**Firm Behavior**

Consider a firm with a technology $Y=F(K,L)$ where $Y=Output$, $K=Capital$, and $L=Labor. $

The price of $Y$ is $P$ (Price).

The price of $K$ is $R$ (Rent).

The price of $L$ is $W$ (Wage).

Therefore, the profit of the firm can be expressed as

$$Π=Revenue-Cost$$

$$=P×F\left(K,L\right)-\left(W×L\right)-\left(R×K\right).$$

In economics, we assume that firms maximize profits. In order to maximize $Π$, we should solve

$$\frac{dΠ}{dK}=P\frac{dF}{dK}-R=0$$

$$\frac{dΠ}{dL}=P\frac{dF}{dL}-W=0.$$

Recall that

$$\frac{dF}{dK}=Marginal Productivity of Capital=MP\_{K}.$$

$$\frac{dF}{dL}=Marginal Productivity of Labor=MP\_{L}.$$

So the profit maximization conditions become

$$MP\_{K}=\frac{R}{P}$$

$$MP\_{L}=\frac{W}{P}.$$

Thus, the firm chooses $(Y,K,L)$ given $(P,W,R)$ to maximize $Π$. This is known as the assumption of competitiveness: Prices are given and Quantities are chosen.

Example: Suppose that $Y=AK^{a}L^{1-a}.$ Then

$$MP\_{L}=(1-a)AK^{a}L^{-a}$$

$$MP\_{K}=aAK^{a-1}L^{1-a}.$$

Note that

$$\left(1-a\right)AK^{a}L^{-a}=\frac{\left(1-a\right)AK^{a}L^{1-a}}{L}=\left(1-a\right)\frac{Y}{L}=\left(1-a\right)AP\_{L}$$

because

$$\frac{Y}{L}=AP\_{L}=Average Productivity of Labor.$$

In a similar fashion

$$\frac{Y}{K}=AP\_{K}=Average Productivity of Capital$$

and this gives us

$$MP\_{L}=(1-a)AP\_{L}=\frac{W}{P}$$

$$MP\_{K}=aAP\_{K}=\frac{R}{P}$$

if the firm is profit maximizing.

**Technology and Income Distribution**

First note that

$$Π=PY-WL-RK.$$

This means

$$PY=Π+WL+RK.$$

In words

$$Value of Output=Entrepreneurial Income+Labor Income+Capital (Rent) Income.$$

Now divide both sides by $Y$ to see

$$1=\frac{Π}{PY}+\frac{WL}{PY}+\frac{RK}{PY}.$$

In words,

$$1=Profit Share+Labor Share+Capital Share.$$

According to the profit maximization

$$MP\_{L}=\frac{W}{P}.$$

Multiply both sides by $L$ to see

$$\frac{MP\_{L}L}{Y}=\frac{WL}{PY}=Labor Share$$

Note, however,

$$MP\_{L}\frac{L}{Y}=\frac{MP\_{L}}{AP\_{L}}.$$

The end result is that

$$\frac{MP\_{L}}{AP\_{L}}=Labor share.$$

In a similar vein,

$$\frac{MP\_{K}}{AP\_{K}}=Capital share.$$

Example: Suppose that $Y=AK^{a}L^{1-a}$. Then

Remark: This result means that the estimation of Cobb and Douglas predicts the income distribution in the US is $Labor Share=1-a=1-\frac{1}{4}=\frac{3}{4}$ and $Capital Share=a=\frac{1}{4}.$ The real income distribution in the US at the time was $Labor Share=74\%$ and $Capital Share=26\%.$

$$Labor share=\frac{MP\_{L}}{AP\_{L}}=1-a$$

$$Capital share=\frac{MP\_{K}}{AP\_{K}}=a.$$

Now let us why MP/AP is so important. Let us first see that MP/AP is actually “the elasticity”. In general, for any couple of variables $(X,Y)$, the $X$ elasticity of $Y$ is

$$ϵ\_{Y,X}=\frac{\%Change in Y}{\%Change in X}=\frac{\frac{dY}{Y}}{\frac{dX}{X}}=\frac{dY}{Y}\frac{X}{dX}=\frac{dY}{dX}\frac{X}{Y}.$$

Now apply this definition of elasticity to output and labor and ask how would 1% increase in employment ($L)$ would affect output ($Y$):

$$ϵ\_{Y,L}=\frac{dY}{dL}\frac{L}{Y}=\frac{\frac{dY}{dL}}{AP\_{L}}=\frac{MP\_{L}}{AP\_{L}}=Labor Share.$$

Of course, this calculation also implies that

$$ϵ\_{Y,K}=\frac{dY}{dK}\frac{K}{Y}=\frac{\frac{dY}{dK}}{AP\_{K}}=\frac{MP\_{K}}{AP\_{K}}=Capital Share.$$

This derivation can also be applied to the CES technology, which includes perfect substitutes, perfect complements, and Cobb-Douglas as special cases.

Recall that

$$Y=\left(\left(aK\right)^{s}+\left(bL\right)^{s}\right)^{\frac{1}{s}}$$

according to the CES technology where $s\in \left(-\infty ,1\right].$ In that case, the labor share can still be calculated by the elasticity formula. To use this, we need the $MP\_{L}$:

$$\frac{dY}{dL}=\frac{1}{s}\left(\left(aK\right)^{s}+\left(bL\right)^{s}\right)^{\frac{1}{s}-1}×s\left(bL\right)^{s-1}b$$

$$=Y\left(\left(aK\right)^{s}+\left(bL\right)^{s}\right)^{-1}\left(bL\right)^{s-1}b$$

$$=YY^{-s}\left(bL\right)^{s-1}b$$

$$=Y^{1-s}\left(bL\right)^{s-1}b$$

$$=(\frac{Y}{L})^{1-s}\left(b\right)^{s-1}b$$

$$=(\frac{Y}{L})^{1-s}\left(b\right)^{s}$$

This is only the marginal productivity. Labor share is

$$Labor share=\frac{MP}{AP}=\frac{\left(\frac{Y}{L}\right)^{1-s}\left(b\right)^{s}}{(\frac{Y}{L})}=\left(\frac{Y}{L}\right)^{-s}\left(b\right)^{s}=\left(AP\_{L}\right)^{-s}\left(b\right)^{s}$$

$$=\left(\frac{b}{AP\_{L}}\right)^{s}$$

In real life, labor Share is decreasing. Moreover $AP\_{L}$ is increasing. Finally, most of the estimates for $s$ is around -2. All these information suggests that $b$ must increase faster than $AP\_{L}$. (Why? Otherwise, labor share would decrease.) But what is an increase in $b$? It gives the same effect as “higher number of workers” (Inspect the CES technology above). So the fast increase in $b$ reduces the elasticity because the “bargaining power of workers go down if there are many workers” given labor and capital are complements, $s<0$.