**Growth (Preliminaries)**

Consider a variable that changes over time. In mathematical terms, let be a variable at time . For convenience, the initial date is set as . This means the variable changes as time changes starting from time . This variable can be population, amount of capital, output, or any variable that goes up or down over time. The following notions are of utmost importance for any variable that depends on time.

**Definition:**  is the change in . Of course, is the change in . The rate of change in is over :

which is the time derivative of .

**Example:** See Table 1 below which reports the population of Turkey in each year starting from 2007 (t=0) until 2015 (t=8). If population increased 1 million people from 2007 to 2008 then

which is the change in population. The change in time is

so that the rate of change from 2007 to 2008 in Turkey’s population is

**Table 1.** Population in Turkey: 2007-2015 (in millions)

|  |  |  |
| --- | --- | --- |
| **t** | **Year** | **Population** |
| **0** | **2007** | **70.586** |
| **1** | **2008** | **71.517** |
| **2** | **2009** | **72.561** |
| **3** | **2010** | **73.722** |
| **4** | **2011** | **74.724** |
| **5** | **2012** | **75.627** |
| **6** | **2013** | **76.667** |
| **7** | **2014** | **77.695** |
| **8** | **2015** | **78.741** |

Source: TUIK web site.

A closely related concept in understanding change over time is the growth rate. Based on the rate of change, we can define the rate of growth as follows.

**Definition:** The rate of growth of is

which is the time derivative of divided by .

**Example:** Recall that the rate of change in population in Turkey in 2007 is

But million people at which says the population in Turkey in 2007 is 70.6 million. If we divide these two numbers we get

This result says that the rate of population growth in Turkey is 1.4%.

Suppose that the rate of growth of does not change over time and it always stays constant. This would mean population growth in Turkey is 1.4% every year. In general, let the constant growth rate of be . Formally,

But we can write the same expression as

Now let us see that if we know the growth rate of then we can compute in open form for each . To do this, first remember a mathematical fact: Derivative of the natural logarithm is

In words, the derivative of the natural logarithm is exactly equal to the growth rate. Therefore, if the growth rate is then

Now integrate both sides by taking the antiderivative yielding

Take the antilogarithm of this equation to see

 stands for the initial value of , or at time .

**Example:** If is the population of Turkey, then according to Table 1 since the initial date corresponds to year 2007. That is to say, the population of Turkey at the initial date (2007) is Now choose implying 1.4% as we computed earlier. Putting these information into

gives the population of Turkey to be

In this formulation means the year is 2007 since it is the initial year. Of course, means the year is 2008. And finally, means year is 2015.

In principle, this equation should approximately give us the population in Turkey. To see whether this is the case or not, let us plot this function with the actual data.

**Figure 1.** Population In Turkey

The two lines show the actual data (Blue) and the computed population (Orange) according to our function They are so close that it is hard to discern one from the other. This close relationship shows that the population in Turkey approximately follows

**Exercises**

1. What is the rate of change of

with respect to .

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with respect to when . What is the growth rate? Does the growth rate change if or ?

1. What is the growth rate of

with respect to when . Does your answer differ change if ?

1. Suppose that the annual growth rate of capital in Turkey is 2%. The amount of capital is 200 (billion dollars) today. How much capital will there be next year, in 2017. How much capital will there be in 2018?
2. Suppose that the annual growth rate of GDP in China is 10%. GDP is 100 (trillion dollars) today. How much will be the Chinese GDP next year, in 2017. How much will the GDP be in 2018?
3. Assume where and constants. Show that the rate of growth is which means the rate of growth is independent of .
4. Let

where the growth rate of is and the growth rate of is . Show that the growth rate of is .

1. Let

where the growth rate of is and the growth rate of is . Show that the growth rate of is .

1. Let

where the growth rate of is and is a constant. Show that the growth rate of is .