Örnek: $x^2y'' - (x+1)y' = 2x+4$ Dif. Denklemin $x_0=-1$ noktası civarında seri çözümünü elde ediniz. (Çözüm serisini 5. derece terimlere kadar yazınız) Ayrıca yakınsaklık yarıçapını ve yakınsaklık aralığınıda belirleyiniz.

Çözüm: $x=0$ Denklemin tekil noktası

$x_0=-1$ adi nokta.

$x=0$'ın $x_0=-1$'e uzaklığı 1 dir. O halde çözüm serisinin yakınsaklık yarıçapı $R=1$ olur.

$|x+1|<1$ olur. Çözüm serisinin yakınsaklık aralığı

$\boxed{-2<x<0}$ Yakınsaklık aralığı

Çözüm $y=\sum_{n=0}^{\infty} a_n(x+1)^n = a_0+a_1(x+1)+a_2(x+1)^2+a_3(x+1)^3+\dots$

$$y'=\sum_{n=1}^{\infty} a_n n(x+1)^{n-1}, \quad y''=\sum_{n=2}^{\infty} a_n n(n-1)(x+1)^{n-2}$$

$$x^2\sum_{n=2}^{\infty} a_n n(n-1)(x+1)^{n-2} - (x+1)\cdot\sum_{n=1}^{\infty} a_n n(x+1)^{n-1} = 2x+4$$

$$x^2 = (x+1)^2 - 2x - 1 = (x+1)^2 - 2(x+1) + 1$$

$$\left[(x+1)^2 - 2(x+1) + 1\right] \sum_{n=2}^{\infty} a_n n(n-1)(x+1)^{n-2} - (x+1) \sum_{n=1}^{\infty} a_n n (x+1)^{n-1} =$$
$$= 2x+4$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x+1)^n - \sum_{n=2}^{\infty} 2a_n n(n-1)(x+1)^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)(x+1)^{n-2}$$
$$- \sum_{n=1}^{\infty} a_n n (x+1)^n = 2x+4$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x+1)^n - \sum_{n=1}^{\infty} 2a_{n+1}(n+1)(n)(x+1)^n + \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)(x+1)^n$$
$$- \sum_{n=1}^{\infty} a_n n (x+1)^n = 2x+4$$
$$= 2(x+1) + 2$$

$$-4a_2(x+1) + 2a_2 + 6a_3(x+1) - a_1(x+1) +$$
$$+ \sum_{n=2}^{\infty} \left[a_n n(n-1) - 2a_{n+1}(n+1)n + a_{n+2}(n+2)(n+1) - a_n n\right](x+1)^n$$
$$= 2(x+1) + 2$$

$$(-4a_2+6a_3-a_1)(x+1)+2a_2+\sum_{n=2}^{\infty}[\;\cdots\;](x+1)^n=$$

$$=2(x+1)+2$$

$$\left.\begin{aligned}-4a_2+6a_3-a_1&=2\\ 2a_2&=2\end{aligned}\right\}\quad \begin{aligned}a_2&=1\\ a_3&=1+\frac{a_1}{6}\end{aligned}$$

$$a_n n(n-1)-2a_{n+1}(n+1)n+a_{n+2}(n+2)(n+1)-a_n\cdot n=0,\; n=2,3,\ldots$$

$$\boxed{a_{n+2}=\frac{2n(n+1)a_{n+1}-n(n-2)a_n}{(n+2)(n+1)}\;;\; n=2,3,4,\ldots}$$

$n=2$ için $a_4=\frac{12a_3-0}{8}=\frac{3a_3}{2}=\frac{3}{2}\left(1+\frac{a_1}{6}\right)=\frac{3}{2}+\frac{a_1}{4}$

$n=3$ için $a_5=\frac{24a_4-3a_3}{20}=\frac{24\left(\frac{3}{4}+\frac{a_1}{8}\right)-3\left(1+\frac{a_1}{6}\right)}{20}$

$$=\frac{18+3a_1-3-\frac{a_1}{2}}{20}$$

$$\boxed{a_5=\frac{3}{4}+\frac{a_1}{60}}\qquad \boxed{a_5=\frac{15}{20}+\frac{a_1}{60}}\;,\;\ldots$$

$$y = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3 + a_4(x+1)^4 + a_5(x+1)^5 + \cdots$$

$$y = a_0 + a_1(x+1) + (x+1)^2 + \left(1 + \frac{a_1}{6}\right)(x+1)^3 + \left(\frac{3}{2} + \frac{a_1}{4}\right)(x+1)^4 + \left(\frac{3}{4} + \frac{a_1}{60}\right)(x+1)^5 + \cdots$$

$$y = a_0 + a_1\left[(x+1) + \frac{(x+1)^3}{6} + \frac{(x+1)^4}{4} + \frac{(x+1)^5}{60} + \cdots\right] + (x+1)^2 + (x+1)^3 + \frac{3}{2}(x+1)^4 + \frac{3}{4}(x+1)^5 + \cdots$$

bulunur.

Örnek: $x^2y'' - (x+1)y' = 2x+4$; $x_0=-1$

$x+1=t$ dersek $\quad \frac{dt}{dx}=1$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}$$

$y'' = \frac{d^2y}{dt^2}$ dur.

$$(t-1)^2\cdot\frac{d^2y}{dt^2} - t\frac{dy}{dt} = 2(t-1)+4$$

$$(t-1)^2\frac{d^2y}{dt^2} - t\frac{dy}{dt} = 2t+2 \qquad t_0=0$$

$$y = \sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \dots$$

$$y' = \sum_{n=1}^{\infty} a_n \cdot n\, t^{n-1}, \quad y'' = \sum_{n=2}^{\infty} a_n n(n-1) t^{n-2}$$

$$(t^2-2t+1)\sum_{n=2}^{\infty} a_n\cdot n(n-1)t^{n-2} - t\sum_{n=1}^{\infty} a_n n\, t^{n-1} = 2t+2$$

$$\sum_{n=2}^{\infty} a_n n(n-1)t^n - \sum_{n=2}^{\infty} 2a_n\cdot n(n-1)t^{n-1} + \sum_{n=2}^{\infty} a_n n(n-1)t^{n-2} -$$

$$- \sum_{n=1}^{\infty} a_n n\, t^n = 2t+2$$

$$\sum_{n=2}^{\infty} a_n n(n-1)\cdot t^n - \sum_{n=1}^{\infty} 2a_{n+1}(n+1)\cdot n\, t^n + \sum_{n=0}^{\infty} a_{n+2}\cdot(n+2)(n+1)\, t^n$$

$$-\sum_{n=1}^{\infty} a_n n t^n = 2t+2$$

$$-$$

$$\underbrace{-\cancel{4}a_2 t + 2a_2 + 6a_3 t - a_1 t}_{} + \sum_{n=2}^{\infty}\Big[a_n n(n-1) - 2a_{n+1}(n+1)n$$

$$+ a_{n+2}(n+2)(n+1) - a_n\cdot n\Big]t^n = 2t+2$$

$$2a_2 = 2 \Rightarrow \boxed{a_2 = 1}$$

$$-\cancel{4}a_2 + 6a_3 - a_1 = 2 \Rightarrow a_3 = \frac{6+a_1}{6} =$$

$$\boxed{a_3 = 1 + \frac{a_1}{6}}$$

$$\boxed{a_{n+2} = \frac{2(n+1)n\, a_{n+1} - n(n-2)\, a_n}{(n+2)(n+1)} \quad ; \; n = 2,3,4,\dots}$$

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots \qquad (t = x+1)$$

$$y = a_0 + a_1 t + t^2 + \left(1 + \frac{a_1}{6}\right)t^3 + \left(\frac{3}{2} + \frac{a_1}{4}\right)t^4 + \left(\frac{3}{4} + \frac{a_1}{60}\right)t^5 + \cdots$$

$$y = a_0 + a_1(x+1) + (x+1)^2 + \left(1 + \frac{a_1}{6}\right)(x+1)^3 + \left(\frac{3}{2} + \frac{a_1}{4}\right)(x+1)^4 + \left(\frac{3}{4} + \frac{a_1}{60}\right)(x+1)^5 + \cdots$$

$$y = a_0 + a_1\left[(x+1) + \frac{(x+1)^3}{6} + \frac{(x+1)^4}{4} + \frac{(x+1)^5}{60} + \cdots\right] + (x+1)^2 + (x+1)^3 + \frac{3}{2}(x+1)^4 + \frac{3}{4}(x+1)^5 + \cdots$$

bulunur.

(Ödev)

Örnek: $(x+2)y''+(x-1)y'-y=0$ Dif. denkleminin $x_0=1$ noktası civarında seri çözümünü elde ediniz. (Çözüm serisini 4. derece terimlere kadar yazınız) Ayrıca yakınsaklık aralığını belirleyiniz.

Çözüm: