THE METHOD OF VARIATION OF PARAMETERS

Consider the following higher order linear DE with constant coefficients;

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = F(x) \quad \dots (1)$$

If the linearly independent solutions of the corresponding homogeneous equation are $y_1, y_2, \dots, y_n$, then the complementary function of (1) is

$$\boxed{y_h = c_1 y_1 + c_2 y_2 + \dots + c_n y_n}$$

The method of variation of parameters is to replace the arbitrary constants $c_1, c_2, \dots, c_n$ in the complementary function by respective functions $c_1(x), \dots, c_n(x)$. Now, we aim to find $c_1(x), c_2(x), \dots, c_n(x)$ to obtain the general solution

$$\boxed{y = c_1(x) y_1 + c_2(x) y_2 + \dots + c_n(x) y_n} \quad \dots (2)$$

If we take the needed derivatives, we will obtain this equation system;

$\rightarrow c_1' y_1 + c_2' y_2 + \dots + c_n' y_n = 0$

$\rightarrow c_1' y_1' + c_2' y_2' + \dots + c_n' y_n' = 0$

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$\rightarrow c_1' y_1^{(n-1)} + c_2' y_2^{(n-1)} + \dots + c_n' y_n^{(n-1)} = \dfrac{F(x)}{a_0}$.

Example: $y'' + y = \tan x$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i$

$y_h = c_1 \cos x + c_2 \sin x$

$\sin x \,/\; c_1' \cos x + c_2' \sin x = 0$

$\cos x \,/\; -c_1' \sin x + c_2' \cos x = \tan x$

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$$c_2' (\underbrace{\sin^2 x + \cos^2 x}_{1}) = \underbrace{\cos x \cdot \tan x}_{\sin x} \Rightarrow c_2' = \sin x$$

$$\boxed{c_2 = -\cos x + k_2}$$

$$c_1' = -c_2' \frac{\sin x}{\cos x} = -\frac{\sin^2 x}{\cos x}$$

$\sin x$

$$c_1 = -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1-\cos^2 x}{\cos x} dx = -\underbrace{\int \sec x \, dx}_{\ln|\sec x + \tan x|} + \underbrace{\int \cos x \, dx}_{\sin x} + k_1$$

$$\boxed{c_1 = -\ln|\sec x + \tan x| + \sin x + k_1}$$

$$y_g = -\cos x \ln|\sec x + \tan x| + \cancel{\cos x \sin x} + k_1 \cos x - \cancel{\sin x \cos x} + k_2 \sin x$$

$$y_g = k_1 \cos x + k_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example: $y'' - 2y' + y = \dfrac{e^x}{x}$

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1$

$y_h = (c_1 + c_2 x) e^x = c_1 e^x + c_2 x e^x$

$$\cancel{c_1' e^x} + \cancel{c_2' x e^x} = 0$$

$$- \quad \cancel{c_1' e^x} + c_2' (e^x + \cancel{x e^x}) = \frac{e^x}{x}$$

$$-c_2' e^x = -\frac{e^x}{x} \Rightarrow c_2' = \frac{1}{x} \Rightarrow \boxed{c_2 = \ln x + k_2}$$

$$c_1' + c_2' x = 0 \Rightarrow c_1' = -c_2' x = -\frac{1}{x} \cdot x = -1 \Rightarrow \boxed{c_1 = -x + k_1}$$

$$\boxed{y_g = -x e^x + k_1 e^x + x e^x \ln x + k_2 x e^x}$$

Example: $y''' + y' = \sec^2 x$

$r^3 + r = 0 \Rightarrow r(r^2+1) = 0 \rightarrow r_1 = 0,\ r_{2,3} = \mp i$

$y_h = c_1 + c_2 \cos x + c_3 \sin x$

$$\left.\begin{aligned} c_1' + c_2' \cos x + c_3' \sin x &= 0 \\ -c_2' \sin x + c_3' \cos x &= 0 \\ -c_2' \cos x - c_3' \sin x &= \sec^2 x \end{aligned}\right\}$$

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$c_1' = \sec^2 x \Rightarrow \boxed{c_1 = \tan x + k_1}$

$$\begin{aligned} \sin x \quad & -c_2' \sin x + \cancel{c_3' \cos x} = 0 \\ \cos x \quad & -c_2' \cos x - \cancel{c_3' \sin x} = \sec^2 x \end{aligned}$$

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$-c_2' \underbrace{(\sin^2 x + \cos^2 x)}_{1} = \underbrace{\cos x . \sec^2 x}_{\sec x} \Rightarrow c_2' = -\sec x$

$\boxed{c_2 = -\ln|\sec x + \tan x| + k_2}$

$c_3' = \underset{-\sec x}{c_2'} \dfrac{\sin x}{\cos x} = -\dfrac{\sin x}{\cos^2 x}$

$c_3 = \displaystyle\int \frac{-\sin x}{\cos^2 x} dx \quad \left.\begin{aligned} \cos x &= t \\ -\sin x\, dx &= dt \end{aligned}\right\} \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\cos x}$

$\boxed{c_3 = -\sec x + k_3}$

$y_g = \tan x + k_1 - \cos x \ln|\sec x + \tan x| + k_2 \cos x - \sec x \sin x$
$+ k_3 \sin x$

$$y'' - 3y' + 2y = \sin(e^{-x})$$

$$r^2 - 3r + 2 = 0 \Rightarrow r_1 = 2,\ r_2 = 1$$

$$-2 \quad -1$$

$$y_h = c_1 e^{2x} + c_2 e^{x}$$

$$\left.\begin{aligned} c_1' e^{2x} + c_2' e^{x} &= 0 \\ 2c_1' e^{2x} + c_2' e^{x} &= \sin(e^{x}) \end{aligned}\right\}$$

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$$2y'' - y' = xe^{x/2}$$

$$2r^2 - r = 0 \Rightarrow r_1 = 0,\ r_2 = \frac{1}{2}$$

$$y_h = c_1 + c_2 e^{x/2}$$

$$\left.\begin{aligned} c_1' + c_2' e^{x/2} &= 0 \\ \frac{1}{2} c_2' e^{x/2} &= \frac{xe^{x/2}}{2} \end{aligned}\right\}$$