Ek-1

Tanım: Eğer $f(x)$, $[a,b]$ aralığında integre edilebilir bir fonksiyon ise, $f(x)$'in a dan b ye belirli integrali,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{i-1}{n}(b-a)\right)\right]$$

limiti olarak hesaplanır. (Alt Riemann Toplamları) $L(f,P_n)$ ile gösterilir.

Örnek: $\int_0^1 x\,dx = \frac{1}{2}$ dir.

$f(x)=x$; $[0,1]$ aralığını n eşit parçaya bölelim.

$$L(f,P_n) = \frac{1-0}{n}\sum_{i=1}^{n} f\left(0+\frac{i-1}{n}(1-0)\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i-1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{i-1}{n}$$

$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right\} = \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right]$$

$$\boxed{L(f,P_n) = \frac{n^2-n}{2n^2}}$$

EK-2

Üst Riemann Toplamları

$$U(f,P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{i}{n}(b-a)\right)$$

örnek: $\int_0^1 x\,dx = \frac{1}{2}$ dir.

$f(x) = x$; $[0,1]$ aralığını n eşit parçaya bölelim.

$$U(f,P_n) = \frac{1-0}{n} \sum_{i=1}^{n} f\left(0 + \frac{i}{n}(1-0)\right)$$

$$= \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \frac{1}{n} \sum_{i=1}^{n} \frac{i}{n}$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \cdot \frac{n(n+1)}{2}$$

$$\boxed{U(f,P_n) = \frac{n^2+n}{2n^2}} \boxed{= \frac{n+1}{2n}}$$

örneğin; $n=11$ için $L(f,P_{11}) = \frac{5}{11}$ Alt Riemann Toplamı için

$n=100$ için $L(f,P_{100}) = \frac{99}{200}$

$$\int_0^1 x\,dx = \lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{n^2-n}{2n^2} = \frac{1}{2}$$ olur.

<u>Ek-3</u>

Üst Riemann Toplamı için;

$n=10$ için $U(f,P_{10}) = [illegible]$

$n=100$ için $U(f,P_{100}) = \frac{101}{200}$

n için $U(f,P_n) = \frac{n+1}{2n}$

$$\int_0^1 x\,dx = \lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$ olur.

<u>Örnek:</u> $\int_0^2 x^2\,dx = \frac{8}{3}$

a) Alt Riemann Toplamı $L(f,P_n) = ?$

b) Üst Riemann Toplamı $U(f,P_n) = ?$

c) $\lim_{n\to\infty} U(f,P_n) = ?$ $\lim_{n\to\infty} L(f,P_n) = ?$

a) $U(f,P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + \frac{i-1}{n}(b-a)\right)$

$$= \frac{2}{n} \sum_{i=1}^{n} f\left(\frac{2(i-1)}{n}\right)$$

Ek-4

$$L(f,P_n) = \frac{2}{n} \cdot \sum_{i=1}^{n} \left(\frac{2(i-1)}{n}\right)^2$$

$$= \frac{8}{n^3}\left[\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right]$$

$$= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right]$$

$$= \frac{8}{n^3} \cdot \left[\frac{2n^3 - 3n^2 + n}{6}\right]$$

$$\boxed{L(f,P_n) = \frac{8n^3 - 12n^2 + 4n}{3n^3}}$$

b) $$U(f,P_n) = \frac{2}{n}\sum_{i=1}^{n} f\left(\frac{2i}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$$

$$= \frac{8}{n^3}\sum_{i=1}^{n} i^2 = \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$U(f,P_n) = \frac{8}{n^3} \cdot \frac{2n^3 + 3n^2 + n}{6} = \boxed{\frac{8n^3 + 12n^2 + 4n}{3n^3}}$$

EK-5

c) $\lim\limits_{n\to\infty} L(f,P_n) = \lim\limits_{n\to\infty} \dfrac{8n^3-12n^2+4n}{3n^3} = \dfrac{8}{3}$

$\lim\limits_{n\to\infty} U(f,P_n) = \lim\limits_{n\to\infty} \dfrac{8n^3+12n^2+4n}{3n^3} = \dfrac{8}{3}$

Ödev: $\displaystyle\int_0^1 (x^3-3x+2)\,dx = \frac{3}{4}$

$L(f,P_n)=?$ $\quad U(f,P_n)=?$

$\lim\limits_{n\to\infty} L(f,P_n)=?$ $\quad \lim\limits_{n\to\infty} U(f,P_n)=?$

Ödev: $\displaystyle\int_0^3 (9-x^2)\,dx$ integralini Riemann Toplamlarını kullanarak bulunuz.

EK-6

Örnek: 1) $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{4-x_i^2}\cdot\Delta x$, $[0,1]$

$$\int_0^1 \sqrt{4-x^2}\,dx$$

2) $\lim_{n\to\infty} \sum_{i=1}^{n} (5x_i^3+\sqrt{x_i}+4)\Delta x$, $[0,\pi]$

$$\int_0^{\pi} (5x^3+\sqrt{x}+4)\,dx$$

3) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n+2i}$ limitini belirli integral olarak ifade ediniz.

Çözüm: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x = \int_a^b f(x)\,dx$

$\Delta x = \frac{2}{n} = \frac{2-0}{n}$

$$\frac{2}{n+2i} = \underbrace{\frac{2}{n}}_{\Delta x}\cdot\frac{1}{1+\frac{2}{n}}$$

$a=0$, $b=2$ olsun.

$f(a+\frac{b-a}{n}\cdot i) = f(\underbrace{\frac{2}{n}i}_{x_i}) \Rightarrow f(x)=\frac{1}{1+x}$ dir.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n+2i} = \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} \frac{1}{1+\frac{2i}{n}} = \int_0^2 \frac{1}{1+x}\,dx = \ln 3$$

EK-7

Örnek: $\lim\limits_{n\to\infty} \frac{1}{n^{16}}\left(1^{15}+2^{15}+3^{15}+\dots+n^{15}\right)=?$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{15}}{n^{16}}=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^{15}}{n^{15}}\cdot\frac{1}{n}$$

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{\left(\frac{i}{n}\right)^{15}}_{f(x_i)}\cdot\frac{1}{n} \qquad : f(x_i)=\left(\frac{i}{n}\right)^{15}$$

$f(x)=x^{15}$ olur.

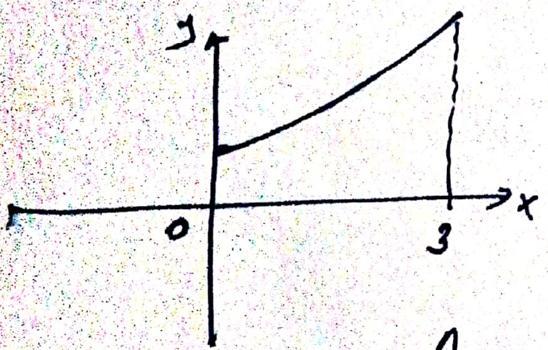
$$=\int_0^1 x^{15}\,dx=\frac{1}{16}$$

EK-8

Örnek! $f(x)=e^x$ fonksiyonunun $[0,3]$ aralığında,

$\lim_{n\to\infty} L(f,P_n)=?$ $\quad \lim_{n\to\infty} U(f,P_n)=?$

$f(x)=e^x$ artan fonksiyondur.



$$\Delta x=\frac{3-0}{n}=\frac{3}{n}$$

$$x_i=a+i\Delta x=\frac{3}{n}(i-1)$$

$$L(f,P_n)=\sum_{i=1}^{n} f(x_i)\cdot\Delta x$$

$$=\sum_{i=1}^{n}\frac{3}{n}\cdot f\left(\frac{3}{n}(i-1)\right)=\frac{3}{n}\sum_{i=1}^{n} e^{\frac{3}{n}(i-1)}$$

$e^{\frac{3}{n}}=r$ dersek

$$S_n=\sum_{i=1}^{n} r^{i-1}=1+r+r^2+r^3+\dots+r^{n-1}$$

$$rS_n=r+r^2+r^3+r^4+\dots+r^n$$

$$S_n-rS_n=1-r^n \Rightarrow S_n=\frac{1-r^n}{1-r} \text{ olur.}$$

Ek-9

$$L(f,P_n) = \frac{3}{n}\,\frac{1-e^{\frac{3}{n}\cdot n}}{1-e^{\frac{3}{n}}} = \frac{3}{n}\cdot\frac{1-e^{3}}{1-e^{\frac{3}{n}}} \text{ olur.}$$

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{3(1-e^{3})}{n(1-e^{\frac{3}{n}})}$$

$$\lim_{n\to\infty} n(1-e^{\frac{3}{n}}) = \infty.0 \text{ Belirsizliği}$$

$$= \lim_{n\to\infty} \frac{1-e^{\frac{3}{n}}}{\frac{1}{n}} = \frac{0}{0} \text{ Belirsizliği}$$

$$= \lim_{n\to\infty} \frac{\frac{3}{n^2}\cdot e^{\frac{3}{n}}}{-\frac{1}{n^2}} = -3$$

$$\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} \frac{3(1-e^{3})}{-3} = e^{3}-1 \text{ bulunur.}$$

Ödev: Benzer şekilde $\lim_{n\to\infty} U(f,P_n) = ?$

Kesinlikle kendiniz yapınız......