Dif. Op. için Ters Sag. Prob. (YL.) 21/06/2021

Final Sınavı

1) (30 p.) $y'(1-\sin x\cos x)+y^2\cos x-y+\sin x=0$ , $y_1=\cos x$

$y=y_1+\frac{1}{z}$ , $\boxed{y=\cos x+\frac{1}{z}}$ dönüşümünü yapalım.

$y'=-\sin x-\frac{z'}{z^2}$ olur.

$\left(-\sin x-\frac{z'}{z^2}\right)(1-\sin x\cos x)+\left(\cos^2 x+\frac{2\cos x}{z}+\frac{1}{z^2}\right)\cos x-\cos x-\frac{1}{z}+\sin x=0$

$\cancel{-\sin x}+\cancel{\sin^2 x\cos x}-\frac{z'}{z^2}+\frac{z'}{z^2}\sin x\cos x+\cancel{\cos^3 x}+\frac{2\cos^2 x}{z}+\frac{\cos x}{z^2}-\cos x-\frac{1}{z}+\cancel{\sin x}$

$z'(1-\sin x\cos x)-2z\cos^2 x+z-\cos x=0$

$$z'+\frac{(1-2\cos^2 x)}{1-\sin x\cos x}z=\frac{\cos x}{1-\sin x\cos x}$$

$$z'-\frac{2\cos 2x}{2-\sin 2x}z=\frac{2\cos x}{2-\sin 2x}$$ Lineer Dif. Denklem

$$\lambda=e^{-\int\frac{2\cos 2x}{2-\sin 2x}}=e^{\ln(2-\sin 2x)}=2-\sin 2x$$

$$\left(z\cdot(2-\sin 2x)\right)'=2\cos x$$

$$z\cdot(2-\sin 2x)=2\sin x+c$$

$$\boxed{z = \frac{2\sin x + c}{2 - \sin 2x}}$$

$$y = \cos x + \frac{1}{z}$$

$$\boxed{y = \cos x + \frac{2 - \sin 2x}{2\sin x + c}}$$ bulunur.

2) (35p). $x^3y''+x^2y'+xy= x\tan^3(\ln x)\ ,\ x>0$

$$x^2y''+xy'+y = \tan^3(\ln x) \quad \text{Euler Dif. Denk.}$$

$x=e^t$ dersek $\quad t=\ln x$

$$y'=\frac{dy}{dt}\cdot e^{-t}\ ,\quad y''=\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)e^{-2t}$$

$$\frac{d^2y}{dt^2}-\cancel{\frac{dy}{dt}}+\cancel{\frac{dy}{dt}}+y=\tan^3 t$$

$$\frac{d^2y}{dt^2}+y=\tan^3 t$$

$$\frac{d^2y}{dt^2}+y=0 \Rightarrow r^2+1=0 \quad r_{1,2}=\mp i$$

$$\boxed{y = C_1\cos t + C_2\sin t} \qquad \begin{matrix} C_1=C_1(t)=? \\ C_2=C_2(t)=? \end{matrix}$$

$$\left.\begin{matrix} C_1'\cos t+C_2'\sin t=0 \\ -C_1'\sin t+C_2'\cos t=\tan^3 t \end{matrix}\right\} \quad C_2'=\cos t\cdot\tan^3 t$$

$$C_2=\int\frac{\sin^3 t}{\cos^2 t}dt+k_2$$

$$-C_1'=\frac{\sin^4 t}{\cos^3 t}$$

$$C_1=-\int\frac{\sin^4 t}{\cos^3 t}dt+k_1$$

$$C_2 = \int \frac{(1-\cos^2 t)\sin t}{\cos^2 t}\,dt + k_2$$

$$C_2 = \int \frac{(1-u^2)}{u^2}\,du + k_2$$

$$C_2 = -\frac{1}{u} - u + k_2$$

$$\boxed{C_2 = -\frac{1}{\cos t} - \cos t + k_2}$$

$$C_1 = -\int \frac{(1-\cos^2 t)^2}{\cos^3 t}\,dt + k_1$$

$$C_1 = -\int \frac{1-2\cos^2 t+\cos^4 t}{\cos^3 t}\,dt + k_1$$

$$C_1 = -\Big[\underbrace{\int \sec^3 t\,dt}_{I} - 2\int \sec t\,dt + \int \cos t\,dt\Big] + k_1$$

$$\boxed{C_1 = -\Big[I - 2\ln(\sec t + \tan t) + \sin t\Big] + k_1}$$

$y = C_1\cos t + C_2\sin t$ de $C_1$ ve $C_2$ yerine yazılır.

$(x = e^t,\ t = \ln x)$ yerine yazılır.

3) (35p.) $y'' - e^{2y} + e^{y} = 0 \quad y(0)=0, \; y'(0)=2$

$y' = p$ dersek, $y'' = p\dfrac{dp}{dy}$ olur.

$$p\cdot\frac{dp}{dy} - e^{2y} + e^{y} = 0$$

$$\int p\,dp + \int(-e^{2y} + e^{y})\,dy = \int 0$$

$$\frac{p^2}{2} - \frac{1}{2}e^{2y} + e^{y} = \frac{C_1}{2}$$

$$p^2 - e^{2y} + 2e^{y} = C_1$$

$$p = \sqrt{e^{2y} - 2e^{y} + C_1}$$

$$2 = \sqrt{1 - 2 + C_1} \Rightarrow -1 + C_1 = 4$$

$$C_1 = 5$$

$$p = \sqrt{e^{2y} - 2e^{y} + 5}$$

$$\underbrace{\int \frac{dy}{\sqrt{e^{2y} - 2e^{y} + 5}}}_{I} = \int dx \Rightarrow \boxed{I = x + C_2}$$

$e^{y} = t$ dersek $e^{y}dy = dt$ olur.

$$I = \int \frac{dt}{t\sqrt{t^2 - 2t + 5}} \qquad t = \frac{1}{u} \text{ dersek } dt = -\frac{1}{u^2}du$$

$$I = -\int \frac{du}{u\cdot\sqrt{\frac{1}{u^2} - \frac{2}{u} + 5}} = -\int \frac{du}{\sqrt{5u^2 - 2u + 1}} = -\frac{1}{\sqrt{5}}\int \frac{du}{\sqrt{u^2 - \frac{2}{5}u + \frac{1}{5}}}$$

$$I = -\frac{1}{\sqrt{5}} \int \frac{du}{\sqrt{\left(u-\frac{1}{5}\right)^2 - \frac{1}{25} + \frac{1}{5}}}$$

$u - \frac{1}{5} = \theta$
$du = d\theta$

$$I = -\frac{1}{\sqrt{5}} \int \frac{du}{\sqrt{\theta^2 + \frac{4}{25}}}$$

$$I = -\frac{1}{\sqrt{5}} \ln\left(\theta + \sqrt{\theta^2 + \frac{4}{25}}\right)$$

$$I = -\frac{1}{\sqrt{5}} \ln\left(u - \frac{1}{5} + \sqrt{\left(u-\frac{1}{5}\right)^2 + \frac{4}{25}}\right)$$

$$I = -\frac{1}{\sqrt{5}} \ln\left(e^{-y} - \frac{1}{5} + \sqrt{\left(e^{-y} - \frac{1}{5}\right)^2 + \frac{4}{25}}\right)$$

$$I = x + c_2$$

$$y(0) = 0 \Rightarrow I = -\frac{1}{\sqrt{5}} \ln\left(1 - \frac{1}{5} + \sqrt{\left(1-\frac{1}{5}\right)^2 + \frac{4}{25}}\right)$$

$$I = -\frac{1}{\sqrt{5}} \ln\left(\frac{4}{5} + \frac{2\sqrt{5}}{5}\right) = c_2 \text{ olur.}$$

$$\boxed{-\frac{1}{\sqrt{5}} \ln\left(\frac{4+2\sqrt{5}}{5}\right) + x = -\frac{1}{\sqrt{5}} \ln\left(e^{-y} - \frac{1}{5} + \sqrt{\left(e^{-y} - \frac{1}{5}\right)^2 + \frac{4}{25}}\right)}$$

Sonuç ✓

Not: $y'' - e^{2y} + e^{y} = 0$ Dif. Denkleminde $y$ bilinmeyen fonksiyon. Bu denklemin Laplace alınamaz....