Örnek: $x^2 y'' - 4xy' + 4y = 0$ Dif. Denkleminin $y(2) = 4$ ve $y'(2) = -1$ koşullarını sağlayan çözümünü bulunuz.

Çözüm: $x = e^t$, $t = \ln x$, $\dfrac{dt}{dx} = \dfrac{1}{x} = \dfrac{1}{e^t} = e^{-t}$

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot e^{-t}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{dt} \cdot \frac{dt}{dx} = \left(\frac{d^2y}{dt^2} \cdot e^{-t} - e^{-t}\frac{dy}{dt}\right) e^{-t}$$

$$y'' = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) e^{-2t}$$

$$e^{2t} \cdot e^{-2t} \cdot \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 4e^{t} \cdot e^{-t}\frac{dy}{dt} + 4y = 0$$

$$\boxed{\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 4y = 0}$$

$r^2 - 5r + 4 = 0$

1 4

$r_1 = 1$, $r_2 = 4$

$y = c_1 e^t + c_2 e^{4t}$

$\boxed{y = c_1 x + c_2 x^4}$

$y(2) = 4 \Rightarrow 4 = 2c_1 + 16c_2$

$y'(2) = -1 \Rightarrow -1 = c_1 + 32c_2$

$c_1 = 3$

$c_2 = -\dfrac{1}{8}$

$\boxed{y = 3x - \dfrac{1}{8}x^4}$ bulunur.

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Örnek: $x^2y'' + 4xy' + 2y = \ln x$

$x = e^t, \quad \frac{dt}{dx} = e^{-t}, \quad \ln x = t$

$y' = \frac{dy}{dt}e^{-t}, \quad y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 4\cdot\frac{dy}{dt} + 2y = t$$

$$\boxed{\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t} \quad \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

$r^2 + 3r + 2 = 0$

$r_1 = -2, \quad r_2 = -1$

$y_H = c_1e^{-2t} + c_2e^{-t}, \quad y_ö = at + b, \quad y_ö' = a, \quad y_ö'' = 0$

$3a + 2at + 2b = t$

$2a = 1, \quad 3a + 2b = 0$

$a = \frac{1}{2}, \quad b = -\frac{3}{4}$

$y_ö = \frac{1}{2}t - \frac{3}{4}$

$y = c_1e^{-2t} + c_2e^{-t} + \frac{1}{2}t - \frac{3}{4}$

$$\boxed{y = \frac{c_1}{x^2} + \frac{c_2}{x} + \frac{1}{2}\ln x - \frac{3}{4}}$$

Örnek: $x^2y'' - 3xy' + 4y = x + x^2\ln x$ Dif. Denk.

Genel çözümünü bulunuz.

Çözüm: $x = e^t,\ t = \ln x,\ \frac{dt}{dx} = e^{-t}$

$y' = e^{-t}\frac{dy}{dt},\ y'' = e^{-2t}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$

$\underbrace{e^{2t}\cdot e^{-2t}}_{1}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) - 3\underbrace{e^t\cdot e^{-t}}_{1}\frac{dy}{dt} + 4y = e^t + e^{2t}\cdot t$

$$\boxed{\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = e^t + te^{2t}}$$

$\boxed{y = y_H + y_Ö}$

$\boxed{y_Ö = y_{Ö1} + y_{Ö2}}$

$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$

$r^2 - 4r + 4 = 0$ $\quad (r-2)^2 = 0 \quad r_{1,2} = 2$

$$\boxed{y_H = (c_1 + c_2 t)e^{2t}}$$

$y_{Ö1} = ke^t,\ y_{Ö1}' = ke^t,\ y_{Ö1}'' = ke^t$

$ke^t - 4ke^t + 4ke^t = e^t$

$ke^t = e^t \Rightarrow \boxed{k = 1}$

$\boxed{y_{Ö1} = e^t}$

$y_{Ö2} = (at+b)t^2e^{2t}$

$y_{Ö2} = (at^3 + bt^2)e^{2t}$

$y_{Ö2}' = (3at^2 + 2bt)e^{2t} + 2(at^3 + bt^2)e^{2t}$

$y_{Ö2}'' = (6at + 2b)e^{2t} + 2(3at^2 + 2bt)e^{2t} + 2(3at^2 + 2bt)e^{2t} + 4(at^3 + bt^2)e^{2t}$

$(6at + 2b)e^{2t} + 4\cancel{(3at^2 + 2bt)e^{2t}} + \cancel{4(at^3 + bt^2)e^{2t}} - \cancel{4(3at^2 + 2bt)e^{2t}}$

$- \cancel{8(2at^3 + bt^2)e^{2t}} + \cancel{4(at^3 + bt^2)e^{2t}} = te^{2t}$

$$6at + 2b = t$$

$$\left.\begin{array}{l} 6a = 1 \\ 2b = 0 \end{array}\right\} a = \frac{1}{6}, \ b = 0$$

$$\boxed{y_{ö2} = \frac{1}{6} t^3 e^{2t}}$$

$$y_ö = y_{ö1} + y_{ö2}$$

$$\boxed{y_ö = e^t + \frac{1}{6} t^3 e^{2t}}$$

$$y = y_H + y_ö$$

$$y = (C_1 + C_2 t)e^{2t} + e^t + \frac{1}{6} t^3 e^{2t}$$

$$x = e^t, \quad t = \ln x$$

$$\boxed{y = (C_1 + C_2 \ln x)x^2 + x + \frac{1}{6}(\ln x)^3 \cdot x^2}$$ bulunur.

Örnek: $x^3y''+x^2y'+xy=x\,\mathrm{Cosec}(\ln x),\ x>0$

Denklemini çözünüz.

Çözüm: $x=e^t,\quad t=\ln x,\quad \frac{dt}{dx}=e^{-t}$

$$y'=e^{-t}\frac{dy}{dt}\ ,\quad y''=e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$x^2y''+xy'+y=\mathrm{Cosec}(\ln x)\quad ;\ x>0$$

$$e^{2t}\cdot e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+e^{t}\cdot e^{-t}\cdot\frac{dy}{dt}+y=\mathrm{Cosec}\,t$$

$$\boxed{\frac{d^2y}{dt^2}+y=\mathrm{Cosec}\,t}$$

$$\frac{d^2y}{dt^2}+y=0$$

$$r^2+1=0\Rightarrow r_{1,2}=\mp i$$

$$\boxed{y=c_1\mathrm{Cos}t+c_2\mathrm{Sin}t}\quad \begin{matrix}c_1=c_1(t)\\ c_2=c_2(t)\end{matrix}$$

$$y'=\underbrace{c_1'\mathrm{Cos}t+c_2'\mathrm{Sin}t}_{=0\ (\text{Kabul})\ (1)}-c_1\mathrm{Sin}t+c_2\mathrm{Cos}t$$

$$y''=-c_1'\mathrm{Sin}t+c_2'\mathrm{Cos}t-c_1\mathrm{Cos}t-c_2\mathrm{Sin}t$$

$$-c_1'\mathrm{Sin}t+c_2'\mathrm{Cos}t\cancel{-c_1\mathrm{Cos}t}\cancel{-c_2\mathrm{Sin}t}\cancel{+c_1\mathrm{Cos}t}\cancel{+c_2\mathrm{Sin}t}=\mathrm{Cosec}\,t$$

$$-c_1'\mathrm{Sin}t+c_2'\mathrm{Cos}t=\mathrm{Cosec}\,t\qquad (2)$$

$$c_1'\mathrm{Cos}t+c_2'\mathrm{Sin}t=0\qquad (1)$$
$$-c_1'\mathrm{Sin}t+c_2'\mathrm{Cos}t=\mathrm{Cosec}\,t\qquad (2)$$

$$c_2' = \frac{\cos t}{\sin t} \Rightarrow c_2 = \int \frac{\cos t}{\sin t}\,dt + k_2$$

$$\boxed{c_2 = \ln|\sin t| + k_2}$$

$$-c_1' = 1 \Rightarrow c_1 = -\int dt + k_1$$

$$\boxed{c_1 = -t + k_1}$$

$$y = c_1 \cos t + c_2 \sin t$$

$$y = (-t + k_1)\cos t + (\ln|\sin t| + k_2)\sin t$$

$$y = k_1 \cos t + k_2 \sin t - t\cos t + \sin t \cdot \ln|\sin t|$$

$$\boxed{y = k_1 \cdot \cos|\ln x| + k_2 \sin|\ln x| - \ln x \cdot \cos(\ln x) + \sin(\ln x) \cdot \ln|\sin(\ln x)|}$$

Ödev: $x^3 y'' + 4x^2 y' = -1$, $x > 0$ çözünüz.

Legendre Diferansiyel Denklemi

$$a_n(ax+b)^n \cdot y^{(n)} + a_{n-1}(ax+b)^{n-1} + \dots + a_1(ax+b) \cdot y' + a_0 y = f(ax+b)$$

şeklindeki denkleme Legendre Dif. Denklemi denir.

Burada $a_0, a_1, \dots, a_n, a, b \in \mathbb{R}$ sabitler

(*) $\boxed{ax+b=e^t}$ dönüşümü veya $\boxed{ax+b=u}$ dönüşümü yapılarak çözülür.

Eğer $ax+b=u$ dönüşümü yapılırsa,

$$\frac{du}{dx}=a$$

$$y' = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = a\frac{dy}{du}$$ olur.

$$y'' = \frac{dy'}{dx} = \frac{dy'}{du}\cdot\frac{du}{dx} = a^2\frac{d^2y}{du^2}, \quad y''' = a^3\frac{d^3y}{du^3}, \dots$$

$$a_n \cdot u^n \cdot a^n \cdot \frac{d^ny}{du^n} + a_{n-1}u^{n-1}a^{n-1} \cdot \frac{d^{n-1}y}{du} + \dots + a_2 u^2 \cdot a^2 \frac{d^2y}{du^2} + a_1 \cdot a u \frac{dy}{du} + a_0 y = f(u)$$

Euler Dif. Denklemine dönüşür.

Sonrada $u=e^t$ dönüşümü yapılarak denklem sabit katsayılı denkleme dönüştürülmüş olur.

Örnek: $(3x+1)^2y''-2(3x+1)y'+6y=[\ln(3x+1)]^2$

Dif. Denk. genel çözümünü bulunuz.

Çözüm: $3x+1=e^t$ dersek $t=\ln(3x+1)$

$$\frac{dt}{dx}=\frac{3}{3x+1}=3\cdot e^{-t}$$

$$y'=\frac{dy}{dx}=\frac{dy}{dt}\cdot 3e^{-t}$$

$$y''=\frac{dy'}{dx}=\frac{dy'}{dt}\cdot 3e^{-t}=\left(\frac{d^2y}{dt^2}\cdot 3e^{-t}-3e^{-t}\frac{dy}{dt}\right)3e^{-t}$$

$$y''=3^2e^{-2t}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$3^2\cdot\underbrace{e^{2t}\cdot e^{-2t}}_{1}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)-2\cdot 3\underbrace{e^{t}\cdot e^{-t}}_{1}\frac{dy}{dt}+6y=t^2$$

$$\boxed{9\frac{d^2y}{dt^2}-15\frac{dy}{dt}+6y=t^2}$$

$$9\frac{d^2y}{dt^2}-15\frac{dy}{dt}+6y=0$$

$9r^2-15r+6=0$ $\quad \Delta=225-216=9>0$

$$r_{1,2}=\frac{15\mp 3}{18} \quad r_1=\frac{2}{3}, \quad r_2=1$$

$$\boxed{y_H=c_1e^{\frac{2}{3}t}+c_2e^{t}}$$

⑧

$y_o = at^2 + bt + c$

$y_o' = 2at + b$

$y_o'' = 2a$

$18a - 30at - 15b + 6at^2 + 6bt + 6C = t^2$

$6at^2 + (-30a + 6b)t + 18a - 15b + 6C = t^2$

$6a = 1 \Rightarrow a = \frac{1}{6}$

$-3a + 6b = 0 \Rightarrow b = \frac{1}{12}$

$18a - 15b + 6C = 0 \Rightarrow c = -\frac{7}{24}$

$$\boxed{y_o = \frac{1}{6}t^2 + \frac{1}{12}t - \frac{7}{24}}$$

$y = y_H + y_o = C_1 e^{\frac{2}{3}t} + C_2 e^t + \frac{1}{6}t^2 + \frac{1}{12}t - \frac{7}{24}$

$$\boxed{y = C_1 \cdot (3x+1)^{2/3} + C_2(3x+1) + \frac{1}{6}\ln(3x+1)^2 + \frac{1}{12}\ln(3x+1) - \frac{7}{24}}$$