$(A-\lambda I)K_3 = K_2$

$$\left.\begin{aligned} 2k_1+k_2-k_3 &= 2 \\ k_2-k_3 &= 1 \end{aligned}\right\} \quad \begin{aligned} &k_3 = s \text{ keyfi} \\ &k_2 = s+1 \text{ olur.} \\ &k_1 = \tfrac{1}{2} \text{ olur.} \end{aligned}$$

$$K_3 = \begin{bmatrix} \frac{1}{2} \\ s+1 \\ s \end{bmatrix} \quad s=1 \text{ olsun} \quad K_3 = \begin{bmatrix} 1/2 \\ 2 \\ 1 \end{bmatrix}$$

$$U_3 = \left(\tfrac{1}{2}K_1 t^2 + K_2 t + K_3\right)e^t$$

$$U_3 = \left\{ \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1/2 \\ 2 \\ 1 \end{bmatrix} \right\} e^t$$

$$U_3 = e^t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2}t^2+2t+2 \\ \frac{1}{2}t^2+t+1 \end{bmatrix} \qquad T = [U_1\ U_2\ U_3]$$

$$T = \begin{bmatrix} 0 & 0 & \frac{1}{2}e^t \\ e^t & (t+2)e^t & (\frac{1}{2}t^2+2t+2)e^t \\ e^t & (t+1)e^t & (\frac{1}{2}t^2+t+1)e^t \end{bmatrix} \qquad X = TC.$$

$$\left.\begin{aligned} x_1 &= \tfrac{1}{2}c_3 e^t \\ x_2 &= c_1 e^t + c_2(t+2)e^t + c_3(\tfrac{1}{2}t^2+2t+2)e^t \\ x_3 &= c_1 e^t + c_2(t+1)e^t + c_3(\tfrac{1}{2}t^2+t+1)e^t \end{aligned}\right\} \text{ Genel çözüm}$$

Örnek:

$$\left.\begin{aligned} y_1' &= y_1 + y_3 \\ y_2' &= y_2 + 2y_3 \\ y_3' &= -6y_1 + 3y_2 + y_3 \end{aligned}\right\}$$ Sistemini çözünüz.

Çözüm: $(A-\lambda I)Y = 0$ $\quad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -6 & 3 & 1 \end{bmatrix}$

$$\left.\begin{aligned} (1-\lambda)k_1 + k_3 &= 0 \\ (1-\lambda)k_2 + 2k_3 &= 0 \\ -6k_1 + 3k_2 + (1-\lambda)k_3 &= 0 \end{aligned}\right\}$$ 2

$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = ?$

$|A-\lambda I| = 0$ $\quad \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ -6 & 3 & 1-\lambda \end{vmatrix} = 0$

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2(1-\lambda) & 1-\lambda & 2 \\ 0 & 3 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$$

$\lambda_{1,2,3} = 1$ (üç katlı kök)

$\lambda_1 = 1$ için

$$\left.\begin{aligned} k_3 &= 0 \\ 2k_3 &= 0 \\ -6k_1 + 3k_2 &= 0 \end{aligned}\right\}$$ $k_3 = 0$, $k_1 = s$ keyfi, $k_2 = 2s$ olur $(s=1)$ $\quad K_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

$$u_1 = e^t \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$(A-\lambda_1 I)K_2 = K_1$ den K_2 bulunur.

$(A-\lambda_1 I)K_3 = K_2$ den K_3 bulunur.

$$u_2 = (K_1 t + K_2)e^t$$

$$u_3 = \left(\frac{1}{2}t^2 K_1 + tK_2 + K_3\right)e^t$$

$$T = [u_1 \; u_2 \; u_3]$$

$$\boxed{Y = TC.}$$

Not: $\lambda_{1,2,3}=\lambda$ üç katlı özdeğerine iki tane lineer bağımsız K_1 ve K_2 öz vektörleri karşılık geliyorsa, ele alınan denklem sisteminin iki çözümü ; $u_1 = K_1 e^{\lambda t}$, $u_2 = K_2 e^{\lambda t}$ olacaktır.

Üçüncü çözüm ise ;

(**) $\boxed{u_3 = \xi t e^{\lambda t} + \eta e^{\lambda t}}$ olacaktır.

Burada $(A-\lambda I)\xi = 0$

$(A-\lambda I)\eta = \xi$ 'den ξ ve η bulunur.

Burada ξ ; K_1 ve K_2 öz vektörlerinin lineer birleşimleri olarak seçilirse ;

(*) $(A-\lambda I)\eta = \xi$ denkleminden η bulunur.

Eğer $\xi = c_1 K_1 + c_2 K_2$ şeklinde alınırsa c_1 ve c_2 öyle seçilebilirki (*) denkleminden η çözülebilir

Örnek: $X' = \begin{bmatrix} 5 & -3 & -2 \\ 8 & -5 & -4 \\ -4 & 3 & 3 \end{bmatrix} X$ sistemini çözünüz.

Çözüm: $(A-\lambda I)K = 0$

$$\left.\begin{aligned} (5-\lambda)k_1 - 3k_2 - 2k_3 &= 0 \\ 8k_1 + (-5-\lambda)k_2 - 4k_3 &= 0 \\ -4k_1 + 3k_2 + (3-\lambda)k_3 &= 0 \end{aligned}\right\}$$

$$|A-\lambda I| = 0 \quad \begin{vmatrix} 5-\lambda & -3 & -2 \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 1-\lambda \\ 8 & -5-\lambda & -4 \\ -4 & 3 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 8 & -5-\lambda & -12 \\ -4 & 3 & 7-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-2\lambda+1) = 0$$

$(1-\lambda)^3 = 0 \Rightarrow \lambda_{1,2,3} = 1$ üç katlı kök

$$\lambda_1 = 1 \text{ için } \left.\begin{aligned} 4k_1 - 3k_2 - 2k_3 &= 0 \\ 8k_1 - 6k_2 - 4k_3 &= 0 \\ -4k_1 + 3k_2 + 2k_3 &= 0 \end{aligned}\right\}$$

$k_2 = s$

$k_3 = t$ olsun. (keyfi)

$k_1 = \dfrac{3s+2t}{4}$

$$K = s\begin{bmatrix} 3/4 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$s=4$, $t=0$ olsun.

$$K_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$ olur

$$u_1 = K_1 e^t = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} e^t$$

$s=0$, $t=2$ olsun.

$$K_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$ olur.

$$u_2 = K_2 e^t = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} e^t$$

$$\xi = c_1 K_1 + c_2 K_2 = c_1\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3c_1 + c_2 \\ 4c_1 \\ 2c_2 \end{bmatrix}$$

olmak üzere 3. çözüm : $u_3 = \xi t e^t + \eta e^t$ dir.
Lineer bağımsız

$(A-\lambda I)\eta = \xi$

$$\left.\begin{aligned} 4\eta_1 - 3\eta_2 - 2\eta_3 &= 3c_1 + c_2 \\ 8\eta_1 - 6\eta_2 - 4\eta_3 &= 4c_1 \\ -4\eta_1 + 3\eta_2 + 2\eta_3 &= 2c_2 \end{aligned}\right\}$$

$$\left[\begin{array}{ccc|c} 4 & -3 & -2 & 3c_1+c_2 \\ 0 & 0 & 0 & -2c_1-2c_2 \\ 0 & 0 & 0 & 3c_1+3c_2 \end{array}\right] \qquad \left.\begin{array}{l} -2c_1-2c_2=0 \\ 3c_1+3c_2=0 \end{array}\right\}$$

$$\boxed{c_2=-c_1}$$

$c_1=1$ olsun, $c_2=-1$ olur.

$$\xi=c_1K_1+c_2K_2=K_1-K_2=\begin{bmatrix}2\\4\\-2\end{bmatrix}$$

$$\xi=\begin{bmatrix}2\\4\\-2\end{bmatrix}$$ olur.

$$(A-\lambda I)\eta=\xi \qquad (\eta=K_3)$$

$$(A-\lambda I)K_3=\xi$$

$$\left.\begin{array}{l} 4k_1-3k_2-2k_3=2 \\ 8k_1-6k_2-4k_3=4 \\ -4k_1+3k_2+2k_3=-2 \end{array}\right\}$$

$k_2=s$, $k_3=t$ keyfi olsun.

$$k_1=\frac{3s+2t+2}{4}$$

$$K_3=\begin{bmatrix}\frac{3s+2t+2}{4}\\ s\\ t\end{bmatrix}$$

$s=6$, $t=0$ olsun.

$$\eta=K_3=\begin{bmatrix}5\\6\\0\end{bmatrix}$$

$$u_3=\xi te^t+\eta e^t$$

$$u_3=t\begin{bmatrix}2\\4\\-2\end{bmatrix}e^t+\begin{bmatrix}5\\6\\0\end{bmatrix}e^t$$

$T = [u_1 \; u_2 \; u_3]$

$$T = \begin{bmatrix} 3e^t & e^t & (2t+5)e^t \\ 4e^t & 0 & (4t+6)e^t \\ 0 & 2e^t & -2te^t \end{bmatrix} \qquad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$X = TC$$

$$\left.\begin{aligned} x_1 &= 3c_1e^t + c_2e^t + c_3(2t+5)e^t \\ x_2 &= 4c_1e^t + c_3(4t+6)e^t \\ x_3 &= 2c_2e^t - 2c_3te^t \end{aligned}\right\}$$

Eğer $x_1(0)=0$, $x_2(0)=2$, $x_3(0)=2$ ise

$c_1 = 8$, $c_2 = 1$, $c_3 = -5$ olur..

$$\left.\begin{aligned} x_1 &= 24e^t + e^t - 5(2t+5)e^t \\ x_2 &= 32e^t - 5(4t+6)e^t \\ x_3 &= 2e^t + 10te^t \end{aligned}\right\} \quad \left.\begin{aligned} x_1 &= -10te^t \\ x_2 &= 2e^t - 20te^t \\ x_3 &= 2e^t + 10te^t \end{aligned}\right\}$$

olur.

Örnek 4: $X' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} X + \begin{bmatrix} e^{-2t} \\ -2e^{t} \end{bmatrix}$

Çözüm: $\boxed{X = X_H + X_ö}$

$X_H = ?$ $\quad X' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} X$

$(A - \lambda I)K = 0$

$(1-\lambda)k_1 + k_2 = 0$

$4k_1 + (-2-\lambda)k_2 = 0$

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$

$\lambda^2 + \lambda - 6 = 0 \quad \lambda_1 = -3 \quad \lambda_2 = 2$ bulunur

$\lambda_1 = -3$ için $\left.\begin{aligned} 4k_1 + k_2 = 0 \\ 4k_1 + k_2 = 0 \end{aligned}\right\}$ $k_1 = 1$ keyfi, $k_2 = -4$ olur $\quad K_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

$u_1 = e^{\lambda_1 t} K_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix} e^{-3t}$ ✓

$\lambda_2 = 2$ için $\left.\begin{aligned} -k_1 + k_2 = 0 \\ 4k_1 - 4k_2 = 0 \end{aligned}\right\}$ $k_1 = k_2 = 1$ olsun. $\quad K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$u_2 = e^{\lambda_2 t} K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$ ✓ $\quad T = \begin{bmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$X = TC.$

$\left.\begin{aligned} x_1 &= c_1 e^{-3t} + c_2 e^{2t} \\ x_2 &= -4c_1 e^{-3t} + c_2 e^{2t} \end{aligned}\right\}$ olur. $(*)$

18 EK-2

$x_o^{o} = ?$

Sabitin değişimi yöntemi
(Sabitlerin)

$c_1 = c_1(t) = ?, \quad c_2 = c_2(t) = ?$

$x_1' = c_1' e^{-3t} + c_2' e^{2t} - 3c_1 e^{-3t} + 2c_2 e^{2t}$

$x_2' = -4c_1' e^{-3t} + c_2' e^{2t} + 12c_1 e^{-3t} + 2c_2 e^{2t}$

$c_1' e^{-3t} + c_2' e^{2t} - 3c_1 e^{-3t} + \cancel{2c_2 e^{2t}} = c_1 e^{-3t} + \cancel{c_2 e^{2t}} - 4c_1 e^{-3t} + \cancel{c_2 e^{2t}} + e^{-2t}$

$-4c_1' e^{-3t} + c_2' e^{2t} + 12c_1 e^{-3t} + \cancel{2c_2 e^{2t}} = 4c_1 e^{-3t} + \cancel{4c_2 e^{2t}} + 8c_1 e^{-3t} - \cancel{2c_2 e^{2t}} - 2e^{t}$

(1) $c_1' e^{-3t} + c_2' e^{2t} = e^{-2t}$

(2) $-4c_1' e^{-3t} + c_2' e^{2t} = -2e^{t}$

1. Yol : $5c_1' e^{-3t} = e^{-2t} + 2e^{t} \Rightarrow c_1 = \frac{1}{5}\int (e^{t} + 2e^{4t})\,dt + k_1$

$c_1 = \frac{1}{5}\left(e^{t} + \frac{1}{2}e^{4t}\right) + k_1$ ✓

$5c_2' e^{2t} = 4e^{-2t} - 2e^{t} \Rightarrow c_2 = \frac{1}{5}\int (4e^{-4t} - 2e^{-t})\,dt + k_2$

$c_2 = \frac{1}{5}\left(-e^{-4t} + 2e^{-t}\right) + k_2$ ✓

c_1 ve c_2 yi Homojen sistemin (*) genel çözümünde yerine yazalım.

$x_1 = \left[\frac{1}{5}\left(e^{t} + \frac{1}{2}e^{4t}\right) + k_1\right]e^{-3t} + \left[\frac{1}{5}\left(-e^{-4t} + 2e^{-t}\right) + k_2\right]e^{2t}$

$x_2 = -4\left[\frac{1}{5}\left(e^{t} + \frac{1}{2}e^{4t}\right) + k_1\right]e^{-3t} + \left[\frac{1}{5}\left(-e^{-4t} + 2e^{-t}\right) + k_2\right]e^{2t}$