial\mathbf{g} \text{ given by } \Phi(\widetilde{F}) = (F(e), -dF_e)$$

is an injective Lie algebra homomorphism for a connected Lie group G. Additionaly, if G is connected and simply connected, then surjectivity assertion is also satisfied.

This chapter also contains the study of transitivity and local controllability properties of linear control systems on Lie groups. We consider the characterization of $L(\Sigma)$, the Lie algebra of a control system Σ on a Lie group G and then some algebraic objects, *the Lie algebra rank condition and ad-rank condition*, are introduced to determine transitivity and local controllability of Σ. In fact, the Lie algebra rank condition characterizes transitivity but unfortunately does not determine controllability since it is just a necessary condition for controllability property. We show this situation giving an example on the Heisenberg Lie group of dimension 3. The other algebraic condition, ad-rank condition, is just a sufficient condition for local controllability as appointed by Ayala and Tirao in [**6**].

CHAPTER IV

Throughout this chapter we study null controllability of linear control systems on a specific class of Lie groups, simply connected and nilpotent Lie groups. By definition, for a given linear control system Σ on a Lie group G, $\Sigma-$null controllable set at the time $t \geq 0$ is

$$\Im(t) = \{x \in G \mid \ \exists u \in \mathcal{U} \text{ such that } x(x,u,t) = 0\}$$

and $\Sigma-$null controllable set is

$$\Im = \underset{t\geq 0}{\cup} \Im(t).$$

We establish a result containing some topological properties of $\Im$, Proposition 4.2. Then using the Theorems 3.18 and 3.15 we extend well-known facts about null controllability property for linear control systems on $\mathbb{R}^n$ to linear control systems on a simply connected and nilpotent Lie group G. In fact, as a main contribution of this chapter we give a global sufficient condition on null controllability of this class of control systems as follows :

THEOREM 4.3 Let Σ_{nil} denote a linear control system on a simply connected and nilpotent Lie group G. If Σ_{nil} satisfies ad-rank condition and if $Spect(ad(X)) \subset \mathbb{C}^{-}$, then $\Im = G$.

CHAPTER V

Let Σ be a linear control system defined on a Lie group G. In order to analize local controllability of Σ we introduce an associated system Σ_D on $\mathbb{R}^n$. In fact, the drift vector field X has a singularity at the identity e of G and by linearization of X, the system Σ induces a new control system Σ_D on $\mathbb{R}^n$ having the form of

$$\Sigma_D : \left\{ \begin{array}{c} \dot{z} = Dz + Bu \ , \ u \in \mathcal{U} \\ z \in \mathbb{R}^n \end{array} \right\}$$

where D denotes the derivation associated with the infinitesimal automorphism X and B is the matrix $(Y_e^1, Y_e^2, ..., Y_e^m)$ constructed with invariant vector fields Y^j, $j = 1, 2, ..., m$, as column vectors. If Σ_D satisfies the Kalman ‘s rank condition, then Σ is locally controllable. More precisely, the complete controllability of Σ_D on $\mathbb{R}^n$ gives us local controllability property of Σ. In fact, if

$$rank(B \ DB \ D^2B \ \cdots \ D^{n-1}B) = \dim(\mathbb{R}^n \ )$$

then Σ satisfies ad-rank condition

$$\dim Span\{Y^j, ad^i(X)(Y^j) \mid \ 1 \leq j \leq m \text{ and } 0 \leq i \leq n-1\} = \dim(G)$$

since $ad^0 = D^0 = Id$ and $ad^i(X)(B) = D^iB$ for all $i \geq 0$.

As indicated in [**6**], this is the sufficient condition for locall controllability of linear control systems on G. With this approach, sufficient condition to the local controllability property for a given linear control system on G can be determined just by some simple computations on the Lie algebra level after finding the matrix D.

Now, suppose G to be simply connected and nilpotent Lie group. As is known, in such a case the exponential map is a diffeomorphism of **g** and G in global sense. Given a system Σ on G

$$\dot{x}(t) = X(x(t)) + \sum_{j=1}^{m} u_j(t) \ Y^j(x(t))$$

there exists a control system $\Sigma_{\log}$ defined on the Lie algebra $\mathbf{g}$ of G as an equivalence of Σ in $\mathbf{g}$. Using a constant control $u = (u_1, u_2, ..., u_m) \in \mathbb{R}^m$, $\Sigma_{\log}$ has the face of

$$\Sigma_{\log} : \left\{ \begin{array}{c} \dot{z} = Dz + \sum_{j=1}^{m} u_j \; d(\log)_{\exp z} \; Y^j(\exp z) \\ z \in \mathbf{g} \end{array} \right\}.$$

So, in order to study Σ we can also use $\Sigma_{\log}$ and as a matter of fact, this construction in many cases allows us to reduce seemingly complicated computations.

CHAPTER VI

This section of the thesis is based on observability property of linear control systems on Lie groups. As indicated in [3], related to the observability property of this class of systems the authors introduce the notion of *linear pair.* Our interest in this chapter is to generalize that concept in a natural way and obtain more general results for general linear pairs where the dynamics of the system is given by a vector field in the normalizer. In fact, we extend all the results appear in [**3**]. By definition, a general linear pair (X, π_K) on G is determined by $X \in norm_{\mathcal{X}^\infty(G)}$ $(\mathbf{g})$ and by a closed Lie subgroup K. This definition extends both the classical pair (C, A) induced by a linear control system Σ on $\mathbb{R}^n$ and the notion of linear pairs (X, h) on a connected Lie group G. Then we establish the following useful lemma to show $ad(X)-$invariance of the Lie algebra $\mathcal{I}$ of I.

Lemma 6.6 Let (X, π_K) be a general linear pair on G. Then,

1.There exist a vector field $Z \in aut(G)$ and a right invariant vector field Y^R on G such that $X = Z + Y$.

2. The linear transformations $ad(Z)$ and $ad(X)$ which are defined on $X^\infty(G)$ agree on the Lie algebra $\mathbf{g}$ of G.

3. $ad(Z)$ is a derivation of $\mathbf{g}$ such that for every $Y \in \mathbf{g}$ and real t

$$\varphi_t(\exp Y) = \exp(e^{t(ad(Z)} \; Y)$$

where $\{\varphi_t\}_{t\in\mathbb{R}}$ is the flow in $Aut(G)$, associated to the vector field Z.

Using the Lemma 6.6 we construct in Theorem 6.8 an algebraic characterization of $\mathcal{I}$. That is, the Lie algebra $\mathcal{I}$ has the face of

$$\mathcal{I} = \bigcap_{i=0}^{n-1} ad^{-i}(X)(\mathcal{K})$$

where $\mathcal{K}$ expresses the Lie algebra of closed Lie subgroup K of G. Then, we extend both local and global observability results appear in [**3**] as follows :

COROLLARY 6.10 (X, π_K) is locally observable if and only if $\mathcal{I} = \{0\}$.

THEOREM 6.12 A general linear pair (X, π_K) is observable (globally) if and only if
i) (X, π_K) is locally observable
ii) $\mathrm{Sing}(Z) \cap K = \{e\}$.

CHAPTER VII

This last chapter contains some computational algorithms to be able to compute some concepts that we need in many steps of the work like derivations of a given Lie algebra, infinitesimal automorphisms of corresponding Lie group G, some algebraic conditions to study local controllability and transitivity properties of linear control systems on Lie groups, called *the ad-rank condition and the Lie algebra rank condition.* The algorithm for finding Lie algebra of all derivations is the most important one. In particular, we construct an original algorithm to determine the Lie algebra $\partial\mathbf{g}$ of all derivations for a Lie algebra $\mathbf{g}$ of the considered type in this chapter. As is seen, derivations play a central role and their Lie algebra is quite useful tool to compute those concepts mentioned above.

By the fact that exponential and its inverse logarithm maps are diffeomorphisms for simply connected and nilpotent Lie groups, we are in such a case able to find each infinitesimal automorphism associated to a given derivation. Particularly, in [**22**] the exponential and logarithm maps are given for lower-dimensional non-Abelian simply connected and nilpotent Lie groups in 3,4,5 and 6 dimensions.

We know from local controllability theorem that if ad-rank condition is satisfied, then the system is locally controllable. Since, $ad(X) : \mathbf{g} \to \mathbf{g}$ is an algebra endomorphism defined by $ad(X)(Y) = [X, Y] = D(Y)$ for every $Y \in \mathbf{g}$, just by using the usual matrix multiplication (of matrix representation of D and invariant vector field Y as a column vector) and iterating it we can construct this sequence to analize local controllability property of linear control systems on Lie groups.