Introduction to Fluid Mechanics

Chapter 4

Basic Equations in Integral Form for a Control Volume

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Main Topics

- ✓ Basic Laws for a System
- ✓ Relation of System Derivatives to the Control Volume Formulation
- ✓ Conservation of Mass
- ✓ Momentum Equation for Inertial Control Volume
- ✓ Momentum Equation for Inertial Control Volume with Rectilinear Acceleration
- ✓ The Angular Momentum Principle
- ✓ The First Law of Thermodynamics
- ✓ The Second Law of Thermodynamics

Basic Laws for a System

✓ **Conservation of Mass**

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0$$

$$M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\text{V}\!\!\!-(\text{system})} \rho \, d\text{V}\!\!\!-$$

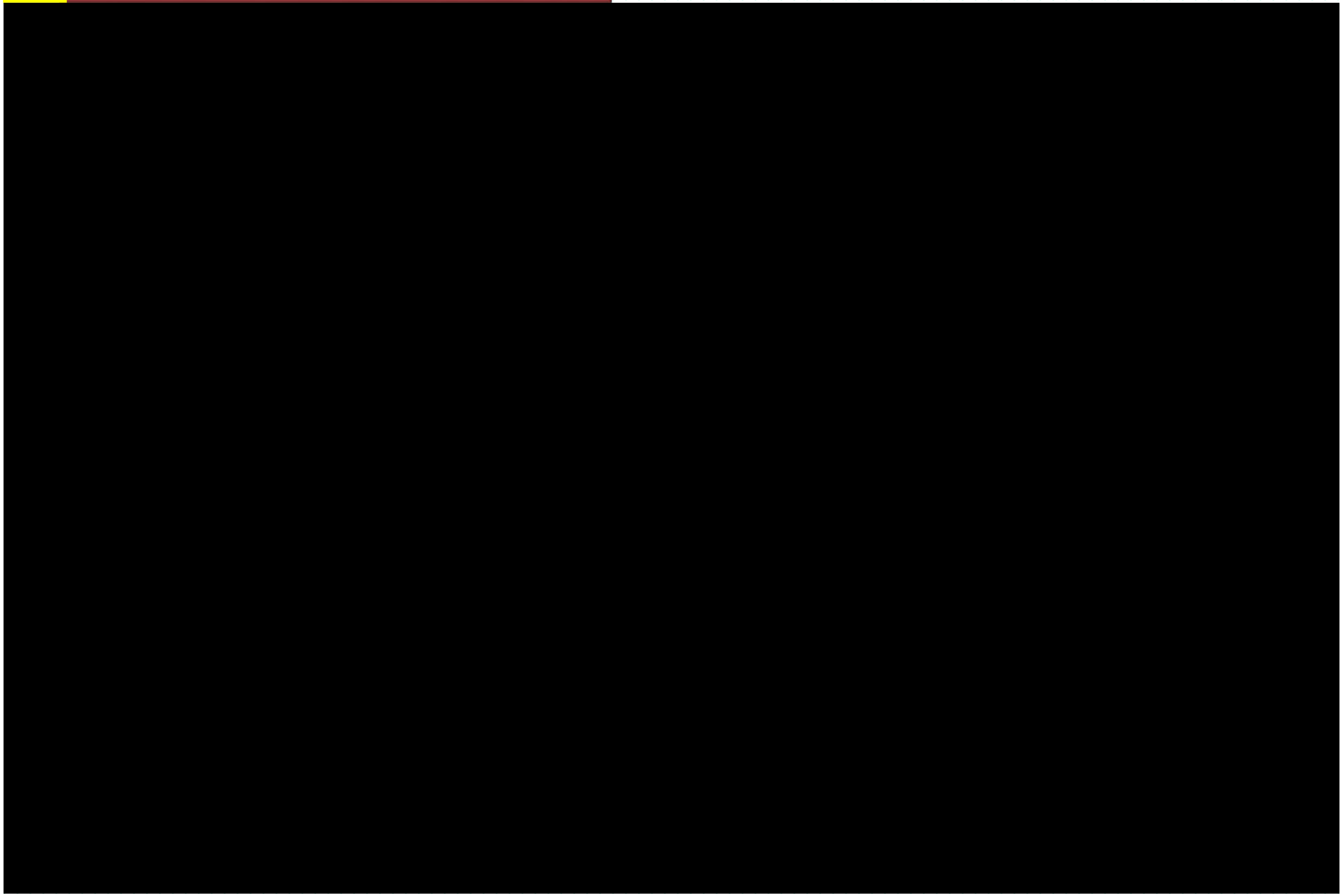
Basic Laws for a System

✓ **Momentum Equation for Inertial Control Volume**

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}}$$

$$\vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V}\, dm = \int_{\mathcal{V}(\text{system})} \vec{V}\, \rho\, d\mathcal{V}$$

Conservation of Momentum



Basic Laws for a System

✓ The Angular Momentum Principle

$$\vec{T} = \left.\frac{d\vec{H}}{dt}\right)_{\text{system}}$$

$$\vec{H}_{\text{system}} = \int_{M(\text{system})} \vec{r} \times \vec{V}\, dm = \int_{\forall(\text{system})} \vec{r} \times \vec{V}\, \rho\, d\forall$$

$$\vec{T} = \vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g}\, dm + \vec{T}_{\text{shaft}}$$

Basic Laws for a System

✓ **The First Law of Thermodynamics**

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{\text{system}}$$

$$E_{\text{system}} = \int_{M(\text{system})} e\, dm = \int_{\mathcal{V}(\text{system})} e\, \rho\, d\mathcal{V}$$

$$e = u + \frac{V^2}{2} + gz$$

Basic Laws for a System

✓The Second Law of Thermodynamics

$$\left.\frac{dS}{dt}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q}$$

$$S_{\text{system}} = \int_{M(\text{system})} s\,dm = \int_{V\!\!\!\!-(\text{system})} s\,\rho\,dV\!\!\!\!-$$

Relation of System Derivatives to the Control Volume Formulation

✓Extensive and Intensive Properties

$$N_{\text{system}} = \int_{M(\text{system})} \eta \, dm = \int_{\forall(\text{system})} \eta \, \rho \, d\forall$$

$$N = M, \ \eta = 1$$

$$N = \vec{P}, \ \eta = \vec{V}$$

$$N = \vec{H}, \ \eta = \vec{r} \times \vec{V}$$

$$N = E, \ \eta = e$$

$$N = S, \ \eta = s$$

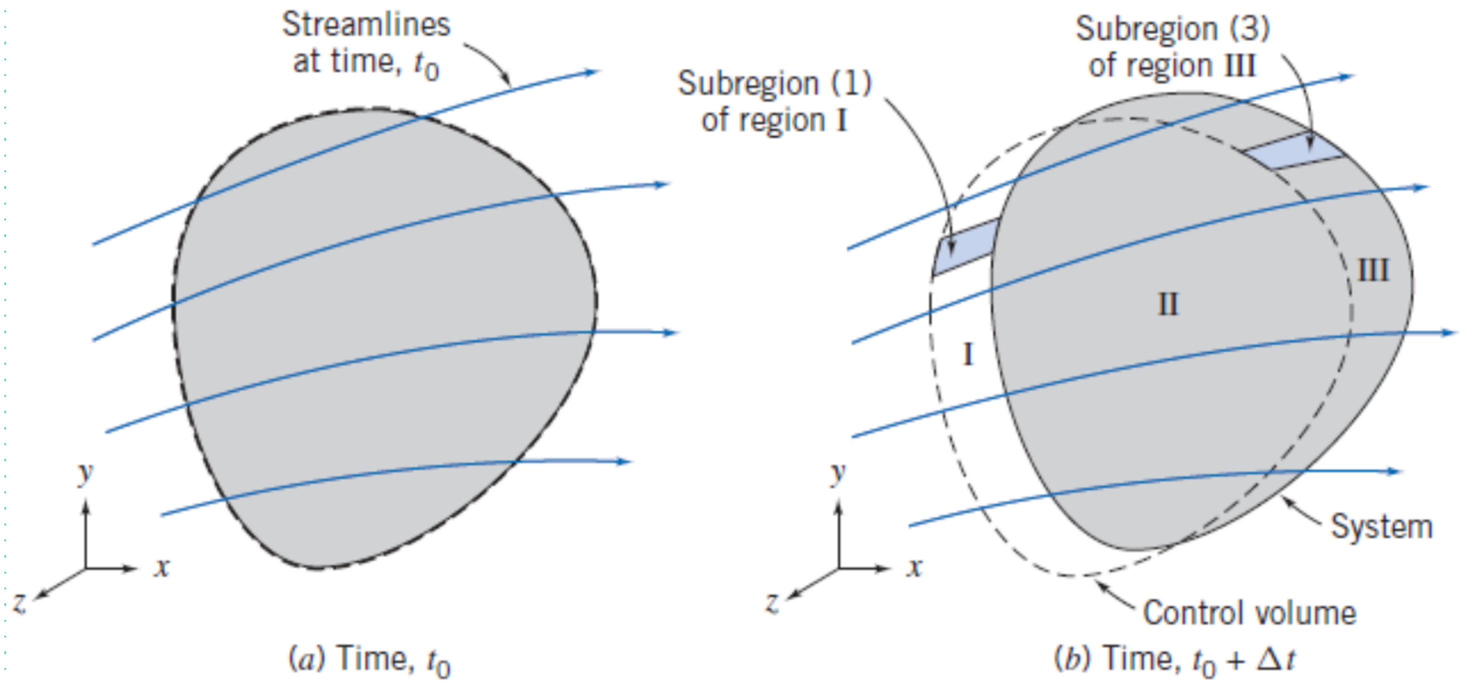


Fig. 4.1 System and control volume configuration.

Derivation

within the control volume at time t_0, is partially out of the control volume at time $t_0+\Delta t$. In fact, three regions can be identified. These are: regions I and II, which together make up the control volume, and region III, which, with region II, is the location of the system at time $t_0+\Delta t$.

the rate of change of N_{system}

$$\left.\frac{dN}{dt}\right)_{\text{system}} \equiv \lim_{\Delta t \to 0} \frac{N_s)_{t_0+\Delta t} - N_s)_{t_0}}{\Delta t}$$

$$N_s)_{t_0+\Delta t} = (N_{\text{II}} + N_{\text{III}})_{t_0+\Delta t} = (N_{\text{CV}} - N_{\text{I}} + N_{\text{III}})_{t_0+\Delta t}$$

$$N_s)_{t_0} = (N_{\text{CV}})_{t_0}$$

$$\left.\frac{dN}{dt}\right)_s = \lim_{\Delta t \to 0} \frac{(N_{\text{CV}} - N_{\text{I}} + N_{\text{III}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{N_{\text{CV}})_{t_0+\Delta t} - N_{\text{CV}})_{t_0}}{\Delta t} = \frac{\partial N_{\text{CV}}}{\partial t} = \frac{\partial}{\partial t} \int_{\text{CV}} \eta \, \rho \, d\forall$$

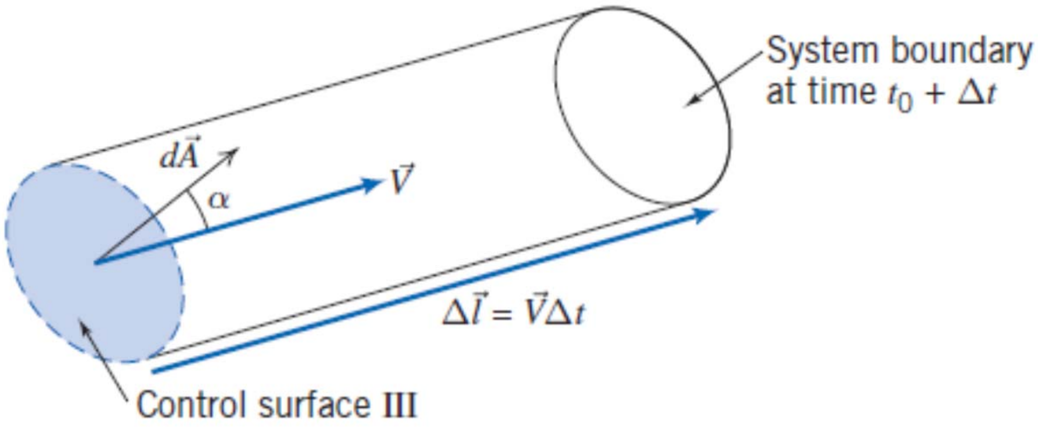


Fig. 4.2 Enlarged view of subregion (3) from Fig. 4.1.

$$dN_{\mathrm{III}})_{t_0+\Delta t} = (\eta\,\rho\,d\mathcal{V})_{t_0+\Delta t}$$

$$dN_{\mathrm{III}})_{t_0+\Delta t} = \eta\,\rho\vec{V}\cdot d\vec{A}\,\Delta t$$

$$\lim_{\Delta t\to 0}\frac{N_{\mathrm{I}})_{t_0+\Delta t}}{\Delta t} = -\int_{\mathrm{CS}_1}\eta\,\rho\vec{V}\cdot d\vec{A}$$

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\,\rho\,d\mathcal{V} + \int_{\mathrm{CS}_1}\eta\,\rho\vec{V}\cdot d\vec{A} + \int_{\mathrm{CS}_{\mathrm{III}}}\eta\,\rho\vec{V}\cdot d\vec{A}$$

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\mathrm{CV}}\eta\,\rho\,d\mathcal{V} + \int_{\mathrm{CS}}\eta\,\rho\vec{V}\cdot d\vec{A}$$

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\,\rho\,d\Psi + \int_{\text{CS}} \eta\,\rho\vec{V}\cdot d\vec{A}$$

Reynolds Transport Theorem

$\left.\frac{dN}{dt}\right)_{\text{system}}$ — the rate of change of the system extensive property N.

$\frac{\partial}{\partial t}\int_{\text{CV}} \eta\,\rho\,d\Psi$ — the rate of change of the amount of property N in the control volume.

$\int_{\text{CS}} \eta\,\rho\vec{V}\cdot d\vec{A}$ — the rate at which property N is exiting the surface of the control volume.

Relation of System Derivatives to the Control Volume Formulation

✓ **Reynolds Transport Theorem**

$$\boxed{\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\,\rho\, d\forall + \int_{\text{CS}} \eta\,\rho\vec{V}\cdot d\vec{A}}$$

Relation of System Derivatives to the Control Volume Formulation

✓ Interpreting the Scalar Product

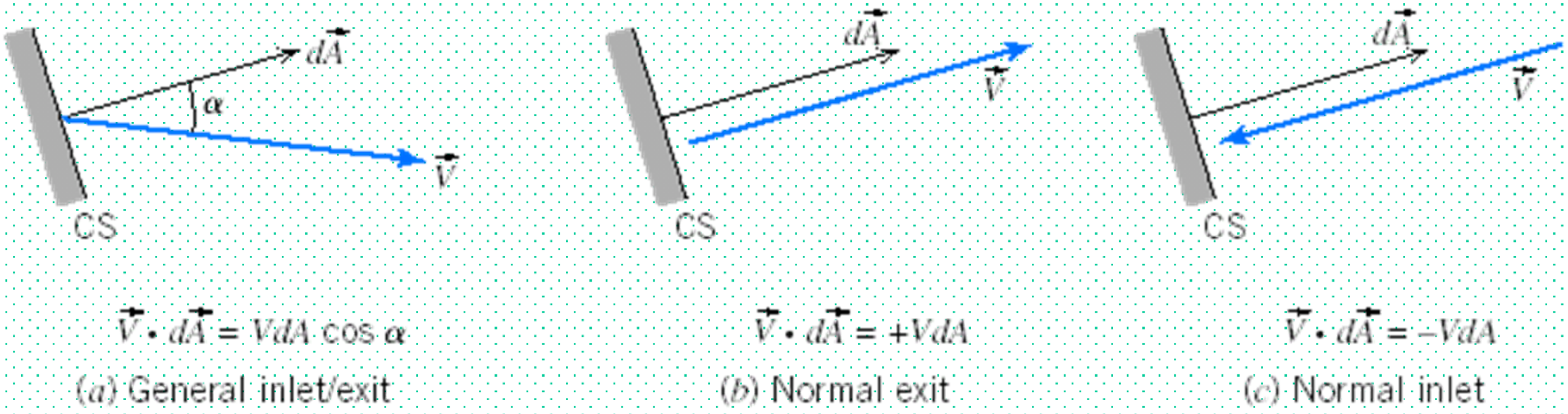


Fig. 4.3 Evaluating the scalar product.

Conservation of Mass

✓ Basic Law, and Transport Theorem

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0$$

$$\left.\frac{dM}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho \, d\forall + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A}$$

Conservation of Mass

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\forall + \int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\forall = -\int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A}$$

Rate of increase of mass in CV	=	Net influx of mass

Conservation of Mass

✓ **Incompressible Fluids**

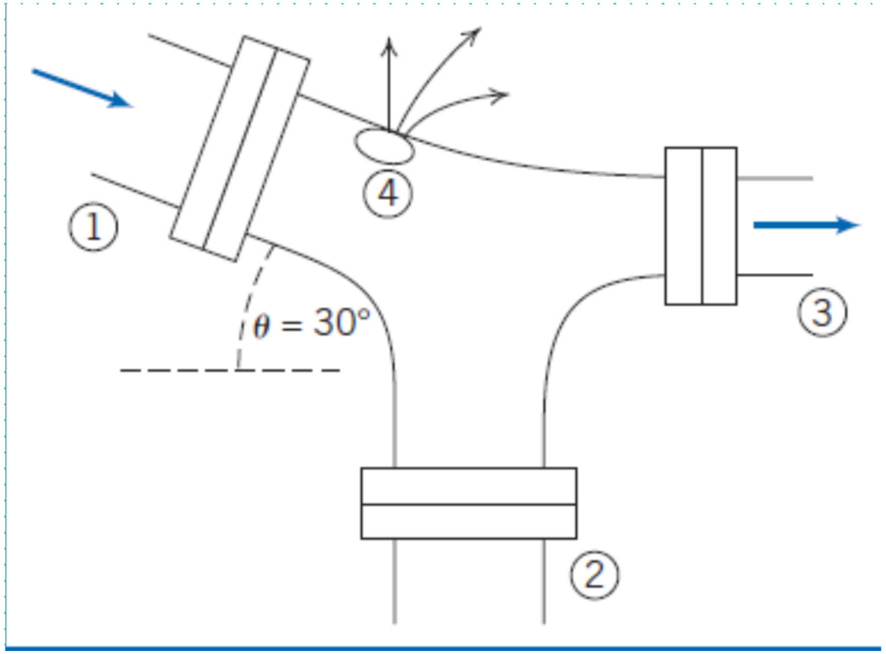
$$\int_{CS} \vec{V} \cdot d\vec{A} = 0$$

✓ **Steady, Compressible Flow**

$$\int_{CS} \rho\vec{V} \cdot d\vec{A} = 0$$

Example 4.1 MASS FLOW AT A PIPE JUNCTION

Consider the steady flow in a water pipe joint shown in the diagram. The areas are: $A_1=A_2=0.2\ m^2$, and $A_3=0.15\ m^2$. In addition, fluid is lost out of a hole at 4, estimated at a rate of $0.1\ m^3/s$. The average speeds at sections 1 and 3 are $V_1=5$ m/s and $V_3=12$ m/s, respectively. Find the velocity at section 2 .

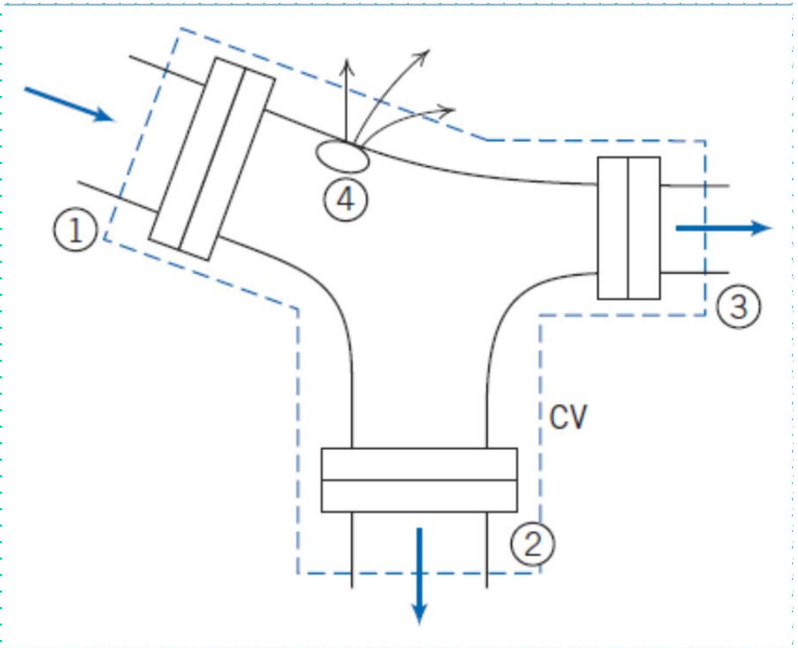


$$A_1 = 0.2\,\mathrm{m}^2 \quad A_2 = 0.2\,\mathrm{m}^2 \quad A_3 = 0.15\,\mathrm{m}^2$$
$$V_1 = 5\,\mathrm{m/s} \quad V_3 = 12\,\mathrm{m/s} \quad \rho = 999\,\mathrm{kg/m^3}$$

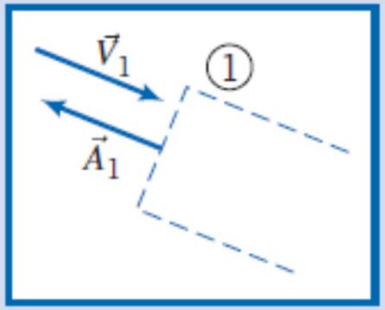
$$\sum_{CS} \vec{V} \cdot \vec{A} = 0$$

Assumptions: (1) Steady flow (given).
(2) Incompressible flow.
(3) Uniform properties at each section.

$$\vec{V}_1 \cdot \vec{A}_1 + \vec{V}_2 \cdot \vec{A}_2 + \vec{V}_3 \cdot \vec{A}_3 + Q_4 = 0$$

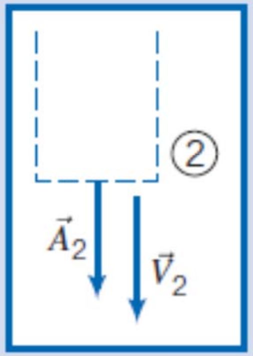


$\vec{V}_1 \cdot \vec{A}_1 = -V_1 A_1$



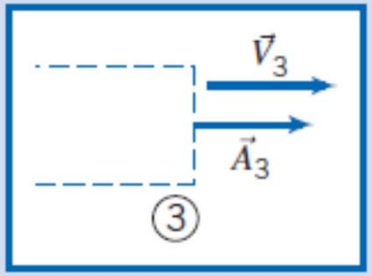
$\left\{ \begin{array}{l} \text{Sign of } \vec{V}_1 \cdot \vec{A}_1 \text{ is} \\ \text{negative at surface } 1 \end{array} \right\}$

$\vec{V}_2 \cdot \vec{A}_2 = +V_2 A_2$



$\left\{ \begin{array}{l} \text{Sign of } \vec{V}_2 \cdot \vec{A}_2 \text{ is} \\ \text{positive at surface } 2 \end{array} \right\}$

$\vec{V}_3 \cdot \vec{A}_3 = +V_3 A_3$



$\left\{ \begin{array}{l} \text{Sign of } \vec{V}_3 \cdot \vec{A}_3 \text{ is} \\ \text{positive at surface } 3 \end{array} \right\}$

$$-V_1A_1 + V_2A_2 + V_3A_3 + Q_4 = 0$$

$$V_2 = \frac{V_1A_1 - V_3A_3 - Q_4}{A_2}$$

$$= \frac{5\,\frac{\text{m}}{\text{s}} \times 0.2\,\text{m}^2 - 12\,\frac{\text{m}}{\text{s}} \times 0.15\,\text{m}^2 - \frac{0.1\,\text{m}^3}{\text{s}}}{0.2\,\text{m}^2}$$

$$= \underbrace{-4.5\,\text{m/s}}_{V_2}$$

Recall that V_2 represents the magnitude of the velocity, which we assumed was outwards from the control volume.
The fact that V_2 is negative means that in fact we have an inflow at location 2 —our initial assumption was invalid.

Momentum Equation for Inertial Control Volume

✓ **Basic Law, and Transport Theorem**

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}}$$

$$\boxed{\left.\frac{d\vec{P}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}\,\rho\, d\forall + \int_{\text{CS}} \vec{V}\rho\vec{V}\cdot d\vec{A}}$$

Momentum Equation for Inertial Control Volume

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}\,\rho\, d\forall + \int_{\mathrm{CS}} \vec{V}\,\rho\vec{V}\cdot d\vec{A}$$

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u\,\rho\, d\forall + \int_{\mathrm{CS}} u\,\rho\vec{V}\cdot d\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} v\,\rho\, d\forall + \int_{\mathrm{CS}} v\,\rho\vec{V}\cdot d\vec{A}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} w\,\rho\, d\forall + \int_{\mathrm{CS}} w\,\rho\vec{V}\cdot d\vec{A}$$

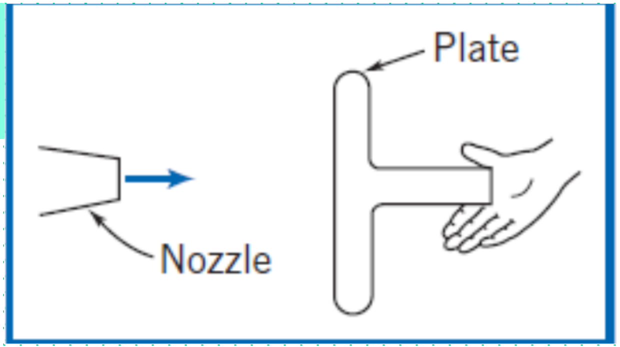
for uniform flow at each inlet and exit

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} u\,\rho\, d\forall + \sum_{\mathrm{CS}} u\,\rho\vec{V}\cdot\vec{A}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} v\,\rho\, d\forall + \sum_{\mathrm{CS}} v\,\rho\vec{V}\cdot\vec{A}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{\mathrm{CV}} w\,\rho\, d\forall + \sum_{\mathrm{CS}} w\,\rho\vec{V}\cdot\vec{A}$$

Example 4.4 CHOICE OF CONTROL VOLUME FOR MOMENTUM ANALYSIS



Water from a stationary nozzle strikes a flat plate as shown. The water leaves the nozzle at 15 m/s; the nozzle area is 0.01 m^2. Assuming the water is directed normal to the plate, and flows along the plate, determine the horizontal force you need to resist to hold it in place.

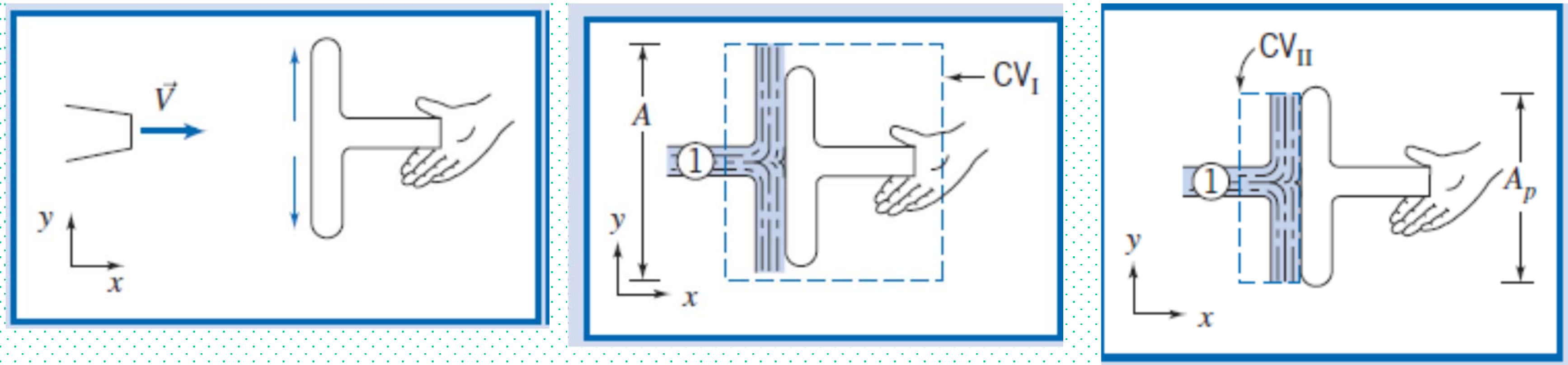
Assmp: (1) Steady flow.
(2) Incompressible flow.
(3) Uniform flow at each section where fluid crosses the CV boundaries.

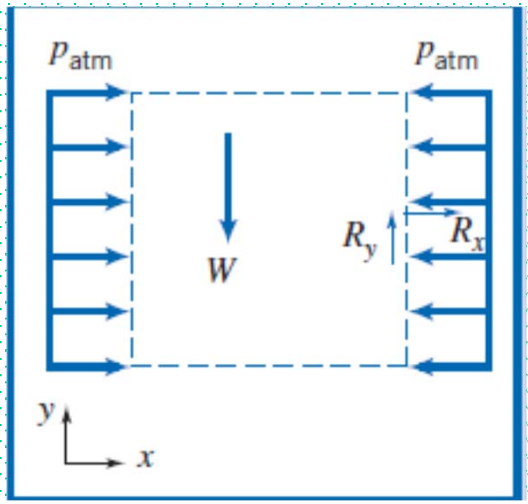
Jet velocity, $\vec{V} = 15\hat{i}\ \mathrm{m/s}$

Nozzle area, $A_n = 0.01\ \mathrm{m^2}$

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\mathrm{CV}} \vec{V}\rho d\mathcal{V} + \int_{\mathrm{CS}} \vec{V}\rho\vec{V}\cdot d\vec{A} \quad \text{and} \quad \frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho\, d\mathcal{V} + \int_{\mathrm{CS}} \rho\vec{V}\cdot d\vec{A} = 0$$

$$\vec{F} = \vec{F}_S + \vec{F}_B = \sum_{CS} \vec{V} \rho \vec{V} \cdot \vec{A} \quad \text{and} \quad \sum_{CS} \rho \vec{V} \cdot \vec{A} = 0$$





$$F_{S_x} + F_{B_x} = \sum_{CS} u \rho \vec{V} \cdot \vec{A}$$

There are no body forces in the x direction, so $F_{Bx} = 0$,

$$F_{S_x} = \sum_{CS} u \rho \vec{V} \cdot \vec{A}$$

$$F_{S_x} = \underbrace{p_{atm}A}_{\substack{\text{force due to atmospheric}\\ \text{pressure acts to right}\\ \text{(positive direction)}\\ \text{on left surface}}} - \underbrace{p_{atm}A}_{\substack{\text{force due to atmospheric}\\ \text{pressure acts to left}\\ \text{(negative direction)}\\ \text{on right surface}}} + \underbrace{R_x}_{\substack{\text{force of your hand on}\\ \text{control volume}\\ \text{(assumed positive)}}}$$

$$R_x = \sum_{CS} u\,\rho\vec{V}\cdot\vec{A} = u\,\rho\vec{V}\cdot\vec{A}|_1 \qquad \{\text{For top and bottom surfaces, } u = 0\}$$

$$R_x = -u_1\,\rho V_1 A_1 \qquad \{\text{At } 1,\ \rho\vec{V}_1\cdot\vec{A}_1 = \rho(-V_1A_1) \text{ since } \vec{V}_1 \text{ and } \vec{A}_1 \text{ are } 180° \text{ apart. Note that } u_1 = V_1\}$$

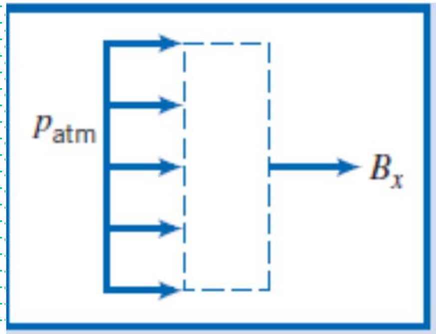
$$R_x = -15\,\frac{\text{m}}{\text{s}} \times 999\,\frac{\text{kg}}{\text{m}^3} \times 15\,\frac{\text{m}}{\text{s}} \times 0.01\,\text{m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}} \qquad \{u_1 = 15\,\text{m/s}\}$$

$$R_x = -2.25\,\text{kN} \qquad \{R_x \text{ acts opposite to positive direction assumed.}\}$$

The horizontal force on your hand is

$$K_x = -R_x = 2.25\,\text{kN} \leftarrow$$

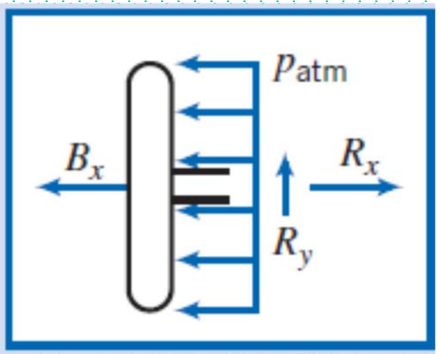
CV_{II} with Horizontal Forces Shown



$$F_{S_x} = \sum_{CS} u \, \rho \vec{V} \cdot \vec{A}$$

$$F_{S_x} = p_{atm} A_p + B_x = u \, \rho \vec{V} \cdot \vec{A}|_1 = -u_1 V_1 A_1 = -2.25 \text{ kN}$$

$$B_x = -p_{atm} A_p - 2.25 \text{ kN}$$



$$\sum F_x = 0 = -B_x - p_{atm} A_p + R_x$$

$$R_x = p_{atm} A_p + B_x$$

$$R_x = p_{atm} A_p + (-p_{atm} A_p - 2.25 \text{ kN}) = -2.25 \text{ kN}$$

Momentum Equation for Inertial Control Volume

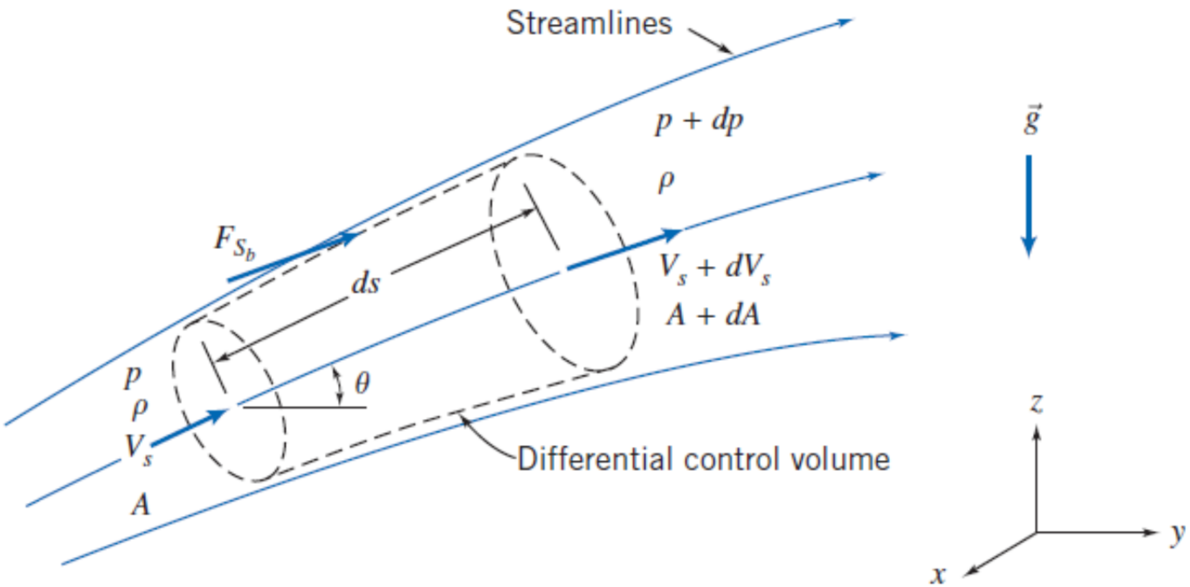
*Differential Control Volume Analysis

a. Continuity Equation

$$\underbrace{\cancel{\frac{\partial}{\partial t}}\int_{\mathrm{CV}} \rho \, d\forall}_{=0(1)} + \int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

Assps: (1) Steady flow.
(2) No flow across bounding streamlines.
(3) Incompressible flow, ρ=constant.

$$(-\rho V_s A) + \{\rho(V_s + dV_s)(A + dA)\} = 0$$



$$\rho(V_s + dV_s)(A + dA) = \rho V_s A$$

$$V_s\,dA + A\,dV_s + dA\,dV_s = 0$$

$$V_s\,dA + A\,dV_s = 0$$

b. Streamwise Component of the Momentum Equation

$$F_{S_s} + F_{B_s} = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}}} \int_{\mathrm{CV}} u_s \, \rho \, d\forall + \int_{\mathrm{CS}} u_s \, \rho \vec{V} \cdot d\vec{A}$$

$$F_{S_s} = pA - (p + dp)(A + dA) + \left(p + \frac{dp}{2}\right) dA$$

$$F_{S_s} = -A \, dp - \frac{1}{2} dp \, dA$$

$$F_{B_s} = \rho g_s \, d\forall = \rho(-g \sin\theta)\left(A + \frac{dA}{2}\right) ds$$

$$F_{B_s} = -\rho g \left(A + \frac{dA}{2}\right) dz$$

$$\int_{\mathrm{CS}} u_s \, \rho \vec{V} \cdot d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)\{\rho(V_s + dV_s)(A + dA)\}$$

$$-\frac{dp}{\rho} - g\,dz = V_s\,dV_s = d\left(\frac{V_s^2}{2}\right)$$

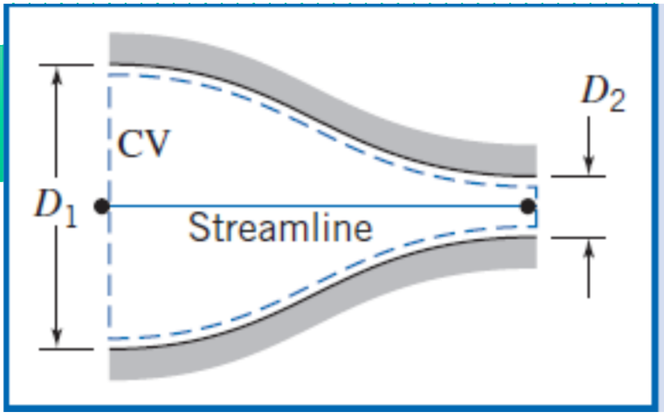
$$\frac{dp}{\rho} + d\left(\frac{V_s^2}{2}\right) + g\,dz = 0$$

$$\boxed{\frac{P}{\rho} + \frac{V_s^2}{2} + gz = \text{constant}}$$

This equation is subject to the restrictions:

- 1. Steady flow.
- 2. No friction.
- 3. Flow along a streamline.
- 4. Incompressible flow.

Example 4.9 NOZZLE FLOW: APPLICATION OF BERNOULLI EQUATION



Water flows steadily through a horizontal nozzle, discharging to the atmosphere. At the nozzle inlet the diameter is D_1; at the nozzle outlet the diameter is D_2. Derive an expression for the minimum gage pressure required at the nozzle inlet to produce a given volume flow rate, Q. Evaluate the inlet gage pressure if D_1=75 mm, D_2=25 mm, and the desired flow rate is 0.02 m^3/s.

Assmp: (1) Steady flow (given).
(2) Incompressible flow.
(3) Frictionless flow.
(4) Flow along a streamline.
(5) $z_1 = z_2$:
(6) Uniform flow at sections 1 and 2

Bernoulli equation along a streamline between 1 and 2 to evaluate p_1.

$$p_{1_g} = p_1 - p_{\mathrm{atm}} = p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho}{2}V_1^2\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right]$$

$$(-\rho V_1 A_1) + (\rho V_2 A_2) = 0 \qquad \text{or} \qquad V_1 A_1 = V_2 A_2 = Q$$

$$p_{1_g} = \frac{\rho Q^2}{2A_1^2}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

$$p_{1_g} = \frac{8\rho\, Q^2}{\pi^2\, D_1^4}\left[\left(\frac{D_1}{D_2}\right)^4 - 1\right]$$

$$p_{1_g} = \frac{8}{\pi^2} \times 1000\frac{\mathrm{kg}}{\mathrm{m}^3} \times \frac{1}{(0.075)^4\mathrm{m}^4} \times Q^2[(3.0)^4 - 1]\frac{\mathrm{N}\cdot\mathrm{s}^2}{\mathrm{kg}\cdot\mathrm{m}} \times \frac{\mathrm{Pa}\cdot\mathrm{m}^2}{\mathrm{N}^2}$$

$$p_{1_g} = 2049.44 \times 10^6\, Q^2\frac{\mathrm{N}\cdot\mathrm{s}^2}{\mathrm{m}^8} \times \frac{\mathrm{Pa}\cdot\mathrm{m}^2}{\mathrm{N}}$$

With $Q = 0.02\,\mathrm{m}^3/\mathrm{s}$, then $p_{1_g} = 819,776\,\mathrm{kPa}$

✓Special Case: Bernoulli Equation

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

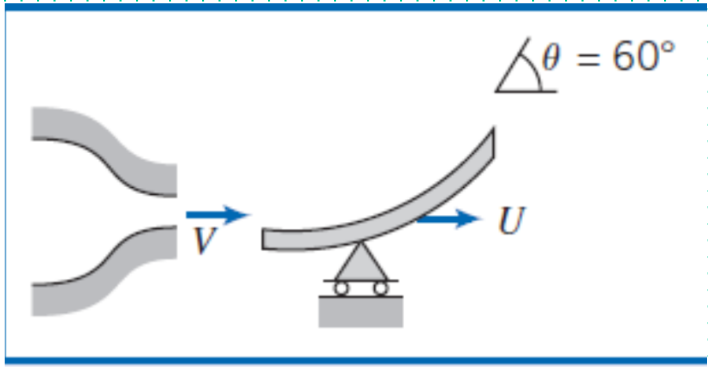
1. Steady Flow
2. No Friction
3. Flow Along a Streamline
4. Incompressible Flow

Momentum Equation for Inertial Control Volume

✓ **Special Case: Control Volume Moving with Constant Velocity**

$$\vec{F} = \vec{F}_s + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\,\rho\,d\forall + \int_{CS} \vec{V}_{xyz}\,\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Example 4.10 VANE MOVING WITH CONSTANT VELOCITY



The sketch shows a vane with a turning angle of 60°. The vane moves at constant speed, U=10 m/s, and receives a jet of water that leaves a stationary nozzle with speed V=30 m/s. The nozzle has an exit area of 0.003 m². Determine the force components that act on the vane.

Assmp: (1) Flow is steady relative to the vane.
(2) Magnitude of relative velocity along the vane is constant
(3) Properties are uniform at sections 1 and 2 .
(4) $F_{Bx}=0$:
(5) Incompressible flow.

The x component of the momentum equation is

$$F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}}\int_{CV} u_{xyz}\,\rho\, d\forall + \int_{CS} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

$$R_x = \int_{A_1} u(-\rho V dA) + \int_{A_2} u(\rho V dA) = +u_1(-\rho V_1 A_1) + u_2(\rho V_2 A_2)$$

$$\int_{A_1} (-\rho V dA) + \int_{A_2} (\rho V dA) = (-\rho V_1 A_1) + (\rho V_2 A_2) = 0$$

$$\rho V_1 A_1 = \rho V_2 A_2$$

$$\boxed{R_x = (u_2 - u_1)(\rho V_1 A_1)}$$

$$R_x = [(V-U)\cos\theta - (V-U)](\rho(V-U)A_1) = (V-U)(\cos\theta - 1)\{\rho(V-U)A_1\}$$

$$= (30-10)\frac{\text{m}}{\text{s}} \times (0.50-1) \times \left(999\frac{\text{kg}}{\text{m}^3}(30-10)\frac{\text{m}}{\text{s}} \times 0.003\,\text{m}^2\right) \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$R_x = -599\text{N}\ \{\text{to the left}\}$$

The y component of the momentum equation is

$$F_{S_y} + F_{B_y} = \overset{=0(1)}{\cancel{\frac{\partial}{\partial t}\int_{CV}}} v_{xyz}\rho \, d\forall + \int_{CS} v_{xyz}\rho \, \vec{V}_{xyz} \cdot d\vec{A}$$

$$R_y - Mg = \int_{CS} v\rho\vec{V} \cdot d\vec{A} = \int_{A_2} v\rho\vec{V} \cdot d\vec{A} \qquad \{v_1 = 0\}$$

$$= \int_{A_2} v(\rho V dA) = v_2(\rho V_2 A_2) = v_2(\rho V_1 A_1)$$

$$= (V - U)\sin\theta\{\rho(V - U)A_1\}$$

$$= (30 - 10)\frac{\text{m}}{\text{s}} \times (0.866) \times \left((999)\frac{\text{kg}}{\text{n}^3}(30 - 10)\frac{\text{m}}{\text{s}} \times 0.003\text{m}^2 \right)$$

$$R_y - Mg = 1.04\,\text{kN} \qquad \{\text{upward}\}$$

$$R_y = 1.04\,\text{kN} + Mg \qquad \{\text{upward}\}$$

$$\vec{R} = -0.599\hat{i} + 1.04\hat{j}\,\text{kN}$$

Momentum Equation for Inertial Control Volume with Rectilinear Acceleration

$$\vec{F}_S + \vec{F}_B - \int_{CV} \vec{a}_{rf}\, \rho\, d\forall =$$

$$\frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\, \rho\, d\forall + \int_{CS} \vec{V}_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations.

$$F_{S_x} + F_{B_x} - \int_{CV} a_{rf_x}\, \rho d\forall = \frac{\partial}{\partial t}\int_{CV} u_{xyz}\, \rho d\forall + \int_{CS} u_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

$$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\, \rho d\forall = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\, \rho d\forall + \int_{CS} v_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

$$F_{S_z} + F_{B_z} - \int_{CV} a_{rf_z}\, \rho d\forall = \frac{\partial}{\partial t}\int_{CV} w_{xyz}\, \rho d\forall + \int_{CS} w_{xyz}\, \rho \vec{V}_{xyz} \cdot d\vec{A}$$

The Angular Momentum Principle

✓ Basic Law, and Transport Theorem

$$\vec{T} = \left.\frac{d\vec{H}}{dt}\right)_{\text{system}}$$

$$\left.\frac{d\vec{H}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{r} \times \vec{V}\,\rho\, d\forall + \int_{\text{CS}} \vec{r} \times \vec{V}\,\rho \vec{V} \cdot d\vec{A}$$

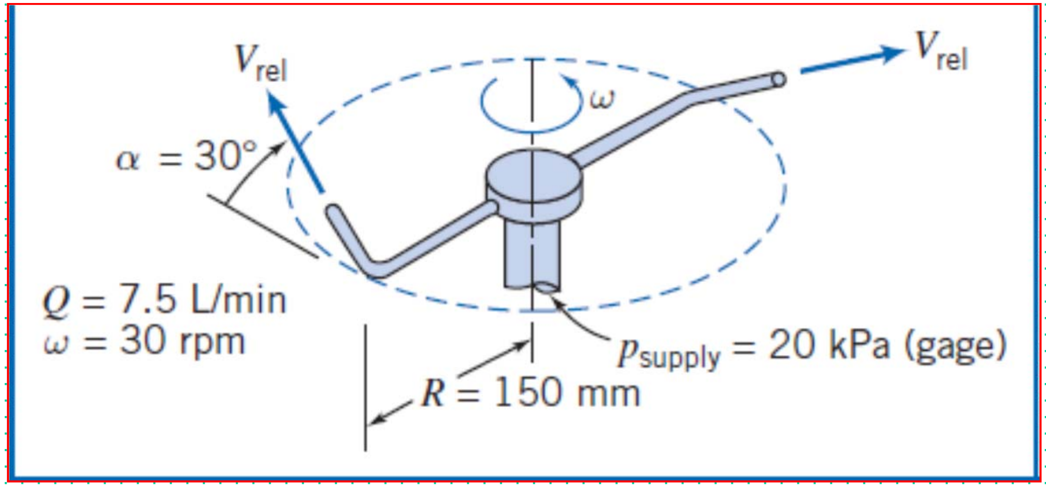
The Angular Momentum Principle

$$\vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g}\, dm + \vec{T}_{\text{shaft}} =$$

$$\frac{\partial}{\partial t} \int_{\text{CV}} \vec{r} \times \vec{V}\, \rho\, d\forall + \int_{\text{CS}} \vec{r} \times \vec{V}\, \rho \vec{V} \cdot d\vec{A}$$

Example 4.14 LAWN SPRINKLER: ANALYSIS USING FIXED CONTROL VOLUME

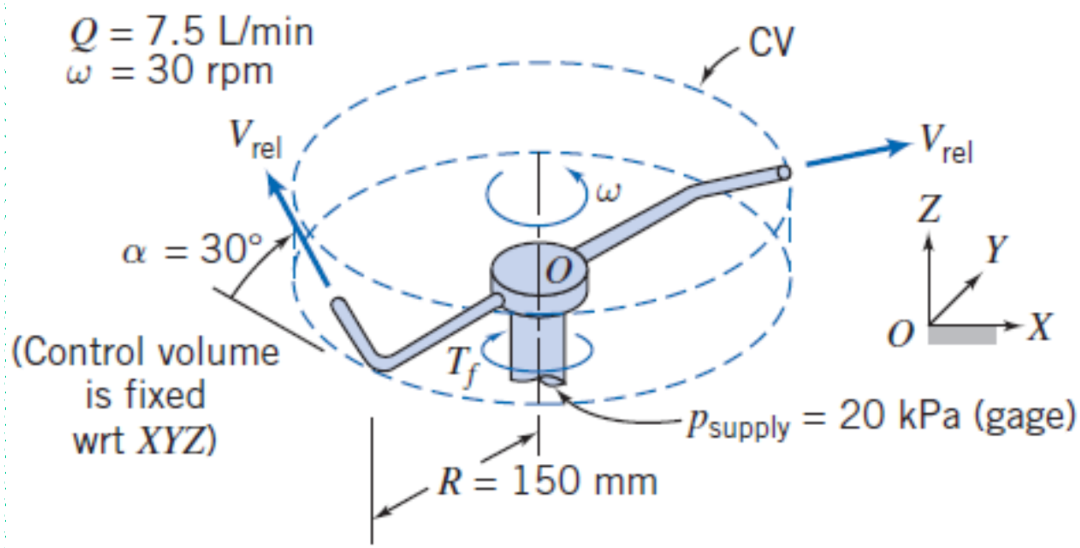
A small lawn sprinkler is shown in the sketch at right. At an inlet gage pressure of 20 kPa, the total volume flow rate of water through the sprinkler is 7.5 liters per minute and it rotates at 30 rpm. The diameter of each jet is 4 mm. Calculate the jet speed relative to each sprinkler nozzle. Evaluate the friction torque at the sprinkler pivot.



Asmt: (1) Incompressible flow.
(2) Uniform flow at each section.
(3) ω= constant:

$$= 0(1)$$

$$\cancel{\frac{\partial}{\partial t}}\int_{CV} \rho\, dV + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

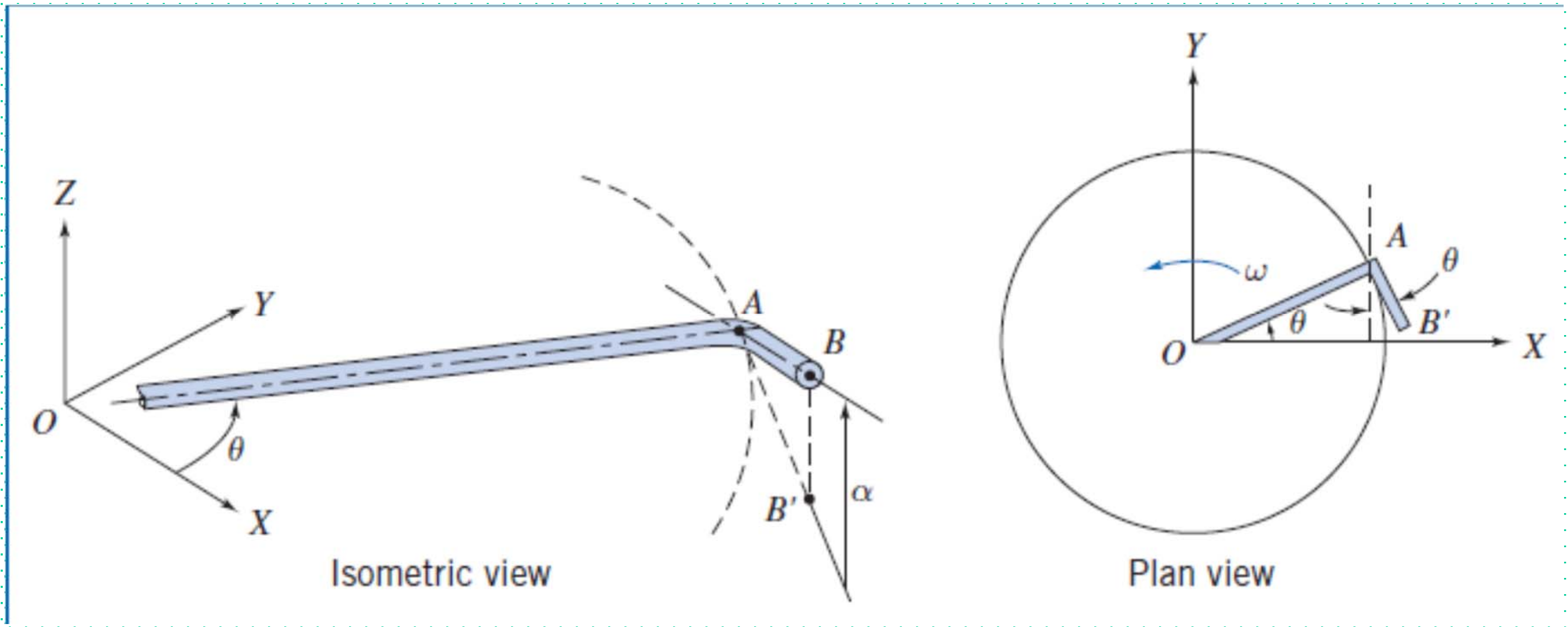


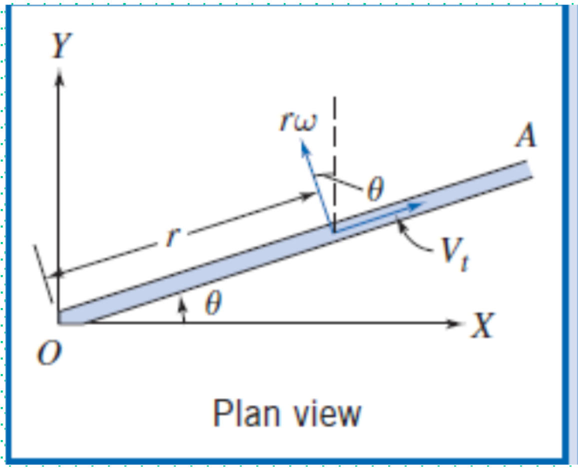
$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g}\, \rho\, dV + \vec{T}_{shaft} = \frac{\partial}{\partial t}\int_{CV} \vec{r} \times \vec{V} \rho\, dV + \int_{CS} \vec{r} \times \vec{V}\, \rho \vec{V} \cdot d\vec{A}$$

$$V_{rel} = \frac{Q}{2A_{jet}} = \frac{Q}{2}\frac{4}{\pi D_{jet}^2}$$

$$= \frac{1}{2} \times 7.5 \frac{L}{min} \times \frac{4}{\pi} \frac{1}{(4)^2\, mm^2} \times \frac{m^3}{1000\, L} \times 10^6 \frac{mm^2}{m^2} \times \frac{min}{60\, s}$$

$$V_{rel} = 4.97\ m/s \longleftarrow$$





$$\vec{V} = \hat{I}\,(V_t \cos\theta - r\omega \sin\theta) + \hat{J}\,(V_t \sin\theta + r\omega \cos\theta)$$

$$\vec{r} = \hat{I}\,r\cos\theta + \hat{J}\,r\sin\theta$$

$$\vec{r}\times\vec{V} = \hat{K}\,(r^2\omega\cos^2\theta + r^2\omega\sin^2\theta) = \hat{K}\,r^2\omega$$

$$\int_{\forall_{OA}} \vec{r} \times \vec{V}\, \rho\, d\forall = \int_O^R \hat{K} r^2 \omega \rho A\, dr = \hat{K} \frac{R^3 \omega}{3} \rho A$$

$$\frac{\partial}{\partial t} \int_{\forall_{OA}} \vec{r} \times \vec{V} \rho\, d\forall = \frac{\partial}{\partial t}\left[\hat{K} \frac{R^3 \omega}{3} \rho A\right] = 0$$

$$\vec{r}_{\text{jet}} = \vec{r}_B \approx \vec{r}\big|_{r=R} = (\hat{I} r \cos\theta + \hat{J} r \sin\theta)\big|_{r=R} = \hat{I} R \cos\theta + \hat{J} R \sin\theta$$

$$\vec{V}_j = \vec{V}_{\text{rel}} + \vec{V}_{\text{tip}} = \hat{I} V_{\text{rel}} \cos\alpha \sin\theta - \hat{J} V_{\text{rel}} \cos\alpha \cos\theta + \hat{K} V_{\text{rel}} \sin\alpha - \hat{I} \omega R \sin\theta + \hat{J} \omega R \cos\theta$$

$$\vec{V}_j = \hat{I}(V_{\text{rel}} \cos\alpha - \omega R) \sin\theta - \hat{J}(V_{\text{rel}} \cos\alpha - \omega R) \cos\theta + \hat{K} V_{\text{rel}} \sin\alpha$$

$$\vec{r}_B \times \vec{V}_j = \hat{I} R V_{\text{rel}} \sin\alpha \sin\theta - \hat{J} R V_{\text{rel}} \sin\alpha \cos\theta - \hat{K} R (V_{\text{rel}} \cos\alpha - \omega R)(\sin^2\theta + \cos^2\theta)$$

$$\vec{r}_B \times \vec{V}_j = \hat{I} R V_{\text{rel}} \sin\alpha \sin\theta - \hat{J} R V_{\text{rel}} \sin\alpha \cos\theta - \hat{K} R (V_{\text{rel}} \cos\alpha - \omega R)$$

$$\int_{\text{CS}} \vec{r} \times \vec{V}_j \rho \vec{V} \cdot d\vec{A} = \left[\hat{I} R V_{\text{rel}} \sin\alpha \sin\theta - \hat{J} R V_{\text{rel}} \sin\alpha \cos\theta - \hat{K} R (V_{\text{rel}} \cos\alpha - \omega R)\right] \rho \frac{Q}{2}$$

$$\int_{CS} \vec{r} \times \vec{V}_j \rho \vec{V} \cdot d\vec{A} = -\hat{K}R(V_{rel}\cos\alpha - \omega R)\rho Q$$

$$-T_f\hat{K} = -\hat{K}R(V_{rel}\cos\alpha - \omega R)\rho Q$$

$$T_f = R(V_{rel}\cos\alpha - \omega R)\rho Q$$

$$\omega R = 30\,\frac{\text{rev}}{\text{min}} \times 150\text{ mm} \times 2\pi\,\frac{\text{rad}}{\text{rev}} \times \frac{\text{min}}{60\text{ s}} \times \frac{\text{m}}{1000\text{ mm}} = 0.471\text{ m/s}$$

$$T_f = 150\text{ mm} \times \left(4.97\,\frac{\text{m}}{\text{s}} \times \cos 30^\circ - 0.471\,\frac{\text{m}}{\text{s}}\right) 999\,\frac{\text{kg}}{\text{m}^3} \times 7.5\,\frac{\text{L}}{\text{min}} \times \frac{\text{m}^3}{1000\text{ L}} \times \frac{\text{min}}{60\text{ s}} \times \frac{\text{N}\cdot\text{s}^3}{\text{kg}\cdot\text{m}} \times \frac{\text{m}}{1000\text{ mm}}$$

$$T_f = 0.0718\text{ N}\cdot\text{m}$$

The First Law of Thermodynamics

✓ **Basic Law, and Transport Theorem**

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{\text{system}}$$

$$\left.\frac{dE}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\,d\forall + \int_{\text{CS}} e\,\rho\vec{V}\cdot d\vec{A}$$

The First Law of Thermodynamics

$$\dot{Q} - \dot{W}_s - \dot{W}_{\text{shear}} - \dot{W}_{\text{other}} =$$

$$\frac{\partial}{\partial t}\int_{\text{CV}} e\,\rho\,d\forall + \int_{\text{CS}}\left(u + pv + \frac{V^2}{2} + gz\right)\rho\vec{V}\cdot d\vec{A}$$

Work Involves

- ✓ **Shaft Work**
- ✓ **Work by Shear Stresses at the Control Surface**
- ✓ **Other Work**

The Second Law of Thermodynamics

✓ **Basic Law, and Transport Theorem**

$$\left.\frac{dS}{dt}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q}$$

$$\left.\frac{dS}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} s\,\rho\,d\forall + \int_{\text{CS}} s\,\rho\vec{V}\cdot d\vec{A}$$

The Second Law of Thermodynamics

$$\frac{\partial}{\partial t}\int_{CV} s\,\rho\,d\forall + \int_{CS} s\,\rho\,\vec{V}\cdot d\vec{A} \geq \int_{CS} \frac{1}{T}\left(\frac{\dot{Q}}{A}\right) dA$$

Introduction to Fluid Mechanics

Chapter 5

Introduction to Differential Analysis of Fluid Motion

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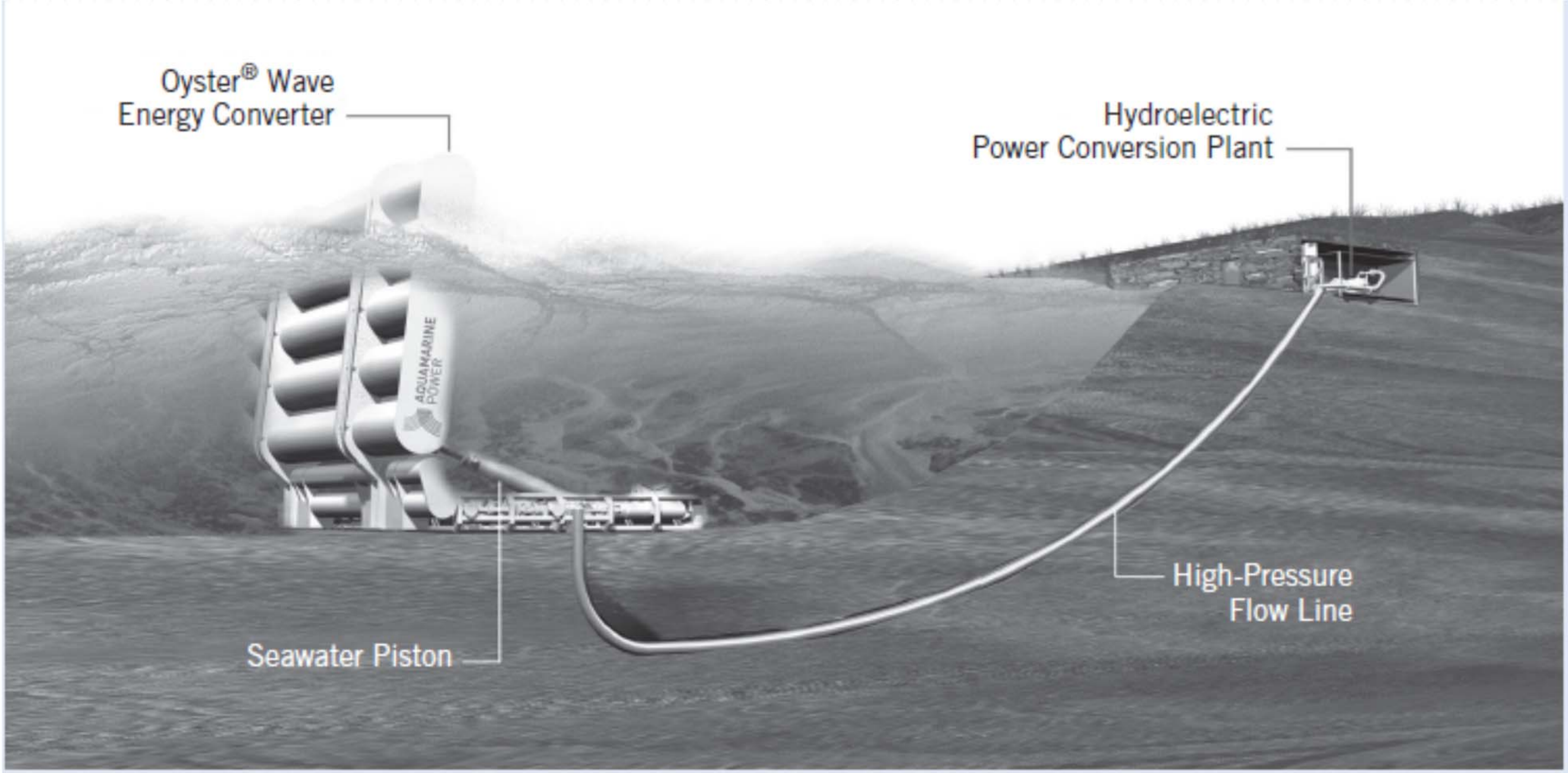


A schematic of Aquamarine's Oyster device

Main Topics

- ✓ Conservation of Mass
- ✓ Stream Function for Two-Dimensional Incompressible Flow
- ✓ Motion of a Fluid Particle (Kinematics)
- ✓ Momentum Equation

Conservation of Mass

✓ Basic Law for a System

$$\frac{\partial}{\partial t}\int_{\mathrm{CV}} \rho \, d\forall + \int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = 0$$

$$\begin{bmatrix}\text{Net rate of mass flux out} \\ \text{through the control surface}\end{bmatrix}$$

$$+\begin{bmatrix}\text{Rate of change of mass} \\ \text{inside the control volume}\end{bmatrix} = 0$$

Conservation of Mass

✓ Rectangular Coordinate System

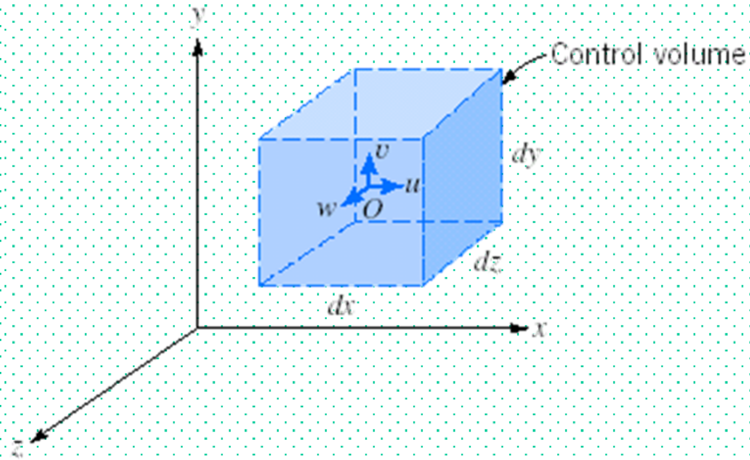


Fig. 5.1 Differential control volume in rectangular coordinates.

$$\rho\Big)_{x+dx/2} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$$

$$u\Big)_{x+dx/2} = u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$$

$$\rho\Big)_{x-dx/2} = \rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$$

$$u\Big)_{x-dx/2} = u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$$

Conservation of Mass

✓ Rectangular Coordinate System

$$\int_{\mathrm{CS}} \rho \vec{V} \cdot d\vec{A} = \left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right] dx\, dy\, dz$$

$$\frac{\partial}{\partial t} \int_{\mathrm{CV}} \rho\, d\forall = \frac{\partial \rho}{\partial t} dx\, dy\, dz$$

Conservation of Mass

✓ **Rectangular Coordinate System**

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

"Continuity Equation"

Conservation of Mass

✓ Rectangular Coordinate System

$$\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$$

"Del" Operator

Conservation of Mass

✓ **Rectangular Coordinate System**

$$\nabla \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

Conservation of Mass

✓ Rectangular Coordinate System

Incompressible Fluid:

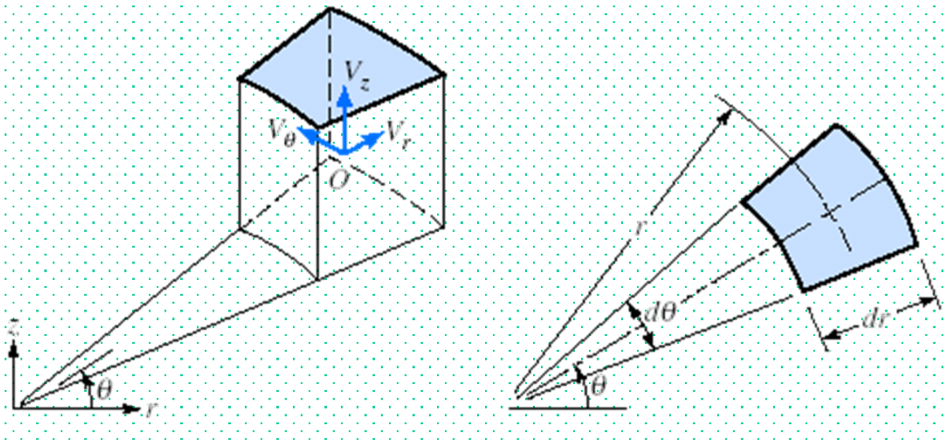
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} = 0$$

Steady Flow:

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \nabla \cdot \rho \vec{V} = 0$$

Conservation of Mass

✓Cylindrical Coordinate System



(*a*) Isometric view (*b*) Projection on $r\theta$ plane

Fig. 5.2 Differential control volume in cylindrical coordinates.

Conservation of Mass

✓ **Cylindrical Coordinate System**

$$\frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta} + \frac{\partial(\rho V_z)}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Conservation of Mass

✓ **Cylindrical Coordinate System**

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{k} \frac{\partial}{\partial z}$$

"Del" Operator

Conservation of Mass

✓ **Cylindrical Coordinate System**

$$\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$$

Conservation of Mass

✓ **Cylindrical Coordinate System**

Incompressible Fluid:

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\partial V_z}{\partial z}=\nabla\cdot\vec{V}=0$$

Steady Flow:

$$\frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta}+\frac{\partial(\rho V_z)}{\partial z}=\nabla\cdot\rho\vec{V}=0$$

Stream Function for Two-Dimensional Incompressible Flow

✓ **Two-Dimensional Flow**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

✓ **Stream Function ψ**

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x}$$

Stream Function for Two-Dimensional Incompressible Flow

- ✓ **Cylindrical Coordinates**

$$\frac{\partial(rV_r)}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0$$

- ✓ **Stream Function $\psi(r,\theta)$**

$$V_r \equiv \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad V_\theta \equiv -\frac{\partial \psi}{\partial r}$$

Motion of a Fluid Particle (Kinematics)

- ✓ Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field
- ✓ Fluid Rotation
- ✓ Fluid Deformation
  - Angular Deformation
  - Linear Deformation

Motion of a Fluid Particle (Kinematics)

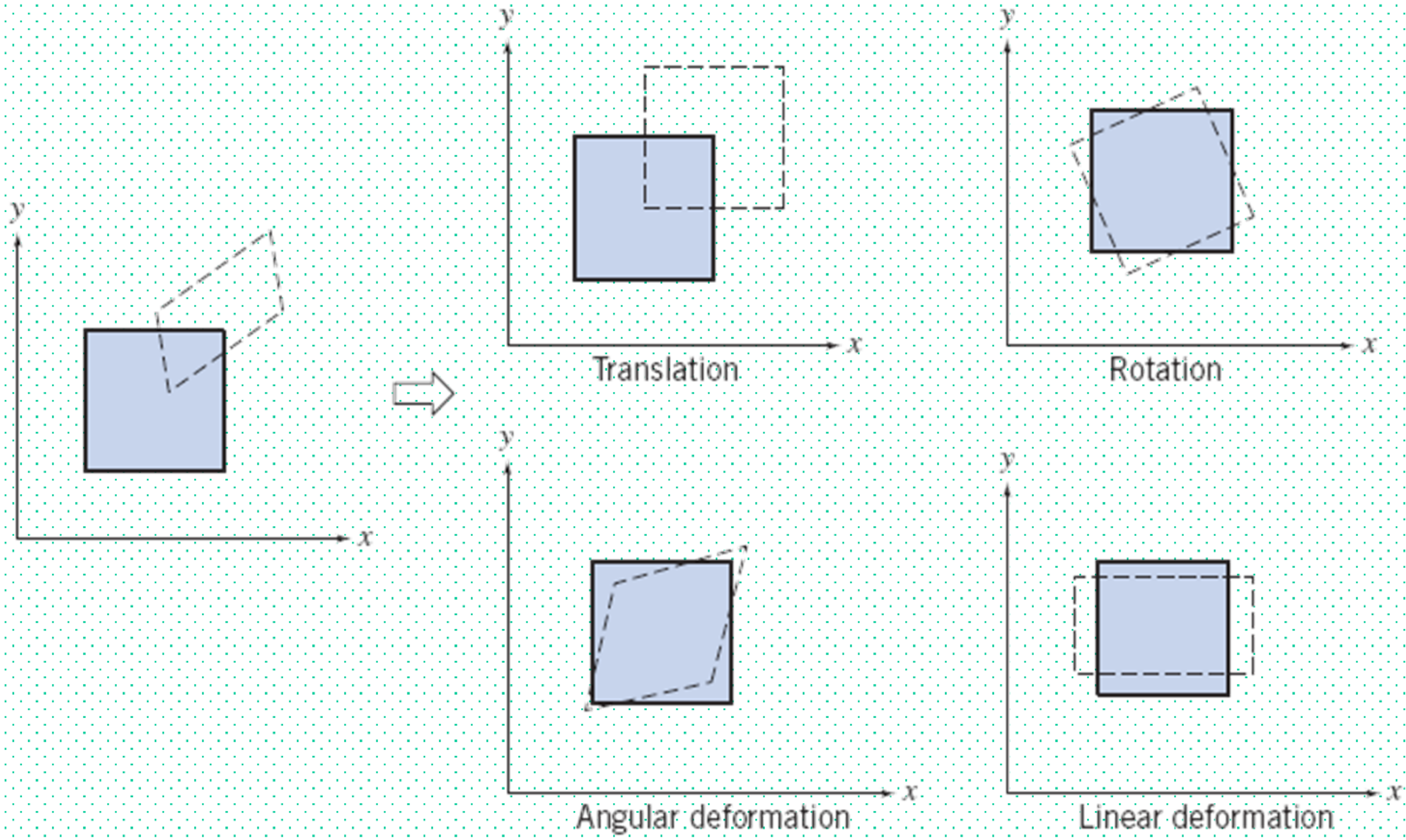


Fig. 5.5 Pictorial representation of the components of fluid motion.

Motion of a Fluid Particle (Kinematics)

✓ **Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field**

$$\frac{D\vec{V}}{Dt} \equiv \vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

Motion of a Fluid Particle (Kinematics)

✓ **Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field**

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\ \text{acceleration}\\ \text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\ \text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\ \text{acceleration}}}$$

Motion of a Fluid Particle (Kinematics)

✓ **Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field**

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}$$

$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

Motion of a Fluid Particle (Kinematics)

✓ **Fluid Translation: Acceleration of a Fluid Particle in a Velocity Field (Cylindrical)**

$$a_{r_p} = V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta^2}{r} + V_z \frac{\partial V_r}{\partial z} + \frac{\partial V_r}{\partial t}$$

$$a_{\theta_p} = V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z \frac{\partial V_\theta}{\partial z} + \frac{\partial V_\theta}{\partial t}$$

$$a_{z_p} = V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z} + \frac{\partial V_z}{\partial t}$$

Motion of a Fluid Particle (Kinematics)

✓ Fluid Rotation

$$\vec{\omega} = \hat{i}\,\omega_x + \hat{j}\omega_y + \hat{k}\omega_z$$

Motion of a Fluid Particle (Kinematics)

✓ Fluid Rotation

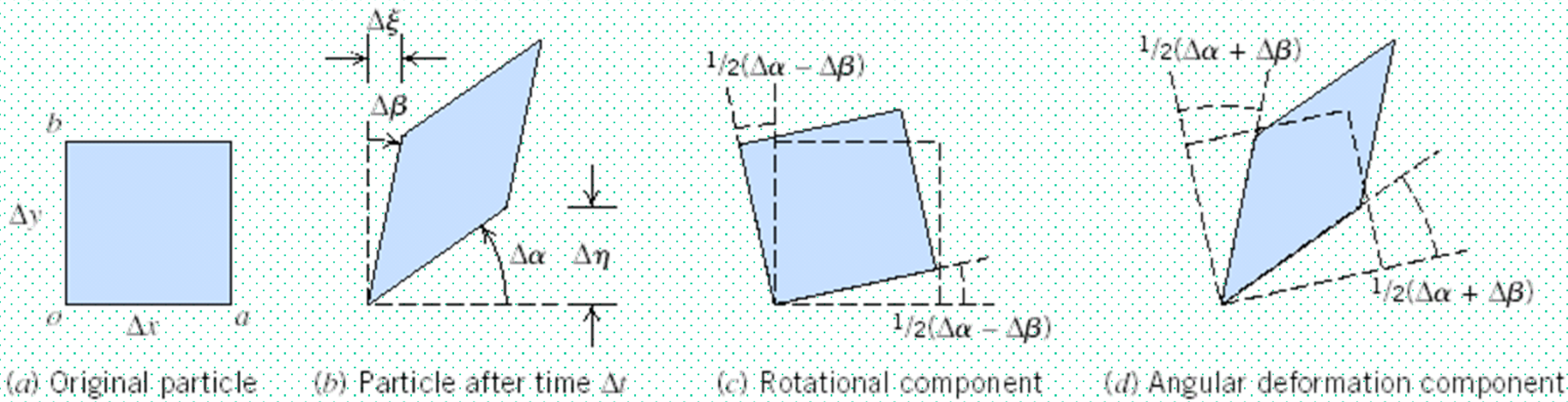


Fig. 5.7 Rotation and angular deformation of perpendicular line segments in a two-dimensional flow.

Motion of a Fluid Particle (Kinematics)

✓ Fluid Rotation

$$\vec{\omega} = \frac{1}{2}\left[\hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V}$$

Motion of a Fluid Particle (Kinematics)

✓ Fluid Deformation:

- **Angular Deformation**

$$\begin{array}{l}\text{Rate of angular} \\ \text{deformation} \\ \text{in } xy \text{ plane}\end{array} = \lim_{\Delta t \to 0} \frac{\left(\dfrac{\partial v}{\partial x} \dfrac{\Delta x}{\Delta x} \Delta t + \dfrac{\partial u}{\partial y} \dfrac{\Delta y}{\Delta y} \Delta t \right)}{\Delta t}$$

$$= \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)$$

Motion of a Fluid Particle (Kinematics)

✓ **Fluid Deformation:**

- **Angular Deformation**

$$\text{Rate of angular deformation in } yz \text{ plane } = \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)$$

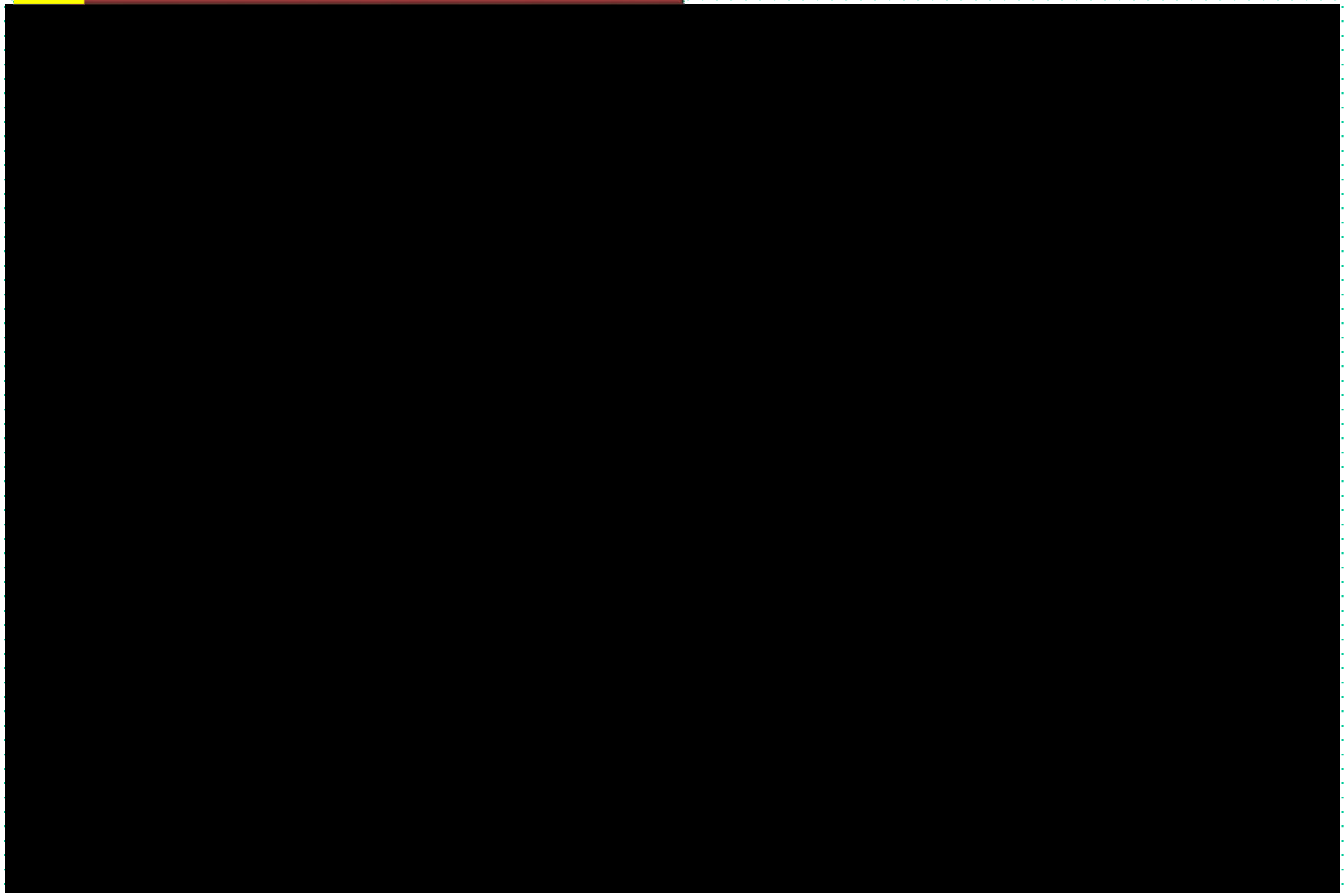
$$\text{Rate of angular deformation in } zx \text{ plane } = \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

Motion of a Fluid Particle (Kinematics)

- **Fluid Deformation:**
  - **Linear Deformation**

$$\text{Volume dilation rate} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V}$$

Conservation of Momentum



Momentum Equation

✓ **Newton's Second Law**

$$d\vec{F} = dm\,\frac{D\vec{V}}{Dt} = dm\left[u\,\frac{\partial \vec{V}}{\partial x} + v\,\frac{\partial \vec{V}}{\partial y} + w\,\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}\right]$$

Momentum Equation

✓ Forces Acting on a Fluid Particle

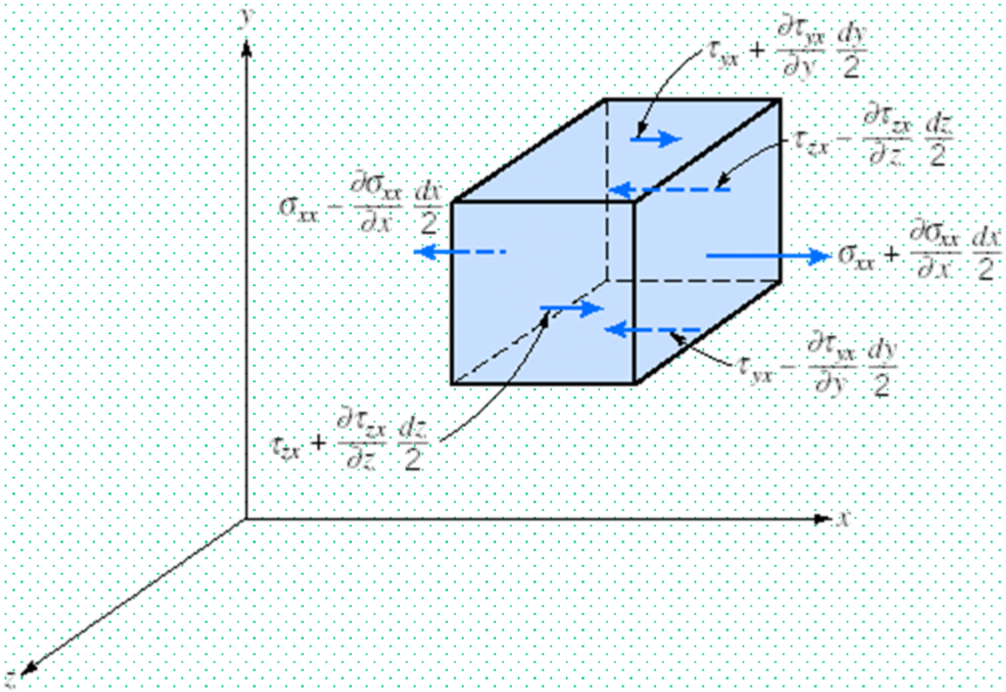


Fig. 5.9 Stresses in the *x* direction on an element of fluid.

Momentum Equation

✓ Forces Acting on a Fluid Particle

$$dF_{B_x} + dF_{S_x} = \left(\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) dx\, dy\, dz$$

$$dF_{B_y} + dF_{S_y} = \left(\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) dx\, dy\, dz$$

$$dF_{B_z} + dF_{S_z} = \left(\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \right) dx\, dy\, dz$$

Momentum Equation

✓ Differential Momentum Equation

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Momentum Equation

✓ **Newtonian Fluid: Navier-Stokes Equations**

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Momentum Equation

✓ **Special Case: Euler's Equation**

$$\rho \frac{D\vec{V}}{Dt} = \rho \vec{g} - \nabla p$$

Computational Fluid Dynamics

✓ Some Applications

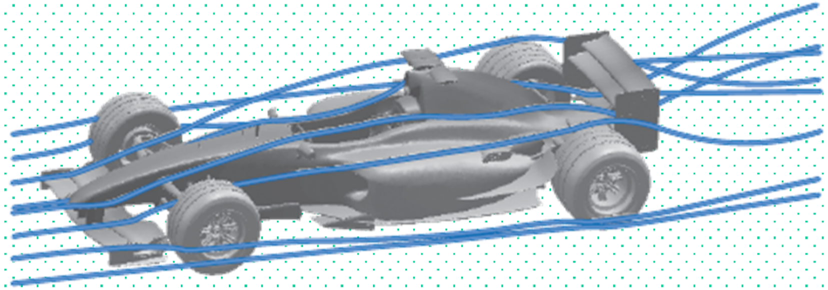


Fig. 5.10 Pathlines around a Formula1 Car (image courtesy of ANSYS, Inc. © 2008).

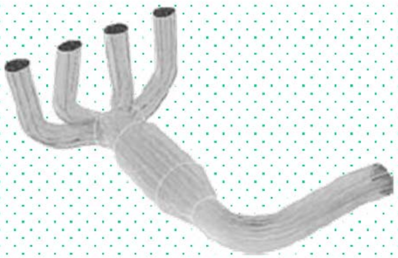


Fig. 5.11 Flow through a catalytic converter (image courtesy of ANSYS, Inc. © 2008).

Fig. 5.12 Static pressure contours for flow through a centrifugal fan (image courtesy of ANSYS, Inc. © 2008).

Computational Fluid Dynamics

✓ **Discretization**

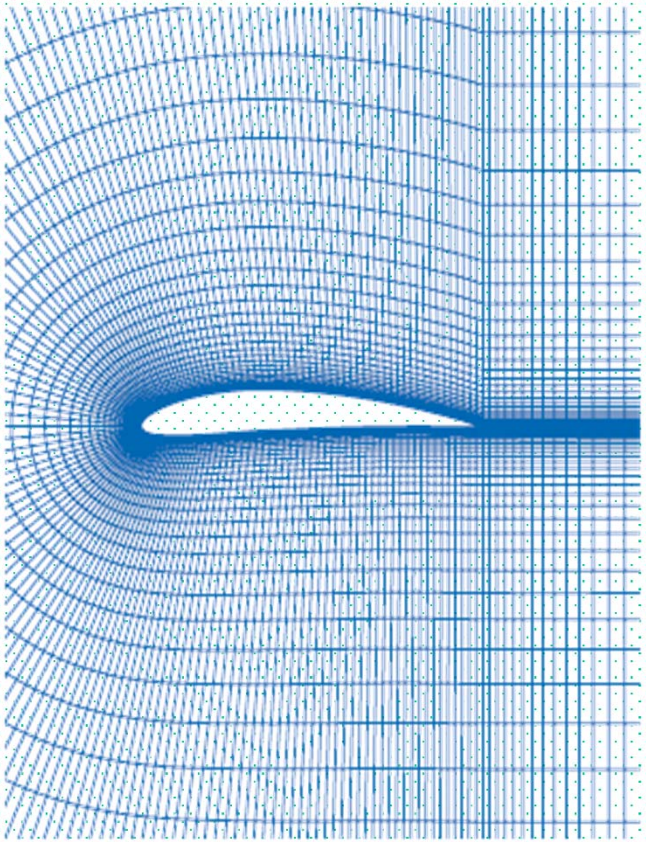


Fig. 5.14 Example of a grid used to solve for the flow around an airfoil.

Introduction to Fluid Mechanics

Chapter 6

Incompressible Inviscid Flow

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Main Topics

- Momentum Equation for Frictionless Flow: Euler's Equation
- Euler's Equation in Streamline Coordinates
- Bernoulli Equation – Integration of Euler's Equation Along a Streamline for Steady Flow
- The Bernoulli Equation Interpreted as an Energy Equation
- Energy Grade Line and Hydraulic Grade Line

Momentum Equation for Frictionless Flow: Euler's Equation

✓ **Euler's Equation**

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p$$

✓ **Continuity**

$$\nabla \cdot \vec{V} = 0$$

Momentum Equation for Frictionless Flow: Euler's Equation

✓ Rectangular Coordinates

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}$$

Momentum Equation for Frictionless Flow: Euler's Equation

✓ **Cylindrical Coordinates**

$$\rho a_r = \rho\left(\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\theta^2}{r}\right) = \rho g_r - \frac{\partial p}{\partial r}$$

$$\rho a_\theta = \rho\left(\frac{\partial V_\theta}{\partial t} + V_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z \frac{\partial V_\theta}{\partial z} + \frac{V_r V_\theta}{r}\right) = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$\rho a_z = \rho\left(\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial V_z}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z}$$

Euler's Equation in Streamline Coordinates

✓ **Along a Streamline (Steady Flow, ignoring body forces)**

$$\frac{1}{\rho}\frac{\partial p}{\partial s} = -V\frac{\partial V}{\partial s}$$

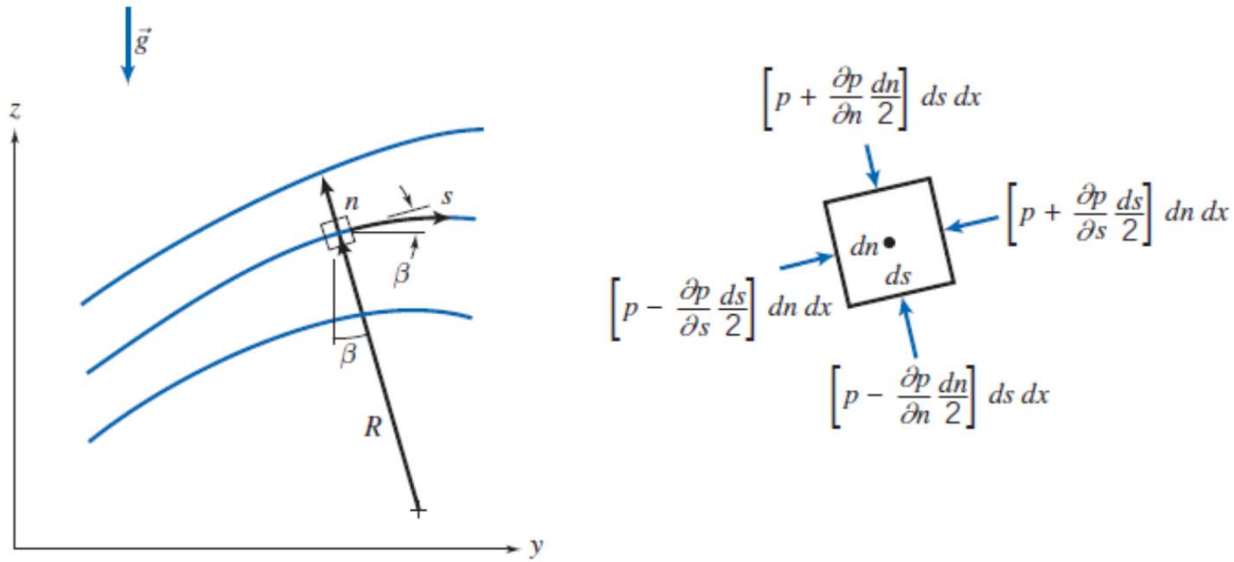


Fig. 6.1 Fluid particle moving along a streamline.

$$\frac{1}{\rho}\frac{\partial p}{\partial s} = -V\frac{\partial V}{\partial s}$$

which indicates that (for an incompressible, inviscid flow) a decrease in velocity is accompanied by an increase in pressure and conversely.

The only force experienced by the particle is the net pressure force, so the particle accelerates toward low-pressure regions and decelerates when approaching high-pressure regions.

✓ Normal to the Streamline (Steady Flow, ignoring body forces)

For steady flow in a horizontal plane, Euler's equation normal to a streamline becomes

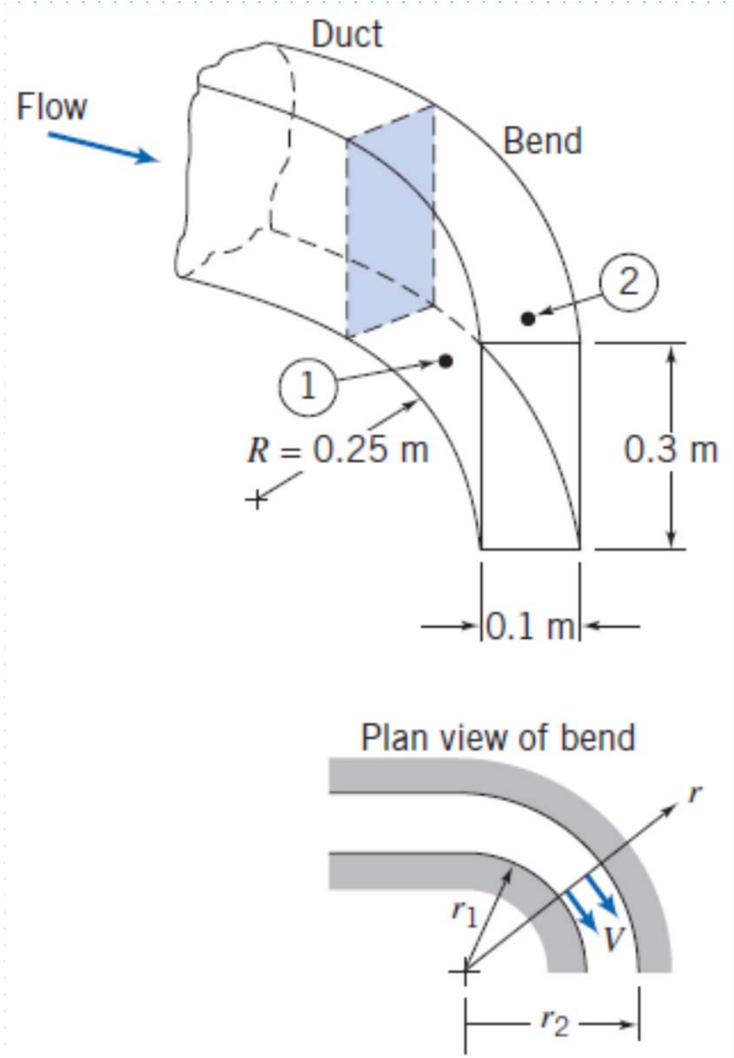
$$\frac{1}{\rho}\frac{\partial p}{\partial n} = \frac{V^2}{R}$$

indicates that pressure increases in the direction outward from the center of curvature of the streamlines.

Because the only force experienced by the particle is the net pressure force, the pressure field creates the centripetal acceleration. In regions where the streamlines are straight, the radius of curvature, R, is infinite so there is *no pressure variation normal to straight streamlines*.

Example 6.1 FLOW IN A BEND

The flow rate of air at standard conditions in a flat duct is to be determined by installing pressure taps across a bend. The duct is 0.3 m deep and 0.1 m wide. The inner radius of the bend is 0.25 m. If the measured pressure difference between the taps is 40 mm of water, compute the approximate flow rate.



Assumptions: (1) Frictionless flow.
(2) Incompressible flow.
(3) Uniform flow at measurement section.

$$p_2 - p_1 = \rho_{H_2O} g \, \Delta h$$

$$\frac{\partial p}{\partial r} = \frac{dp}{dr} = \frac{\rho V^2}{r}$$

$$dp = \rho V^2 \frac{dr}{r}$$

$$p_2 - p_1 = \rho V^2 \ln r \Big]_{r_1}^{r_2} = \rho V^2 \ln \frac{r_2}{r_1}$$

$$V = \left[\frac{p_2 - p_1}{\rho \ln(r_2/r_1)}\right]^{1/2}$$

$$\Delta p = p_2 - p_1 = \rho_{H_2O} g \Delta h$$

$$V = \left[\frac{\rho_{H_2O} g \Delta h}{\rho \ln(r_2/r_1)}\right]^{1/2}$$

$$V = \left[999 \frac{\text{kg}}{\text{m}^3} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 0.04 \text{ m} \times \frac{\text{m}^3}{1.23 \text{ kg}} \times \frac{1}{\ln(0.35 \text{ m}/0.25 \text{ m})}\right]^{1/2}$$

$$= 30.8 \text{ m/s}$$

$$Q = VA = 30.8 \frac{\text{m}}{\text{s}} \times 0.1 \text{ m} \times 0.3 \text{ m}$$

$$Q = 0.924 \text{ m}^3/\text{s}$$

Bernoulli Equation – Integration of Euler's Equation Along a Streamline for Steady Flow

- ✓ **Euler's Equation in Streamline Coordinates (assuming Steady Flow)**

$$\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} = V\frac{\partial V}{\partial s}}$$

$$\frac{\partial p}{\partial s}ds = dp \qquad \text{(the change in pressure along } s\text{)}$$

$$\frac{\partial z}{\partial s}ds = dz \qquad \text{(the change in elevation along } s\text{)}$$

$$\frac{\partial V}{\partial s}ds = dV \qquad \text{(the change in speed along } s\text{)}$$

Bernoulli Equation – Integration of Euler's Equation Along a Streamline for Steady Flow

✓ **Integration Along *s* Coordinate**

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

Bernoulli Equation – Integration of Euler's Equation Along a Streamline for Steady Flow

✓ **Bernoulli Equation**

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

1. **Steady Flow**
2. **No Friction**
3. **Flow Along a Streamline**
4. **Incompressible Flow**

Bernoulli Equation – Integration of Euler's Equation Along a Streamline for Steady Flow

- ✓ **Static, Stagnation, and Dynamic Pressures (Ignore Gravity)**

$$p_0 = p + \frac{1}{2}\rho V^2$$

Stagnation **Static** **Dynamic**

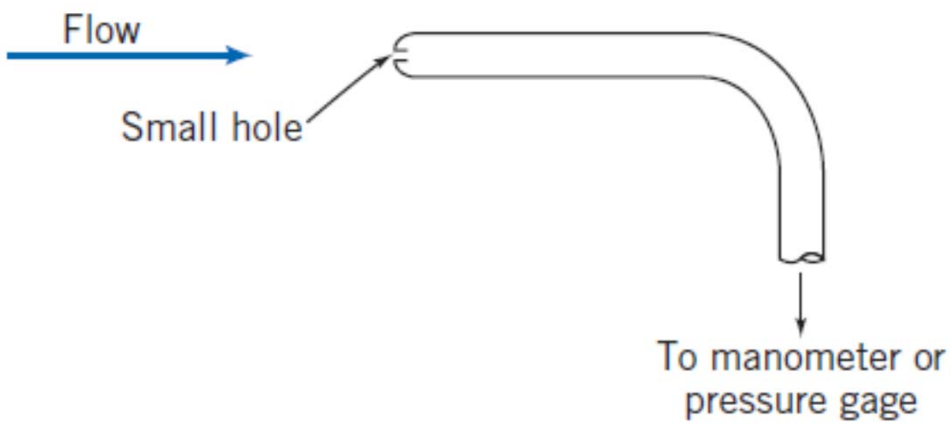


Fig. 6.3 Measurement of stagnation pressure.

Stagnation pressure is measured in the laboratory using a probe with a hole that faces directly upstream. Such a probe is called a stagnation pressure probe, or pitot tube. The measuring section must be aligned with the local flow direction.

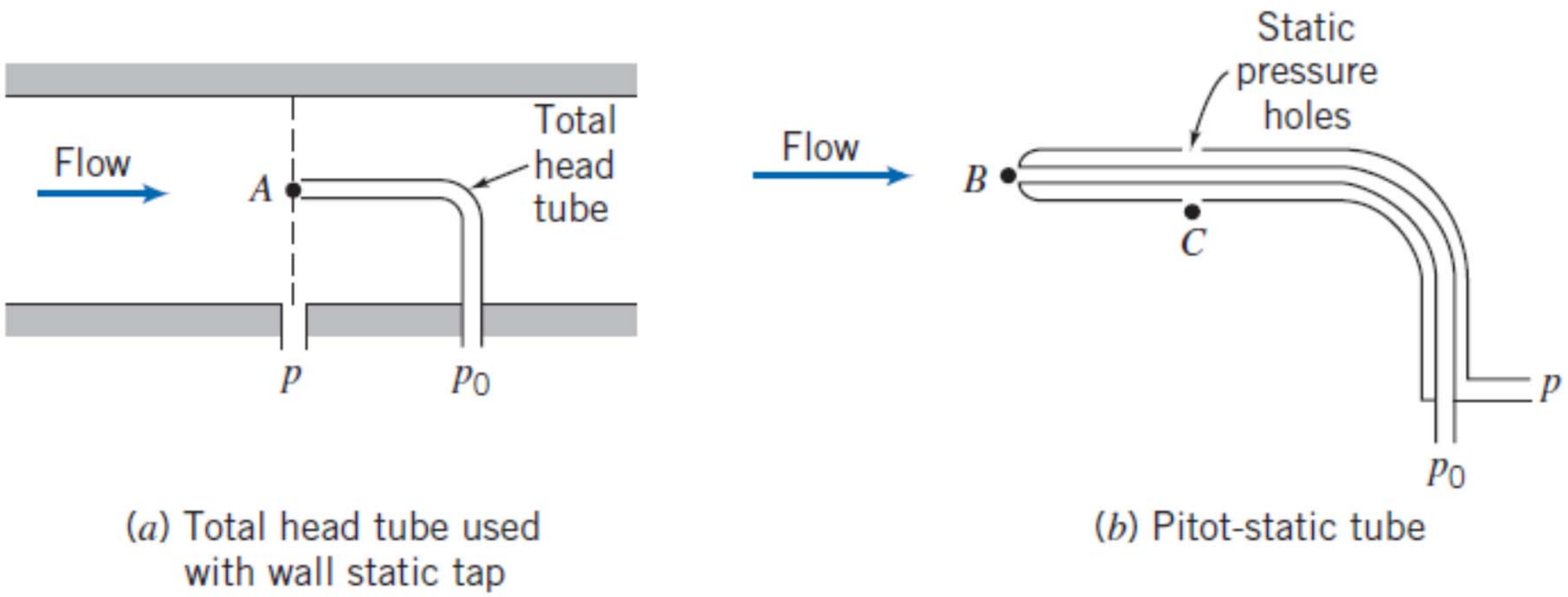
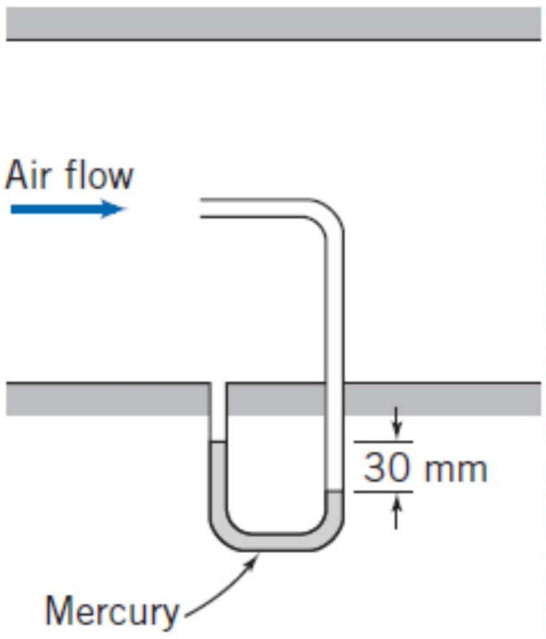


Fig. 6.4 Simultaneous measurement of stagnation and static pressures.

The static pressure corresponding to point A is read from the wall static pressure tap. The stagnation pressure is measured directly at A by the total head tube.

Two probes often are combined, as in the pitot-static tube. The inner tube is used to measure the stagnation pressure at point B, while the static pressure at C is sensed using the small holes in the outer tube.

Example 6.2 PITOT TUBE



A pitot tube is inserted in an air flow to measure the flow speed. The tube is inserted so that it points upstream into the flow and the pressure sensed by the tube is the stagnation pressure. The static pressure is measured at the same location in the flow, using a wall pressure tap. If the pressure difference is 30 mm of mercury, determine the flow speed.

Assumptions:
(1) Steady flow.
(2) Incompressible flow.
(3) Flow along a streamline.
(4) Frictionless deceleration along stagnation streamline.

Writing Bernoulli's equation along the stagnation streamline (with $\Delta z=0$) yields

$$\frac{p_0}{\rho} = \frac{p}{\rho} + \frac{V^2}{2}$$

$$V = \sqrt{\frac{2(p_0 - p)}{\rho_{air}}}$$

$$p_0 - p = \rho_{Hg} g h = \rho_{H_2O} g h SG_{Hg}$$

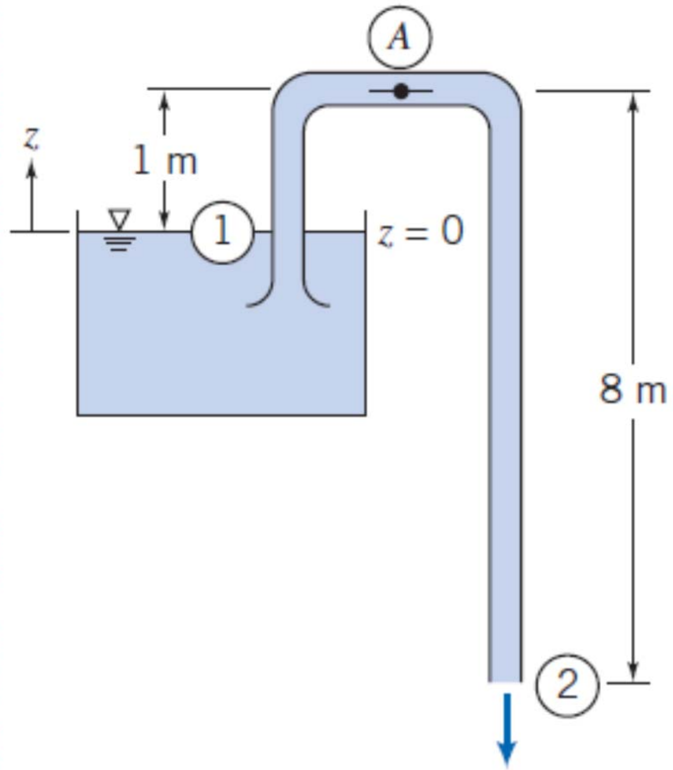
$$V = \sqrt{\frac{2\rho_{H_2O} g h SG_{Hg}}{\rho_{air}}}$$

$$= \sqrt{2 \times 1000 \frac{\text{kg}}{\text{m}^3} \times 9.81 \frac{\text{m}}{\text{s}^2} \times 30 \text{ mm} \times 13.6 \times \frac{\text{m}^3}{1.23 \text{ kg}} \times \frac{1 \text{ m}}{1000 \text{ mm}}}$$

$$V = 80.8 \text{ m/s}$$

Example 6.4 FLOW THROUGH A SIPHON

A U-tube acts as a water siphon. The bend in the tube is 1 m above the water surface; the tube outlet is 7 m below the water surface. The water issues from the bottom of the siphon as a free jet at atmospheric pressure. Determine (after listing the necessary assumptions) the speed of the free jet and the minimum absolute pressure of the water in the bend.



Assumptions:
(1) Neglect friction.
(2) Steady flow.
(3) Incompressible flow.
(4) Flow along a streamline.
(5) Reservoir is large compared with pipe.

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

Since $\text{area}_{\text{reservoir}} \gg \text{area}_{\text{pipe}}$, then $V_1 \approx 0$. Also $p_1 = p_2 = p_{\text{atm}}$, so

$$gz_1 = \frac{V_2^2}{2} + gz_2 \qquad \text{and} \qquad V_2^2 = 2g(z_1 - z_2)$$

$$V_2 = \sqrt{2g(z_1 - z_2)} = \sqrt{2 \times 9.81 \frac{\text{m}}{\text{s}^2} \times 7 \text{ m}} = 11.7 \text{ m/s}$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_A}{\rho} + \frac{V_A^2}{2} + gz_A$$

$$\frac{p_A}{\rho} = \frac{p_1}{\rho} + gz_1 - \frac{V_2^2}{2} - gz_A = \frac{p_1}{\rho} + g(z_1 - z_A) - \frac{V_2^2}{2}$$

$$p_A = p_1 + \rho g(z_1 - z_A) - \rho\frac{V_2^2}{2}$$

$$= 1.01 \times 10^5 \frac{\text{N}}{\text{m}^2} + 999\frac{\text{kg}}{\text{m}^3} \times 9.81\frac{\text{m}}{\text{s}^2} \times (-1\text{ m})\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$-\frac{1}{2} \times 999\frac{\text{kg}}{\text{m}^3} \times (11.7)^2\frac{\text{m}^2}{\text{s}^2} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$p_A = 22.8\text{ kPa (abs) or } -78.5\text{ kPa (gage)}$$

The Bernoulli Equation Interpreted as an Energy Equation

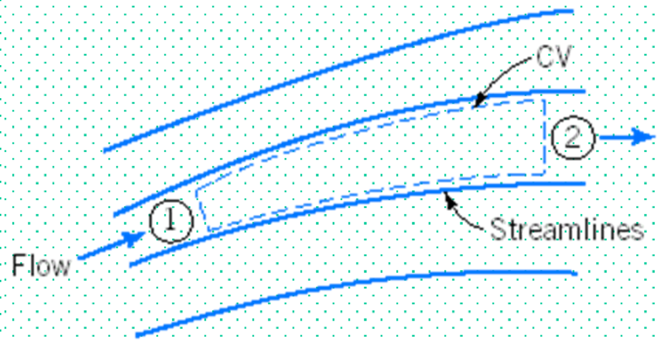


Fig. 6.5 Flow through a stream tube.

The Bernoulli Equation Interpreted as an Energy Equation

✓ Basic Equation

$$\dot{Q} - \underbrace{\cancel{\dot{W}_s}}_{=0(1)} - \underbrace{\cancel{\dot{W}_{\text{shear}}}}_{=0(2)} - \underbrace{\cancel{\dot{W}_{\text{other}}}}_{=0(3)} = \underbrace{\cancel{\frac{\partial}{\partial t}}}_{=0(4)} \int_{\text{CV}} e\,\rho\, d\forall + \int_{\text{CS}} (e + pv)\,\rho \vec{V} \cdot d\vec{A}$$

$$e = u + \frac{V^2}{2} + gz$$

1. No Shaft Work
2. No Shear Force Work
3. No Other Work
4. Steady Flow
5. Uniform Flow and Properties

The Bernoulli Equation Interpreted as an Energy Equation

Hence

$$p_1 v_1 + \frac{V_1^2}{2} + g z_1 = p_2 v_2 + \frac{V_2^2}{2} + g z_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm} \right)$$

Assumption 6: Incompressible

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + g z_2 + \left(u_2 - u_1 - \frac{\delta Q}{dm} \right)$$

Assumption 7:

$$(u_2 - u_1 - \delta Q/dm) = 0$$

The Bernoulli Equation Interpreted as an Energy Equation

✓ **"Energy Equation"**

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

1. No Shaft Work
2. No Shear Force Work
3. No Other Work
4. Steady Flow
5. Uniform Flow and Properties
6. Incompressible Flow
7. $u_2 - u_1 - \delta Q/\delta m = 0$

Energy Grade Line and Hydraulic Grade Line

✓ **Energy Equation**

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z = H = \text{constant}$$

Energy Grade Line and Hydraulic Grade Line

- ✓ **Energy Grade Line (*EGL*)**

$$\frac{p}{\rho g} + \frac{V^2}{2g} + z$$

- ✓ **Hydraulic Grade Line (*HGL*)**

$$\frac{p}{\rho g} + z$$

Energy Grade Line and Hydraulic Grade Line

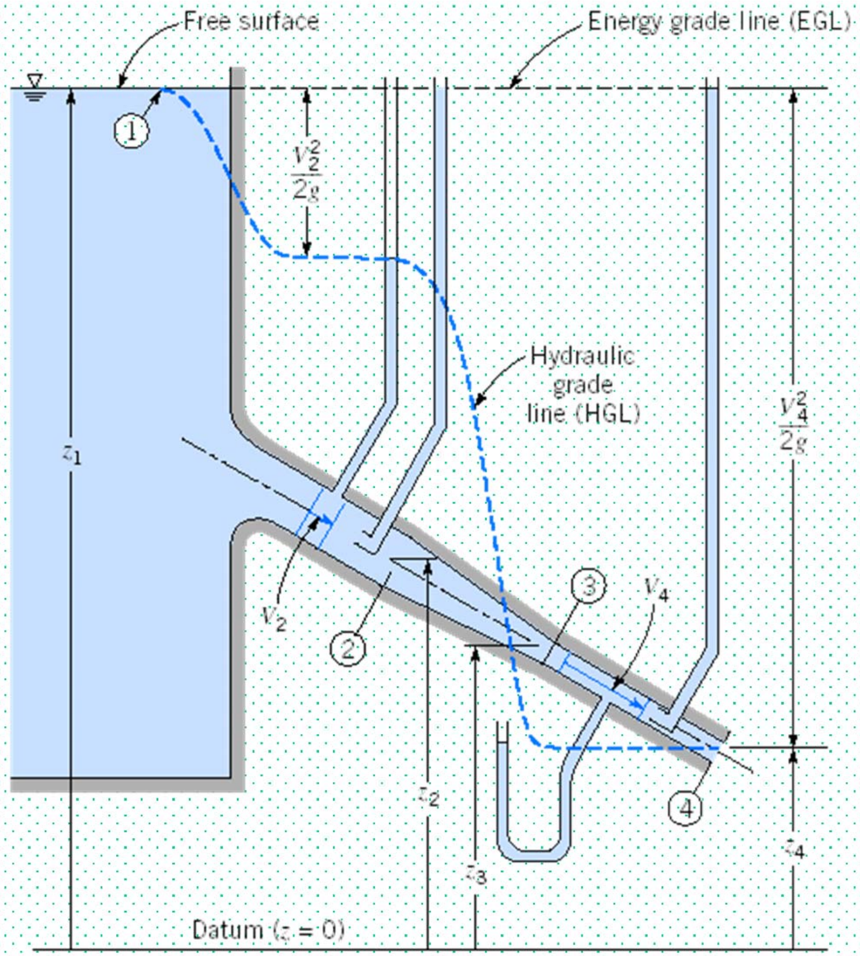


Fig. 6.6 Energy and hydraulic grade lines for frictionless flow.

Irrotational Flow

✓ **Irrotationality Condition**

$$\nabla \times \vec{V} = 0$$

Irrotational Flow

✓ **Velocity Potential**

$$\vec{V} = -\nabla\phi$$

$$u = -\frac{\partial\phi}{\partial x}, \qquad v = -\frac{\partial\phi}{\partial y}$$

$$V_r = -\frac{\partial\phi}{\partial r} \qquad V_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

Irrotational Flow

✓ **Velocity Potential *automatically* satisfies Irrotationality Condition**

$$\nabla \times \nabla\phi \equiv 0 \text{ for all } \phi$$

Irrotational Flow

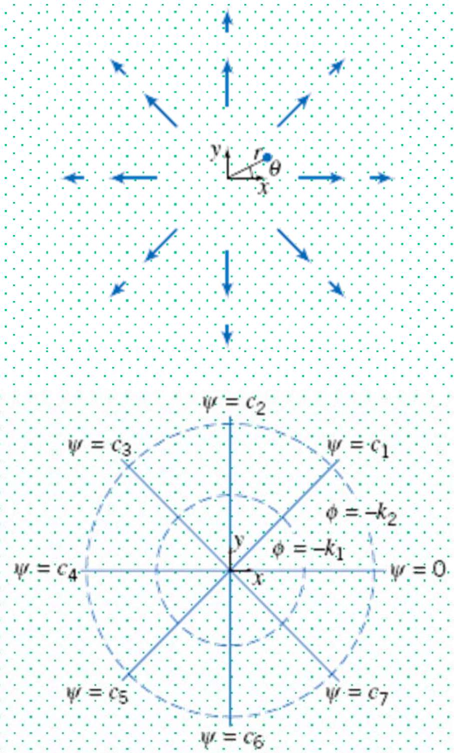
✓ **2D Incompressible, Irrotational Flow**

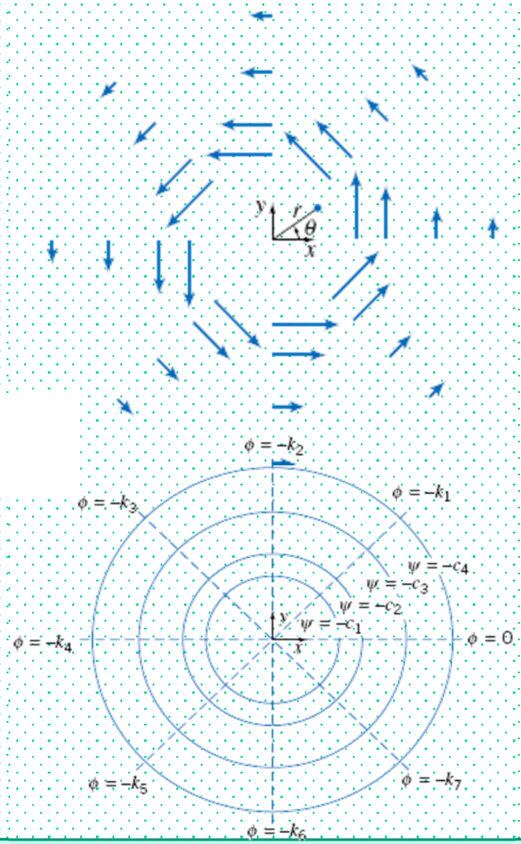
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0$$

Irrotational Flow

✓ **Elementary Plane Flows**





Irrotational Flow

- ✓ **Superposition**