

Tekel matrisi:
Tekel eleman matrisi:

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(Sehli)

Bir Matrisin Normal Formu

Bir A matrisini gözönüne alalım. $\tilde{A} = r \cdot 0$ olsun.
Bu durumda A ya elemanter matris işlemleri uygulanarak, A matrisi;

I_r , $[I_r \ 0]$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ şekillerinden birine

dönüştür. Buna A matrisinin Normal Formu (Sehli) denir.

Örnek: $A_2 = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$

Elemanter matrisler.

Bir B matrise, bir elemanter satır işlemi uygulanarak elde edilen matrise elemanter satır matrisi denir.

Bir B matrise bir elemanter sütun dönüşümü uygulanarak elde edilen matrise elemanter sütun matrisi denir.

Örnek: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = K_{23}$

$$H_{31}(K) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ K & 0 & 1 \end{bmatrix}$$

Örnek $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$

$H_2(-4)$

$H_2(-4) = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$H_2(-4) \cdot A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$

$A \sim B$ olsun. H_1 : 1. satır içine karşılık gelen ekvanter

H_2 : 2. " " " " "

H_5 : 5. " " " " "

K_1 : 1. sütun " " " " " Sütun matrisi

K_2 : 2. " " " " "

K_t : t. " " " " "

olsun.

$(H_5 H_4 \dots H_1) A (K_1 K_2 \dots K_t) = B$

$PAQ = B$

olsun.

Not: A ve B gibi iki matris $PAQ = B$ olacak şekilde tekil olmayan P ve Q matrisleri varsa denir.

Örnek:

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

matrisin Normal Formu

Birleştirilecek olan P ve Q matrislerini bulunuz?

$PAQ = N$ ile N matrisini yaparak doğruluğunu gösteriniz.

Çözüm:

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$H_{21}(-2)$

$H_{31}(-1)$

$K_{21}(-2)$

$K_{31}(1)$

$K_{41}(-2)$

$K_{42}(1), K_3(1/3)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = [I_3 \ 0] = N$$

$P = ?$ AM bulunması

$$I_3 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = P$$

$Q = ?$ NA bulunması

$$I_4 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 1/3 & -4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = Q$$

$PAQ = N$ bulunur. (doğru)

Özellik: Eğer A tekil olmayan n . mertebeden bir
kare matris ise $PAQ=I$ olacak şekilde P ve
 Q tekil olmayan matrisler vardır.

$$PAQ=I \Rightarrow A=P^{-1}Q^{-1} \text{ dir.}$$

Smith Normal Form

Her boyutlu $A(\lambda)$ polinom matrisi elementer
dönüşümler sonucu;

$$\begin{bmatrix} f_1(\lambda) & 0 & 0 & \dots & 0 \\ 0 & f_2(\lambda) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & f_n(\lambda) \end{bmatrix} = N(\lambda) \text{ matrisine dönüşür.}$$

Burada $f_1(\lambda) \mid f_2(\lambda) \mid \dots \mid f_{i+1}(\lambda)$ ve
tam bölür.

$$A(\lambda) \sim N(\lambda)$$

$$P(\lambda).A(\lambda).Q(\lambda)=N(\lambda)$$

Örnek: $A(\lambda) = \begin{bmatrix} \lambda & \lambda-1 \\ \lambda+1 & \lambda^2 \end{bmatrix}$ Polinom matrisinin Smith
formuna bulunuz.

$$A(\lambda) = \begin{bmatrix} \lambda & \lambda-1 \\ \lambda+1 & \lambda^2 \end{bmatrix} \sim \begin{bmatrix} 1 & \lambda-1 \\ -\lambda^2+\lambda+1 & \lambda^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ -\lambda^2+\lambda+1 & \lambda^3-\lambda^2+1 \end{bmatrix} \sim$$

$\underbrace{K_{12}(-1)}_{K_{21}(-\lambda+1)} \quad H_{21}(\lambda^2-\lambda-1)$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & \lambda^3-\lambda^2+1 \end{bmatrix} = N(\lambda)$$

$$P \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ \lambda^2+\lambda & 1 \end{bmatrix} = P$$

$$Q \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\lambda+1 \\ -1 & \lambda \end{bmatrix} = Q$$

$$P \cdot A \cdot Q = N \quad \text{olur.}$$

Örnek:

$$A(\lambda) = \begin{bmatrix} \lambda+2 & \lambda+1 & \lambda+3 \\ \lambda^3+2\lambda^2+\lambda & \lambda^3+\lambda^2+\lambda & 2\lambda^3+3\lambda^2+\lambda \\ \lambda^2+3\lambda+2 & \lambda^2+2\lambda+1 & 3\lambda^2+6\lambda+3 \end{bmatrix}$$

matrisin Smith Normal formunu bulunuz.

$\underbrace{K_{12}(-1)}$

$$A(\lambda) \sim \begin{bmatrix} 1 & \lambda+1 & \lambda+3 \\ \lambda^2 & \lambda^3+\lambda^2+\lambda & 2\lambda^3+3\lambda^2+\lambda \\ \lambda+1 & \lambda^2+2\lambda+1 & 3\lambda^2+6\lambda+3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ \lambda^2 & \lambda & \lambda^3+\lambda \\ \lambda+1 & 0 & 2\lambda^2+2\lambda \end{bmatrix} \sim$$

$K_{21}(-\lambda-1), K_{31}(-\lambda-3) \quad H_{21}(-\lambda^2), H_{31}(-\lambda-1)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & \lambda^3+1 \\ 0 & 0 & 2\lambda^2+2\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 2\lambda^2+2\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2+1 \end{bmatrix} = \text{NAF}$$

$K_{32}(-\lambda^2-1) \quad K_3\left(\frac{1}{2}\right) \quad \text{bulunur.}$

Linear sabit katsayılı Diferansiyel Denklemler Sisteminin Smith Normal formu yardımıyla

Çözümü :

$$\left. \begin{aligned} \frac{dx_1}{dt} &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + h_1(t) \\ \frac{dx_2}{dt} &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + h_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + h_n(t) \end{aligned} \right\}$$

$$\boxed{\frac{dX}{dt} = AX + H}$$

$$D = \frac{d}{dt}$$

$$DX = AX + H$$

$$(DI - A)X = H$$

polinom matris
(B)

$$\boxed{BX = H} \text{ olur.}$$

$$\boxed{BX=H}$$

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B matrisinin Smith Normal formu N. olur.

$$\boxed{PBQ=N}$$
 olur.

Sistemin Çözümü $\boxed{X=QY}$ olur.

0 zaman $X=QY$ denklemini sağlar.

$$BX=H$$

$$BQY=H$$

$$PBQY=PH$$

N

$\boxed{NY=PH}$ olur. Bu eşitlikten Y bulunur.

Sonuçta $\boxed{X=QY}$ den Sistemin Çözümü elde edilir.

Örnek:
$$\left. \begin{aligned} \frac{dx}{dt} + x + y &= 1 \\ \frac{dy}{dt} - x + 3y &= -1 \end{aligned} \right\} \text{Sistemi Smith Normal formu yardımıyla çözümler.}$$

$$X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad A = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}, \quad H = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left. \begin{aligned} (D+1)x + y &= 1 \\ -x + (D+3)y &= -1 \end{aligned} \right\}$$

$$\begin{bmatrix} D+1 & 1 \\ -1 & D+3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\underbrace{\hspace{10em}}_{DI-A=B}$$

$$\boxed{BX=H}$$

$$B = \begin{bmatrix} D+1 & 1 \\ -1 & D+3 \end{bmatrix}$$

matrisman Smith Normal Formulas bulakim.

K_{12}

$$\sim \begin{bmatrix} 1 & D+1 \\ D+3 & -1 \end{bmatrix} \xrightarrow{K_{21}(-D-1)} \sim \begin{bmatrix} 1 & 0 \\ D+3 & -D^2-4D-4 \end{bmatrix} \xrightarrow{H_{21}(-D-3)} \sim \begin{bmatrix} 1 & 0 \\ 0 & D^2+4D+4 \end{bmatrix} \xrightarrow{H_2(-1)} = N$$

$$P \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{H_{21}(-D-3)} \sim \begin{bmatrix} 1 & 0 \\ -D-3 & 1 \end{bmatrix} \xrightarrow{H_2(-1)} \sim \begin{bmatrix} 1 & 0 \\ D+3 & -1 \end{bmatrix} = P$$

$$Q \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{H_{21}(-D-3)} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \xrightarrow{H_2(-1)} \sim \begin{bmatrix} 0 & 1 \\ 1 & -D-1 \end{bmatrix} = Q$$

$$NY = PH$$

$$\begin{bmatrix} 1 & 0 \\ 0 & D^2 + 4D + 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ D+3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ (D^2 + 4D + 4)y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\boxed{y_1 = 1}$$

$$(D^2 + 4D + 4)y_2 = 4$$

$$y'' + 4y' + 4y = 4$$

$$y_H = C_1 e^{-2t} + C_2 t e^{-2t}$$

$$y_0^v = 1$$

$$y_2 = y_H + y_0^v$$

$$\boxed{y_2 = C_1 e^{-2t} + C_2 t e^{-2t} + 1}$$

$$X' = AY$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -D-1 \end{bmatrix} \begin{bmatrix} 1 \\ C_1 e^{-2t} + C_2 t e^{-2t} + 1 \end{bmatrix}$$

$$x = C_1 e^{-2t} + C_2 t e^{-2t} + 1$$

$$y = (C_1 - C_2) e^{-2t} + C_2 t e^{-2t} \quad \left. \vphantom{y = (C_1 - C_2) e^{-2t} + C_2 t e^{-2t}} \right\} \text{bulunur.}$$

Ödev:

$$\left. \begin{aligned} \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} + x + y &= e^t \\ \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{dx}{dt} &= e^{-t} \end{aligned} \right\}$$

Sistemi Smith Normal Formu yardımıyla

çözünüz. $\left(\begin{array}{l} \text{Sonuç: } x = -e^t - 2e^{-t} + c_1 \\ y = 2e^t + e^{-t} - c_1 \end{array} \right)$

Çözüm: $Bx' = H$

$$(D^2 + D + 1)x + (D^2 + 1)y = e^t$$

$$(D^2 + D)x + D^2y = e^{-t}$$

$$B = \begin{bmatrix} D^2 + D + 1 & D^2 + 1 \\ D^2 + D & D^2 \end{bmatrix}, \quad H = \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}, \quad X' = \begin{bmatrix} x \\ y \end{bmatrix} = ?$$

$$H_{12}(-1)$$

$$\sim \begin{bmatrix} 1 & 1 \\ D^2 + D & D^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ D^2 + D & -D \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix} = N$$

$$K_{21}(-1)$$

$$H_{21}(-D^2 - D)$$

$$K_2(-1)$$

$$P_2 \begin{bmatrix} 1 & -1 \\ -D^2 - D & D^2 + D + 1 \end{bmatrix},$$

$$Q_2 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

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$$BX = H$$

$$NY = PH$$

$$X = QY$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -0^2-0 & 0^2+0+1 \end{bmatrix} \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$$

$$y_1 = e^t - e^{-t}$$

$$0y_2 = -e^t - e^t + e^{-t} - e^{-t} + e^{-t}$$

$$y_2' = -2e^t + e^{-t} \Rightarrow y_2 = -2e^t + e^{-t} + C_1$$

$$X = QY$$

$$x_2 = -e^t - 2e^t + C_1$$

$$y = 2e^t + e^{-t} + C_1$$

bulunur.

Ödevi $\frac{dx_1}{dt} = x_1 + 2x_2 + x_3$

$$\frac{dx_2}{dt} = x_1 + 3x_2 + 2x_3$$

$$\frac{dx_3}{dt} = -2x_2$$

Sistemi Smith Normal

Formu yardımıyla çözümler.