Özdeğerler ve Özvektörler

1) $A=\begin{bmatrix}-1 & 2 & -4\\ 0 & 4 & 2\\ 0 & -1 & 1\end{bmatrix}$, $X'=AX$ sistemini çözünüz.

Çözüm: $(A-\lambda I)K=0$

$$\left.\begin{aligned}(-1-\lambda)K_1+2K_2-4K_3&=0\\ (4-\lambda)K_2+2K_3&=0\\ -K_2+(1-\lambda)K_3&=0\end{aligned}\right\}$$

$|A-\lambda I|=0$ $\quad\begin{vmatrix}-1-\lambda & 2 & -4\\ 0 & 4-\lambda & 2\\ 0 & -1 & 1-\lambda\end{vmatrix}=0$

$=(-1-\lambda)\left[\lambda^2-5\lambda+6\right]=0$

$\lambda_1=-1,\ \lambda_2=2,\ \lambda_3=3$ bulunur. (Özdeğerler)

$\lambda_1=-1$ için $\left.\begin{aligned}2K_2-4K_3&=0\\ 5K_2+2K_3&=0\\ -K_2+2K_3&=0\end{aligned}\right\}$ $K_2=0,\ K_3=0$ olur. $K_1=s\ (s\neq 0)$

$K_1=s\begin{bmatrix}1\\0\\0\end{bmatrix}$ $s=1$ için $K_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$

$U_1=\begin{bmatrix}1\\0\\0\end{bmatrix}e^{-t}=\begin{bmatrix}e^{-t}\\0\\0\end{bmatrix}$ ✓

$\lambda_2=2$ için $\left.\begin{aligned}-3K_1+2K_2-4K_3&=0\\ 2K_2+2K_3&=0\\ -K_2-K_3&=0\end{aligned}\right\}$ $K_2=s$ keyfi, $K_3=-s$ olur. $K_1=2s$ olur.

$K_2=s\begin{bmatrix}2\\1\\-1\end{bmatrix}$ $s=1$ için

$K_2=\begin{bmatrix}2\\1\\-1\end{bmatrix}$

$U_2=\begin{bmatrix}2\\1\\-1\end{bmatrix}e^{2t}=\begin{bmatrix}2e^{2t}\\e^{2t}\\-e^{2t}\end{bmatrix}$ ✓

$\lambda_3 = 3$ için
$$\left.\begin{aligned} -4k_1 + 2k_2 - 4k_3 &= 0 \\ k_2 + 2k_3 &= 0 \\ -k_2 - 2k_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} &k_3 = s \text{ keyfi} \\ &k_2 = -2s \\ &k_1 = -2s \text{ olur} \end{aligned}$$

$K_3 = s\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ $s=1$ olsun. $K_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$$U_3 = \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} e^{3t} = \begin{bmatrix} -2e^{3t} \\ -2e^{3t} \\ e^{3t} \end{bmatrix} \checkmark$$

$$T = \begin{bmatrix} e^{-t} & 2e^{2t} & -2e^{3t} \\ 0 & e^{2t} & -2e^{3t} \\ 0 & -e^{2t} & e^{3t} \end{bmatrix} \text{ ; Temel Matris}$$

$$C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$X = TC$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} e^{-t} & 2e^{2t} & -2e^{3t} \\ 0 & e^{2t} & -2e^{3t} \\ 0 & -e^{2t} & e^{3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 &= c_1 e^{-t} + 2c_2 e^{2t} - 2c_3 e^{3t} \\ x_2 &= c_2 e^{2t} - 2c_3 e^{3t} \\ x_3 &= -c_2 e^{2t} + c_3 e^{3t} \end{aligned}\right\} \text{ Genel Çözüm}$$

Ödev:

$$\left.\begin{aligned} x_1' &= x_1 + x_2 - 2x_3 \\ x_2' &= -x_1 + 2x_2 + x_3 \\ x_3' &= x_2 - x_3 \end{aligned}\right\}$$ Sistemin genel çözümünü bulunuz.

Ödev:

$$X' = \begin{bmatrix} 4 & -3 & -2 \\ 10 & -7 & -4 \\ -8 & 6 & 4 \end{bmatrix} X$$ sistemini çözünüz.

Teorem: Bir A kare matrisinin özdeğerleri birbirinden farklı ise, bu özdeğerlere karşılık gelen özvektörler lineer bağımsızdır.

Not: Özdeğerler katlı olduğunda özvektörler lineer bağımlı veya lineer bağımsız olabilir.

Örnek:

$$\left.\begin{aligned} y_1' &= 2y_2 - 3y_3 \\ y_2' &= -2y_1 + 4y_2 - 3y_3 \\ y_3' &= -2y_1 + 2y_2 - y_3 \end{aligned}\right\}$$ Sistemini çözünüz.

Çözüm: $(A-\lambda I)K=0$

$$\left.\begin{aligned} -\lambda k_1 + 2k_2 - 3k_3 &= 0 \\ -2k_1 + (4-\lambda)k_2 - 3k_3 &= 0 \\ -2k_1 + 2k_2 + (-1-\lambda)k_3 &= 0 \end{aligned}\right\} \quad |A-\lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 2 & -3 \\ -2 & 4-\lambda & -3 \\ -2 & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & -3 \\ 2-\lambda & 4-\lambda & -3 \\ 0 & 2 & -1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 2 & -3 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$= (2-\lambda)^2(-1-\lambda) = 0 \qquad \lambda_{1,2} = 2, \quad \lambda_3 = -1$$

$\lambda_1 = 2$ için
$$\left.\begin{aligned} -2k_1 + 2k_2 - 3k_3 &= 0 \\ -2k_1 + 2k_2 - 3k_3 &= 0 \\ -2k_1 + 2k_2 - 3k_3 &= 0 \end{aligned}\right\}$$ $\boxed{r=1}$ $\boxed{n=3}$

$n-r = 3-1 = 2$ tane bilinmeyen keyfi seçilir.

$k_2 = s$, $k_3 = 2t$ keyfi seçelim.

$k_1 = s - 3t$ olur.

$$K_1^* = s\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{K_1} + t\underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{K_2}$$

$$K_1=\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad K_2=\begin{bmatrix}-3\\0\\2\end{bmatrix} \text{ olur.}$$

$\lambda_3=-1$ için

$$\left.\begin{array}{l} k_1+2k_2-3k_3=0\\ -2k_1+5k_2-3k_3=0\\ -2k_1+2k_2 \quad =0\end{array}\right\}$$

$$\begin{bmatrix}1&2&-3&0\\-2&5&-3&0\\-2&2&0&0\end{bmatrix}\sim\begin{bmatrix}1&2&-3&0\\0&9&-9&0\\0&6&-6&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1&0\\0&1&-1&0\\0&0&0&0\end{bmatrix}$$

$r=2,\ n=3 \quad n-r=1$ tane keyfi seçilir.

$$\left.\begin{array}{l}k_1-k_3=0\\k_2-k_3=0\end{array}\right\} \begin{array}{l}k_3=s\\k_1=s\\k_2=s\end{array} \quad s=1$$

$$K_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$u_1=\begin{bmatrix}1\\1\\0\end{bmatrix}e^{2t}, \quad u_2=\begin{bmatrix}-3\\0\\2\end{bmatrix}e^{2t}, \quad u_3=\begin{bmatrix}1\\1\\1\end{bmatrix}e^{-t}$$

$$T=\begin{bmatrix}e^{2t}&-3e^{2t}&e^{-t}\\e^{2t}&0&e^{-t}\\0&2e^{2t}&e^{-t}\end{bmatrix} \qquad X=TC.$$

$$\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}e^{2t}&-3e^{2t}&e^{-t}\\e^{2t}&0&e^{-t}\\0&2e^{2t}&e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

$y_1=c_1e^{2t}-3c_2e^{2t}+c_3e^{-t}, \quad y_3=2c_2e^{2t}+c_3e^{-t}$

$y_2=c_1e^{2t}+c_3e^{-t}$

2) Kökler kompleks olsun.

$\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$ olsun.

$$U=e^{(\alpha+i\beta)t}\underbrace{\begin{bmatrix} \ \end{bmatrix}}_{K_1}$$

gerekli işlemler yapılarak çözüm bulunur.

Örnek: $X'=\begin{pmatrix}2 & 8\\ -1 & -2\end{pmatrix}X$ sistemini çözünüz

$(2-\lambda)k_1+8k_2=0$

$-k_1+(-2-\lambda)k_2=0$

$|A-\lambda I|=0$

$$\begin{vmatrix}2-\lambda & 8\\ -1 & -2-\lambda\end{vmatrix}=0$$

$\lambda^2+4=0 \Rightarrow \boxed{\lambda_{1,2}=\mp 2i}$ bulunur.

$\lambda=2i$ için

$(2-2i)k_1+8k_2=0$

$-k_1+(-2-2i)k_2=0$

$k_2=1$ keyfi

$k_1=-2-2i$ olur. $K_1=\begin{bmatrix}-2-2i\\ 1\end{bmatrix}$

$$U=e^{2it}\begin{bmatrix}-2-2i\\ 1\end{bmatrix}=(\cos 2t+i\sin 2t)\cdot\begin{bmatrix}-2-2i\\ 1\end{bmatrix}$$

$$U=\begin{bmatrix}-2\cos 2t+2\sin 2t\\ \cos 2t\end{bmatrix}+i\begin{bmatrix}-2\cos 2t-2\sin 2t\\ \sin 2t\end{bmatrix}$$

$$T=\begin{bmatrix} -2\cos 2t+2\sin 2t & -2\cos 2t-2\sin 2t \\ \cos 2t & \sin 2t \end{bmatrix}$$

$$X=TC$$

$$x_1=c_1(-2\cos 2t+2\sin 2t)+c_2(-2\cos 2t-2\sin 2t)$$

$$x_2=c_1\cos 2t+c_2\sin 2t$$

Örnek: $X'=\begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}X$ sistemini çözünüz

$(2-\lambda)k_1-k_2=0$

$2k_1+(4-\lambda)k_2=0$

$|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix}=\lambda^2-6\lambda+10=0$$

$\lambda=3+i$ için

$\lambda_{1,2}=3\mp i$ ✓

$(-1-i)k_1-k_2=0$

$k_1=1$ keyfi

$k_2=-1-i$ olur.

$$u=e^{(3+i)t}\cdot\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

$$K=\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

$$u=e^{3t}\cdot(\cos t+i\sin t)\cdot\begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

$$u = \underbrace{\begin{bmatrix} e^{3t}\cos t \\ -e^{3t}\cos t + e^{3t}\sin t \end{bmatrix}}_{u_1} + i \underbrace{\begin{bmatrix} e^{3t}\sin t \\ -e^{3t}\sin t - e^{3t}\cos t \end{bmatrix}}_{u_2}$$

$$T = \begin{bmatrix} e^{3t}\cos t & e^{3t}\sin t \\ -e^{3t}(\cos t - \sin t) & -e^{3t}(\cos t + \sin t) \end{bmatrix}$$

$$X = TC.$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 &= c_1 e^{3t}\cos t + c_2 e^{3t}\sin t \\ x_2 &= -c_1 e^{3t}(\cos t - \sin t) - c_2 e^{3t}(\cos t + \sin t) \end{aligned}\right\}$$

bulunur.

Örnek:

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_1 \\ \frac{dx_2}{dt} &= 2x_1 + x_2 - 2x_3 \\ \frac{dx_3}{dt} &= 3x_1 + 2x_2 + x_3 \end{aligned}\right\}$$ Sistemini çözünüz.

$$\left.\begin{aligned} (1-\lambda)k_1 \qquad\qquad &= 0 \\ 2k_1 + (1-\lambda)k_2 - 2k_3 &= 0 \\ 3k_1 + 2k_2 + (1-\lambda)k_3 &= 0 \end{aligned}\right\} \begin{aligned} (A-\lambda I)K &= 0 \\ |A-\lambda I| &= 0 \end{aligned}$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)\cdot\left[\lambda^2 - 2\lambda + 5\right] = 0$$

$\lambda_1 = 1 \quad \lambda_{2,3} = 1 \mp 2i$ bulunur.

$\lambda_1 = 1$ için $\left.\begin{aligned} 2k_1 - 2k_3 &= 0 \\ 3k_1 + 2k_2 &= 0 \end{aligned}\right\}$ $k_3 = s$ keyfi, $k_1 = s$ olur, $k_2 = -\frac{3s}{2}$

$$K_1 = s\begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}$$

$s=1$ için

$$K_1 = \begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix} \qquad u_1 = e^t\begin{bmatrix} 1 \\ -3/2 \\ 1 \end{bmatrix} \checkmark$$

$\lambda_2 = 1+2i$

$$\left.\begin{aligned} -2ik_1 &= 0 \\ 2k_1 - 2ik_2 - 2k_3 &= 0 \\ 3k_1 + 2k_2 - 2ik_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} &k_1 = 0 \\ &k_2 = 1 \text{ keyfi} \\ &k_3 = -i \text{ olur.} \end{aligned}$$

$$K_2 = \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$$U = e^{(1+2i)t} \cdot \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = e^{t} \cdot (\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix}$$

$$U = \begin{bmatrix} 0 \\ e^t \cos 2t \\ e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} 0 \\ e^t \sin 2t \\ -e^t \cos 2t \end{bmatrix}$$

$$T = \begin{bmatrix} e^t & 0 & 0 \\ -\frac{3}{2}e^t & e^t \cos 2t & e^t \sin 2t \\ e^t & e^t \sin 2t & -e^t \cos 2t \end{bmatrix} \qquad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$X = TC$.

$$\left.\begin{aligned} x_1 &= c_1 e^t \\ x_2 &= -\frac{3}{2}c_1 e^t + c_2 e^t \cos 2t + c_3 e^t \sin 2t \\ x_3 &= c_1 e^t + c_2 e^t \sin 2t - c_3 e^t \cos 2t \end{aligned}\right\} \text{ bulunur.}$$