

$$19. h(t) = f^{(n)}(t) \Rightarrow$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - \dots - f^{(n-1)}(0)$$

$$20. h(t) = (-1)^n \underset{t}{f}^{(n)}(t) \Rightarrow \mathcal{L}\{(-1)^n \underset{t}{f}^{(n)}(t)\} = F^{(n)}(s)$$

$$\hookrightarrow \mathcal{L}\{f(t)\} = F(s) \text{ ise}$$

$$\mathcal{L}\{(-1)^n t^n f(t)\} = F^{(n)}(s) \text{ dir.}$$

$$\lim_{t \rightarrow 0} \frac{f(t)}{t} \text{ ise}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s) ds$$



# Bazı Özel Fonksiyonların Laplace Dönüşümleri

## 1.1) Basamak Fonksiyonu

Birim basamak fonksiyonu (Heaviside fonksiyonu)

$c > 0$  olmak üzere,

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

şeklinde tanımlanır. Birim basamak fonksiyonunun Laplace

dönüşümü

$$L\{u_c(t)\} = \int_0^{\infty} e^{-st} u_c(t) dt = \int_0^c e^{-st} \cdot 0 dt + \int_c^{\infty} e^{-st} \cdot 1 dt$$

$$= \lim_{R \rightarrow \infty} \int_c^R e^{-st} dt = \frac{e^{-cs}}{s}, \quad s > 0$$

şeklinde dir. Bir basamak fonksiyonu yardımı ile,

$$y = h(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

formundaki bir fonksiyon;

$$h(t) = u_c(t) \cdot f(t-c)$$

şeklinde yazılabilir.



$c > 0$  sabit olmak üzere,  $h(t) = u_c(t) \cdot f(t-c)$  formunda  
bir fonksiyonun Laplace dönüşümü, eğer  $s > a \geq 0$  için  
 $F(s) = \mathcal{L}\{f(t)\}$  mevcut ise,

$$\mathcal{L}\{u_c(t) \cdot f(t-c)\} = e^{-cs} \mathcal{L}\{f(t)\} = e^{-cs} F(s), \quad s > a$$

ve

$$\mathcal{L}^{-1}\{e^{-cs} F(s)\} = u_c(t) \cdot f(t-c) \quad \checkmark$$

dir.

$$\mathcal{L}\{e^{ct} \cdot f(t)\} = F(s-c), \quad s > a+c$$

$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct} f(t) \quad \checkmark$$

Örnek:

$$f(t) = \begin{cases} \cos t, & 0 \leq t < \frac{\pi}{3} \\ \cos t + \sin(t - \frac{\pi}{3}), & t \geq \frac{\pi}{3} \end{cases}$$

funksiyonunun  $\mathcal{L}\{f(t)\} = ?$  bulunur.

Çözüm:  $f(t)$  fonksiyonu;

$$f(t) = \cos t + h(t) = \cos t + \begin{cases} 0, & 0 \leq t < \frac{\pi}{3} \\ \sin(t - \frac{\pi}{3}), & t \geq \frac{\pi}{3} \end{cases}$$

şeklinde yazılabilir.



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$h(t)$  fonksiyonunda ;

$$u_{\frac{\pi}{3}}(t) = \begin{cases} 0, & 0 \leq t < \frac{\pi}{3} \\ 1, & t \geq \frac{\pi}{3} \end{cases}$$

olmak üzere

$$h(t) = u_{\frac{\pi}{3}}(t) \cdot \sin\left(t - \frac{\pi}{3}\right)$$

şeklinde yazılabilir.  $f(t)$  fonksiyonu,

$$f(t) = \cos t + u_{\frac{\pi}{3}}(t) \sin\left(t - \frac{\pi}{3}\right)$$

formunda yazılabilir. Buradan Laplace dönüşümün lineerliği-  
de kullanılırsa,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos t\} + \mathcal{L}\left\{u_{\frac{\pi}{3}}(t) \cdot \sin\left(t - \frac{\pi}{3}\right)\right\}$$

$$= \mathcal{L}\{\cos t\} + e^{-\frac{\pi}{3}s} \mathcal{L}\{\sin t\}$$

olar.

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+1} + e^{-\frac{\pi}{3}s} \frac{1}{s^2+1}$$

şeklinde bulunur.



Örnek:  $F(s) = \frac{3-7e^{-3s}}{s^2}$  ise  $\mathcal{L}^{-1}\{F(s)\} = ?$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{3}{s^2}\right\} + \mathcal{L}^{-1}\left\{-\frac{7e^{-3s}}{s^2}\right\}$$

$$= 3 \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + 7 \mathcal{L}^{-1}\left\{-\frac{e^{-3s}}{s^2}\right\}$$

$$f(t) = 3t - 7u_3(t) \cdot (t-3)$$

İmpuls Fonksiyonu (Dirac delta fonksiyonu)

$$\delta(t-t_0) = 0, \quad t \neq t_0$$

$$\int_{-\infty}^{\infty} \delta(t-t_0) dt = 1, \quad \int_{-\infty}^{\infty} f(t) \cdot \delta(t-t_0) dt = f(t_0)$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as}$$

$$\textcircled{A} \quad \mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}, \quad t_0 > 0$$

$$\mathcal{L}\{\delta(t)\} = \lim_{t_0 \rightarrow 0} e^{-st_0} = 1$$

$$\Rightarrow \mathcal{L}\{\delta(t)\} = 1$$

örnek:  $g(t) = 2\delta(t-8) + 5\delta(t+2)$   $\mathcal{L}\{g(t)\} = ?$

$$\mathcal{L}\{g(t)\} = 2 \mathcal{L}\{\delta(t-8)\} + 5 \mathcal{L}\{\delta(t+2)\}$$

$$= 2e^{-8s} + 5e^{-(-2)s} = 2e^{-8s} + 5e^{2s} \quad \text{bulunur.}$$



## Başlangıç Değer Probleminin Çözümü

(Initial value problem)

Örnek:  $y'' + y' + y = \delta(t-2)$ ,  $y(0)=0$ ,  $y'(0)=0$

başlangıç değer problemi çözümü.

Çözüm: Ele alınan dif. denklemin sağ tarafındaki  $\delta(t-2)$  fonksiyonu sürekli bir fonksiyon olmadığından, sağ yanlı dif. denklemin Laplace dönüşümü ile çözümü aranır. Dif. denklemin her iki tarafına Laplace dönüşümü uygulanırsa;

$$L\{\delta(t-t_0)\} = e^{-st_0}$$

$$L\{y'' + y' + y\} = L\{\delta(t-2)\}$$

$$L\{y''\} + L\{y'\} + L\{y\} = e^{-2s}$$

$$s^2 Y(s) - s y(0) - y'(0) + s Y(s) - y(0) + Y(s) = e^{-2s}$$

$$Y(s) = \frac{e^{-2s}}{s^2 + s + 1} = e^{-2s} \frac{1}{s^2 + s + 1} = e^{-2s} F(s)$$

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2 + s + 1}\right\} = L^{-1}\left\{\frac{1}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\}$$

$$= L^{-1}\left\{\frac{\frac{\sqrt{3}}{2} \cdot \frac{2}{\sqrt{3}}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\} = \frac{2}{\sqrt{3}} L^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{(s + \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\right\}$$



$$= \frac{2}{\sqrt{3}} \mathcal{L}^{-1} \left\{ \frac{\frac{\sqrt{3}}{2}}{(s - (-\frac{1}{2}))^2 + (\frac{\sqrt{3}}{2})^2} \right\}$$

$\mathcal{L}^{-1} \{ F(s-c) \} = e^{ct} \cdot f(t)$  dönüşümü dikkate alınırsa,

$c = -\frac{1}{2}$  olur. ve

$$f(t) = \frac{2}{\sqrt{3}} e^{-\frac{1}{2}t} \sin \frac{\sqrt{3}}{2} t$$

bulunur. Diğer taraftan,

$$\mathcal{L}^{-1} \{ e^{-cs} F(s) \} = u_c(t) \cdot f(t-c)$$

dönüşümü dikkate alınırsa,

$$y = \mathcal{L}^{-1} \{ Y(s) \} = \mathcal{L}^{-1} \{ e^{-2s} F(s) \} = u_2(t) \frac{2}{\sqrt{3}} e^{-\frac{1}{2}(t-2)} \sin \frac{\sqrt{3}}{2} (t-2)$$

şeklinde aranan çözüm bulunur.



(Ödev)  
Örnek:

$$p(t) = \begin{cases} 1, & 3 \leq t < 15 \\ 0, & 0 \leq t < 3 \text{ ve } t \geq 15 \end{cases}$$

olmak üzere,

$$y'' + y' + y = p(t), \quad y(0) = 0, \quad y'(0) = 0$$

Başlangıç değer problemi çözümler.

Çözüm:  $p(t) = u_3(t) - u_{15}(t)$  olur.



YAP  
onet!

$$1) f(t) = \begin{cases} 2, & 0 \leq t < 6 \\ t, & t \geq 6 \end{cases}$$

$$y'' + 3y' + 2y = f(t)$$

$$L\{y'' + 3y' + 2y\} = L\{f(t)\}, \quad \begin{cases} y(0) = 0, y'(0) = -2 \end{cases}$$

$$f(t) = t \cdot u_6(t) - 2u_6(t) + 2$$

$$L\{y''\} + 3L\{y'\} + 2L\{y\} = L\{t u_6(t)\} - 2L\{u_6(t)\} + L\{2\}$$

$$s^2 \tilde{y}(s) - \underbrace{s y(0)}_0 - \underbrace{y'(0)}_{-2} + 3s \tilde{y}(s) + 3 \underbrace{y(0)}_0 + 2 \tilde{y}(s) = -\frac{d}{ds} \left( \frac{e^{-6s}}{s} \right) - 2 \frac{e^{-6s}}{s} + \frac{2}{s}$$

$$\tilde{y}(s) [s^2 + 3s + 2] - 2 = -\frac{6e^{-6s} \cdot s - e^{-6s}}{s^2} - \frac{2e^{-6s}}{s} + \frac{2}{s}$$

$$\tilde{y}(s) = \frac{6e^{-6s} \cdot \frac{1}{s} + e^{-6s} \cdot \frac{1}{s^2} - \frac{2e^{-6s}}{s} + \frac{2}{s}}{s^2 + 3s + 2}$$

$$y(t) = L^{-1} \left\{ \frac{6e^{-6s}}{s(s+2)(s+1)} \right\} + L^{-1} \left\{ \frac{e^{-6s}}{s^2(s+2)(s+1)} \right\} - L^{-1} \left\{ \frac{2e^{-6s}}{s(s+1)(s+2)} \right\} + L^{-1} \left\{ \frac{2}{s(s+1)(s+2)} \right\}$$

$$\frac{1}{s(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}$$

$$1 = As^2 + 3As + 2A + Bs^2 + Bs + Cs^2 + 2Cs$$

$$1 = (A+B+C)s^2 + (3A+B+2C)s + 2A$$



(2)

$$A+B+C=0$$

$$3A+B+2C=0$$

$$2A=1 \Rightarrow A=1/2$$

$$B+C=-\frac{1}{2}$$

$$B+2C=-\frac{3}{2}$$

$$C=-1$$

$$B=\frac{1}{2}$$

$$\mathcal{L}^{-1} \left\{ 4e^{-6s} \left( \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{s+1} \right) \right\}$$

$$= 2 \cdot u_6(t) + 2u_6(t) \cdot e^{-2(t-6)} - u_6(t) \cdot e^{-(t-6)}$$

$$= u_6(t) \left( 2 + e^{-2(t-6)} - e^{-(t-6)} \right) \quad (3)$$

$$\frac{1}{s^2(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+1}$$

$$1 = As^3 + 3As^2 + 2As + Bs^2 + 3Bs + 2B + Cs^3 + Cs^2 + Ds^3 + 2Ds^2$$

$$1 = (A+C+D)s^3 + (3A+B+C+2D)s^2 + (2A+3B)s + 2B$$

$$\left. \begin{array}{l} A+C+D=0 \\ 3A+B+C+2D=0 \\ 2A+3B=0 \\ 2B=1 \end{array} \right\} \begin{array}{l} B=\frac{1}{2}, A=-\frac{3}{4} \\ C+D=\frac{3}{4} \\ C+2D=\frac{7}{4} \end{array} \quad \left. \begin{array}{l} D=1 \\ C=-\frac{1}{4} \end{array} \right\} \quad (5)$$

$$\mathcal{L}^{-1} \left\{ e^{-6s} \left( -\frac{3}{4} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s^2} - \frac{1}{4} \cdot \frac{1}{s+2} + \frac{1}{s+1} \right) \right\}$$

$$= -\frac{3}{4} u_6(t) + \frac{1}{2} u_6(t) \cdot (t-6) - \frac{1}{4} u_6(t) \cdot e^{-2(t-6)} + u_6(t) \cdot e^{-(t-6)} \quad (5)$$



(3)

$$y(t) = u_6(t) \left( 2 + e^{-2(t-6)} - e^{-(t-6)} \right) + u_6(t) \left( -\frac{3}{4} + \frac{1}{2}(t-6) - \frac{1}{4} e^{-2(t-6)} + e^{-(t-6)} \right) - 2 \mathcal{L}^{-1} \left\{ \frac{1}{2} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{s+1} \right\}$$

$$y(t) = u_6(t) \left( \frac{5}{4} + \frac{1}{2}(t-6) + \frac{3}{4} e^{-2(t-6)} \right) - 2 \left( \frac{1}{2} + \frac{1}{2} e^{-2t} - e^{-t} \right)$$

$$y(t) = u_6(t) \left( \frac{5}{4} + \frac{1}{2}(t-6) + \frac{3}{4} e^{-2(t-6)} \right) - 1 - e^{-2t} + 2e^{-t}$$



## Heaviside Fonksiyonu (Birim Basamak Fonksiyonu)

$c \geq 0$  olmak üzere

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

şeklinde tanımlanır.

$$L\{u_c(t)\} = \frac{e^{-cs}}{s}, \quad s > 0 \text{ dir.}$$

örneğin,

$$y = h(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases} \text{ fonksiyonu}$$

$h(t) = u_c(t) \cdot f(t-c)$  şeklinde yazılır.

$$\boxed{L\{u_c(t) \cdot f(t-c)\} = e^{-cs} \cdot L\{f(t)\} = e^{-cs} \cdot F(s), \quad s > 0}$$

$$L^{-1}\{e^{-cs} \cdot F(s)\} = u_c(t) \cdot f(t-c) \text{ dir.}$$

$$L\{e^{ct} \cdot f(t)\} = F(s-c), \quad s > a+c.$$

$$L^{-1}\{F(s-c)\} = e^{ct} f(t)$$



$$\mathcal{L}^{-1}\{F(s-c)\} = e^{ct} \cdot f(t) \quad \mathcal{L}^{-1}\{e^{-cs} \cdot F(s)\} = u_c(t) f(t-c), \quad \mathcal{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathcal{L}\{(-1)^n f(t)\} = F^{(n)}(s)$$

Örnek  $y'' + y' - 2y = 6u_2(t)$ ,  $y(0) = y'(0) = 1$

Örnek  $y'' + y = f(t)$ ,  $y(0) = 2$ ,  $y'(0) = 1$

$$f(t) = \begin{cases} 0 & , 0 \leq t < \frac{\pi}{2} \\ 1 & , \frac{\pi}{2} \leq t < \pi \\ 0 & , \pi \leq t < \infty \end{cases}$$

$$f(t) = u_{\frac{\pi}{2}}(t) - u_{\pi}(t)$$

Örnek



# TEMEL LAPLACE DÖNÜŞÜMLERİ

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \implies F(s) = \mathcal{L}\{f(t)\}$$

$$1. f(t) = 1 \implies \mathcal{L}\{1\} = \frac{1}{s} \quad (s > 0)$$

$$2. f(t) = t \implies \mathcal{L}\{t\} = \frac{1}{s^2} \quad (s > 0)$$

$$3. f(t) = t^n \quad (n = 1, 2, \dots) \implies \mathcal{L}\{t^n\} = \frac{(n)!}{s^{n+1}} \quad (s > 0)$$

$$4. f(t) = t^p \quad (p > -1) \implies \mathcal{L}\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}} \quad (s > 0)$$

$$5. f(t) = e^{at} \implies \mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad (s > a)$$

$$6. f(t) = \sin at \implies \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2} \quad (s > 0)$$

$$7. f(t) = \cos at \implies \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2} \quad (s > 0)$$

$$8. f(t) = \sinh at \implies \mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2} \quad (s > |a|)$$

$$9. f(t) = \cosh at \implies \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2} \quad (s > |a|)$$

$$10. f(t) = e^{at} \sin bt \implies \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2} \quad (s > a)$$

$$11. f(t) = e^{at} \cos bt \implies \mathcal{L}\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2 + b^2} \quad (s > a)$$

$$12. f(t) = t^n e^{at} \quad (n = 1, 2, \dots) \implies \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \quad (s > a)$$

$$13. f(t) = u_c(t) \implies \mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \quad (s > 0)$$

$$14. h(t) = u_c(t)f(t-c) \implies \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$$

$$15. h(t) = e^{ct}f(t) \implies \mathcal{L}\{e^{ct}f(t)\} = F(s-c)$$

$$16. h(t) = f(ct) \implies \mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right) \quad (c > 0)$$

$$17. h(t) = \int_0^t f(t-\tau)g(\tau)d\tau \implies \mathcal{L}\left\{\int_0^t f(t-\tau)g(\tau)d\tau\right\} = F(s)G(s)$$

$$18. f(t) = \delta(t-c) \implies \mathcal{L}\{\delta(t-c)\} = e^{-cs}$$