Solutions of Application Questions 3

a. $y_h = c_1 e^{x} + c_2 e^{-x} + c_3 e^{\frac{2x}{3}} + c_4 e^{-\frac{2x}{3}} + c_5 e^{5x} + c_6 e^{-6x}$

b. $y_h = c_1 + c_2 x + c_3 e^{-2x} + c_4 x e^{-2x} + c_5 \cos 3x + c_6 \sin 3x$

c. $y_h = (c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x + c_5 \cos x + c_6 \sin x$

d. $y_h = (c_1 + c_2 x + c_3 x^2) e^{2x} + c_4 e^{-2x} + e^{2x} (c_5 \cos 3x + c_6 \sin 3x)$

e. $(c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x$

f. $y_h = e^{-x} \left((c_1 + c_2 x) \cos 3x + (c_3 + c_4 x) \sin 3x \right) + c_5 e^{2x} + c_6 e^{-2x}$

g. $y_h = e^{2x} \left[(c_1 + c_2 x + c_3 x^2) \cos x + (c_4 + c_5 x + c_6 x^2) \sin x \right]$

h. $y_h = c_1 e^{\sqrt{2}x} + (c_2 + c_3 x) \cos 2x + (c_4 + c_5 x) \sin 2x + c_6 e^{\sqrt{3}x}$

i. $y_h = c_1 \cos x + c_2 \sin x + c_3 \cos 2x + c_4 \sin 2x + c_5 \cos 3x + c_6 \sin 3x$

j. $y_h = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + e^{x} (c_5 \cos x + c_6 \sin x)$

k. $y_h = c_1 e^{2x} + (c_2 + c_3 x) e^{-x} + (c_4 + c_5 x + c_6 x^2) e^{-3x}$

l. $y_h = (c_1 + c_2 x + c_3 x^2) \cos \frac{3x}{2} + (c_4 + c_5 x + c_6 x^2) \sin \frac{3x}{2}$

m. $y_h = e^{-3x} \left[(c_1 + c_2 x + c_3 x^2) \cos 5x + (c_4 + c_5 x + c_6 x^2) \sin 5x \right]$

n. $y_h = e^{x} (c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x) + c_3 \cos \sqrt{2} x + c_4 \sin \sqrt{2} x + (c_5 + c_6 x) e^{-\sqrt{2}x}$

1 a. $r^3 - 3r^2 - r + 3 = 0$

$r^2(r-3) - (r-3) = 0 \Rightarrow (r^2 - 1)(r-3) = 0 \Rightarrow \left. \begin{array}{l} r_1 = 1 \\ r_2 = -1 \\ r_3 = 3 \end{array} \right\} y_h = c_1 e^{x} + c_2 e^{-x} + c_3 e^{3x}$

b.
$$\begin{array}{l|l} r^3 + 2r^2 - 4r - 8 & r - 2 \\ \underline{-\;r^3 - 2r^2} & \overline{r^2 + 4r + 4} \\ \quad 4r^2 - 4r - 8 & \\ \underline{-\;4r^2 - 8r} & \\ \quad 4r - 8 & \end{array}$$

$(r-2)(r+2)^2 = 0$

$r_1 = 2, \quad r_2 = r_3 = -2$

$y_h = c_1 e^{2x} + (c_2 + c_3 x) e^{-2x}$

c. $r^3 - 3r^2 + 3r - 1 = 0$

$$(r^3 - 1) - 3r(r-1) = 0 \Rightarrow (r-1)(\overbrace{r^2 + r + 1 - 3r}^{r^2 - 2r + 1}) = 0 \Rightarrow (r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$$

$$y_h = (c_1 + c_2 x + c_3 x^2) e^x$$

d. $r^5 + 9r^3 = 0 \Rightarrow r^3(r^2 + 9) = 0 \Rightarrow r_1 = r_2 = r_3 = 0$

$r_{4,5} = \mp 3i$

$$y_h = (c_1 + c_2 x + c_3 x^2) + c_4 \cos 3x + c_5 \sin 3x$$

e. $r^4 - 4r^3 + 29r^2 = 0$

$r^2(r^2 - 4r + 29) = 0$

$\Delta = 16 - 4.1.29 = -100$

$r_1 = r_2 = 0$

$r_{3,4} = \dfrac{4 \mp \sqrt{-100}}{2} = 2 \mp 5i$

$$y_h = c_1 + c_2 x + e^{2x}(c_3 \cos 5x + c_4 \sin 5x)$$

f. $r^3 - r^2 - r + 1 = 0$

$r^2(r-1) - (r-1) = 0 \Rightarrow (r^2 - 1)(r-1) = 0 \Rightarrow r_1 = 1 = r_2 = 1$

$r_3 = -1$

$$y_h = (c_1 + c_2 x) e^x + c_3 e^{-x}$$

g. $r^3 - 5r^2 + 6r = 0 \Rightarrow r(r^2 - 5r + 6) = 0 \Rightarrow r_1 = 0,\ r_2 = 3,\ r_3 = 2$

$-3 \quad -2$

$$y_h = c_1 + c_2 e^{3x} + c_3 e^{2x}$$

Solutions of Application Questions 3

a. $y_h = c_1 e^{x} + c_2 e^{-x} + c_3 e^{\frac{2x}{3}} + c_4 e^{-\frac{2x}{3}} + c_5 e^{5x} + c_6 e^{-6x}$

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m. $y_h = e^{-3x}\left[(c_1 + c_2 x + c_3 x^2)\cos 5x + (c_4 + c_5 x + c_6 x^2)\sin 5x\right]$

n. $y_h = e^{x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x) + c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x + (c_5 + c_6 x)e^{-\sqrt{2}x}$

1) a. $r^3 - 3r^2 - r + 3 = 0$

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$$(r^3-1) - 3r(r-1) = 0 \Rightarrow (r-1)(\overbrace{r^2+r+1-3r}^{r^2-2r+1}) = 0 \Rightarrow (r-1)^3 = 0 \Rightarrow r_1 = r_2 = r_3 = 1$$

$$y_h = (c_1 + c_2 x + c_3 x^2)e^x$$

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$$y_h = (c_1 + c_2 x)e^x + c_3 e^{-x}$$

g. $r^3 - 5r^2 + 6r = 0 \Rightarrow r(\underset{-3\ -2}{r^2 - 5r + 6}) = 0 \Rightarrow r_1 = 0,\ r_2 = 3,\ r_3 = 2$

$$y_h = c_1 + c_2 e^{3x} + c_3 e^{2x}$$

2018 - 2nd exam

1) Using the transformation $y = -\frac{u'}{u}$ $(u \neq 0)$, change the D.E $y' - y^2 + \frac{2}{x^2} = 0$ into Euler D.E. Then find the general solution of it.

$$y = -\frac{u'}{u} \Rightarrow y' = -\frac{[u''u - (u')^2]}{u^2}$$

$$-\frac{u''}{u} + \frac{(u')^2}{u^2} - \frac{(u')^2}{u^2} + \frac{2}{x^2} = 0 \Rightarrow \frac{u''}{u} = \frac{2}{x^2}$$

$$x^2 u'' - 2u = 0$$

$$\left.\begin{array}{l} x = e^t \\ u' = e^{-t} Dy \\ u'' = e^{-2t} D(D-1) y \end{array}\right\} \begin{array}{l} e^{2t} e^{-2t} D(D-1) y - 2u = 0 \\ [D^2 - D - 2] u = 0 \end{array}$$

$$\left.\begin{array}{l} r^2 - r - 2 = 0 \\ \quad -2 \quad +1 \\ r_1 = 2,\ r_2 = -1 \end{array}\right\} \begin{array}{l} u = c_1 e^{2t} + c_2 e^{-t} \\ \boxed{u = c_1 x^2 + \frac{c_2}{x}} \end{array}$$

2) Find the general solution of $y'' - 2y' + y = \frac{e^x}{\sqrt{1-x^2}}$ using variation of parameters.

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \qquad y_h = c_1 e^x + c_2 x e^x$

$c_1' e^x + c_2' x e^x = 0$

$c_1' e^x + c_2' e^x + c_2' x e^x = \frac{e^x}{\sqrt{1-x^2}}$

$-$

$c_2' = \frac{1}{\sqrt{1-x^2}} \Rightarrow c_2 = \arcsin x + k_2$

$c_1' = -c_2' x = -\frac{x}{\sqrt{1-x^2}} \Rightarrow c_1 = -\int \frac{x\,dx}{\sqrt{1-x^2}} \qquad 1 - x^2 = t^2$

$-2x\,dx = 2t\,dt$

$c_1 = -\int \frac{-t\,dt}{t} = t = \sqrt{1-x^2} + k_1$

$y = e^x \sqrt{1-x^2} + k_1 e^x + x e^x \arcsin x + k_2 x e^x$

3) Determine that which of $y_1(x)=e^x$ and $y_2(x)=-e^x$ functions is a particular solution for $y'+y+3e^{-x}y^2=e^x$

$y_1'=e^x \Rightarrow e^x+e^x+3e^{-x}.e^{2x} \neq e^x$

$y_2'=-e^x \Rightarrow -e^x-e^x+3e^{-x}.e^{2x}=e^x$ ✓

$$\left.\begin{aligned} y&=-e^x+\frac{1}{u} \\ y'&=-e^x-\frac{u'}{u^2}\end{aligned}\right\} -e^x-\frac{u'}{u^2}-e^x+\frac{1}{u}+3e^{-x}\left(e^{2x}-\frac{2e^x}{u}+\frac{1}{u^2}\right)=e^x$$

$$\cancel{-e^x}-\frac{u'}{u^2}-\cancel{e^x}+\frac{1}{u}+\cancel{3e^x}+\frac{6}{u}+\frac{3e^{-x}}{u^2}=\cancel{e^x}$$

$$-\frac{u'}{u^2}+\frac{7}{u}+\frac{3e^{-x}}{u^2}=0 \Rightarrow u'-7u=-3e^{-x} \quad \text{L.D.E}$$

$$\lambda(x)=e^{\int -7dx}=e^{-7x}$$

$$u=e^{7x}\left[\int e^{-7x}(-3e^{-x})dx+c\right]=e^{7x}\left[\frac{3}{8}e^{-8x}+c\right]=\frac{3}{8}e^{-x}+ce^{7x}$$

4) Find the general solution of $y'' + 2y' + y = \cos^2 x$ using undetermined coefficients.

$\frac{1}{2} + \frac{\cos 2x}{2}$

$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0$, $r_1 = r_2 = -1$ $\Big\}$ $y_h = c_1 e^{-x} + c_2 x e^{-x}$

$y_{p1} = a$, $y_{p1}' = 0 = y_{p1}''$ $\Big\}$ $0 + 2.0 + a = \frac{1}{2}$, $a = \frac{1}{2}$ $\Big\}$ $\boxed{y_{p1} = \frac{1}{2}}$

$$y_{p2} = A\cos 2x + B\sin 2x$$
$$y_{p2}' = -2A\sin 2x + 2B\cos 2x$$
$$y_{p2}'' = -4A\cos 2x - 4B\sin 2x$$

$$-4A\cos 2x - 4B\sin 2x - 4A\sin 2x + 4B\cos 2x + A\cos 2x + B\sin 2x = \frac{\cos 2x}{2}$$

$3/\ -3A + 4B = \frac{1}{2}$

$4/\ -4A - 3B = 0$

$-25A = \frac{3}{2} \Rightarrow A = -\frac{3}{50}$ $\quad B = \frac{4}{50}$

$y_{p2} = -\frac{3}{50}\cos 2x + \frac{4}{50}\sin 2x$

$y_g = y_h + y_{p1} + y_{p2}$

Working Questions II

1 $y'' + y = \cot x$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i \qquad y_h = c_1 \cos x + c_2 \sin x$

$$\begin{array}{l} \sin x / \quad c_1' \cos x + c_2' \sin x = 0 \\ \cos x / \quad -c_1' \sin x + c_2' \cos x = \cot x \\ \hline + \end{array}$$

$$c_2' = \frac{\cos^2 x}{\sin x} \Rightarrow c_2 = \int \frac{(1-\sin^2 x)}{\sin x} dx \Rightarrow c_2 = \int (\operatorname{cosec} x - \sin x)\, dx$$

$$\boxed{c_2 = \ln|\operatorname{cosec} x - \cot x| + \cos x + k_2}$$

$$c_1' = -c_2' \tan x = -\frac{\cos^2 x}{\sin x} \cdot \frac{\sin x}{\cos x} = -\cos x$$

$$\boxed{c_1 = -\sin x + k_1}$$

$$y = k_1 \cos x + k_2 \sin x - \cancel{\sin x \cos x} + \sin x \ln|\operatorname{cosec} x - \cot x| + \cancel{\sin x \cos x}$$

2. $y''' - 2y'' - y' + 2y = -40\cos^2 x - 2e^x + 20$

$$= -\cancel{20} - 20\cos 2x - 2e^x + \cancel{20}$$

$r^3 - 2r^2 - r + 2 = 0 \Rightarrow r^2(r-2) - (r-2) = 0 \Rightarrow (r-1)(r+1)(r-2) = 0$

$r_1 = 1, \; r_2 = -1, \; r_3 = 2$

$y_h = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}$

$$\left.\begin{aligned}
y_{p1} &= A\cos 2x + B\sin 2x \\
y'_{p1} &= -2A\sin 2x + 2B\cos 2x \\
y''_{p1} &= -4A\cos 2x - 4B\sin 2x
\end{aligned}\right\}$$

$$y'''_{p1} = 8A\sin 2x - 8B\cos 2x$$

$$8A\sin 2x - 8B\cos 2x + 8A\cos 2x + 8B\sin 2x + 2A\sin 2x - 2B\cos 2x$$
$$+ 2A\cos 2x + 2B\sin 2x = -20\cos 2x$$

$$\left.\begin{aligned} 10A + 10B &= 0 \\ 10A - 10B &= -20 \end{aligned}\right\} \begin{aligned} A &= -1 \\ B &= 1 \end{aligned} \qquad \boxed{y_{p1} = -\cos 2x + \sin 2x}$$

$$\left.\begin{aligned}
y_{p2} &= kxe^x \\
y'_{p2} &= ke^x + kxe^x \\
y''_{p2} &= 2ke^x + kxe^x \\
y'''_{p2} &= 3ke^x + kxe^x
\end{aligned}\right\} \quad \begin{aligned} &(3k + \cancel{xk} - 4k - \cancel{2xk} - k - \cancel{kx} + \cancel{2kx})e^x = -2e^x \\ &2k = -2 \Rightarrow k = -1 \\ &\boxed{y_{p2} = -xe^x} \end{aligned}$$

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} - \cos 2x + \sin 2x - xe^x$$

3. $x^3 y'' + 4x^2 y' = -1 \quad \Rightarrow x^2 y'' + 4xy' = \frac{-1}{x}$

$$\left.\begin{aligned}
x &= e^t \\
y' &= e^{-t} Dy \\
y'' &= e^{-2t} D(D-1)y
\end{aligned}\right\} \quad \begin{aligned} &e^{2t} \cdot e^{-2t} D(D-1)y + 4e^t \cdot e^{-t} Dy = -e^{-t} \\ &(D^2 + 3D)y = -e^{-t} \end{aligned}$$

$$\begin{aligned} r(r+3) &= 0 \\ r_1 &= 0 \\ r_2 &= -3 \end{aligned} \qquad \left.\begin{aligned} y_p &= ke^{-t} \\ y'_p &= -ke^{-t} \\ y''_p &= ke^{-t} \end{aligned}\right\} \left.\begin{aligned} &(k - 3k)e^{-t} = -e^{-t} \\ &-2k = -1 \Rightarrow k = \tfrac{1}{2} \end{aligned}\right\} y_p = \frac{1}{2}e^{-t}$$

$$\boxed{y_h = c_1 + c_2 e^{-3t}} \qquad \boxed{y = c_1 + \frac{c_2}{x^3} + \frac{1}{2x}}$$

4. $y'' + y = \sec t$

$$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i \qquad y_h = c_1 \cos t + c_2 \sin t$$

$$\begin{array}{l|l} \sin t & c_1' \cos t + c_2' \sin t = 0 \\ \cos t & -c_1' \sin t + c_2' \cos t = \sec t \\ \hline + & \end{array}$$

$$c_2' = 1 \Rightarrow \boxed{c_2 = t + k_2}$$

$$c_1' = -c_2' \tan t = -\tan t \Rightarrow c_1 = \ln|\cos t| + k_1$$

$$y = k_1 \cos t + k_2 \sin t + \cos t \ln|\cos t| + t \sin t$$

5. $y'' + y' - 2y = x^2 e^x$

$$\left.\begin{array}{l} r^2 + r - 2 = 0 \\ \quad +2 \quad -1 \end{array}\right\} \left.\begin{array}{l} r_1 = -2 \\ r_2 = 1 \end{array}\right\} \boxed{y_h = c_1 e^{-2x} + c_2 e^x}$$

$$y_p = (ax^2 + bx + c)xe^x = (ax^3 + bx^2 + cx)e^x$$
$$y_p' = (3ax^2 + 2bx + c)e^x + (ax^3 + bx^2 + cx)e^x$$
$$y_p'' = (6ax + 2b)e^x + 2(3ax^2 + 2bx + c)e^x + (ax^3 + bx^2 + cx)e^x$$

$$(6ax + 2b) + 2(3ax^2 + 2bx + c) + \cancel{(ax^3 + bx^2 + cx)} + (3ax^2 + 2bx + c)$$
$$\cancel{+(ax^3 + bx^2 + cx)} - 2(ax^3 + bx^2 + cx) = x^2$$

$$\left.\begin{array}{l} 9ax^2 + (6b + 6a)x + (3c + 2b) = x^2 \\ a = \frac{1}{9} \;,\; b = -\frac{1}{9} \;,\; c = +\frac{2}{27} \end{array}\right\} \quad y_p = \left(\frac{1}{9}x^3 - \frac{1}{9}x^2 + \frac{2}{27}x\right)e^x$$

6. $y'' - y = \dfrac{-2e^x}{e^x+1}$

$r^2 - 1 = 0 \Rightarrow r_1 = 1,\ r_2 = -1$

$y_h = c_1 e^x + c_2 e^{-x}$

$$\left.\begin{aligned} c_1' e^x + c_2' e^{-x} &= 0 \\ c_1' e^x - c_2' e^{-x} &= \frac{-2e^x}{e^x+1} \end{aligned}\right\}$$

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$$2c_1' \cancel{e^x} = \frac{-2e^x}{e^x+1}$$

$$c_1 = -\int \frac{dx}{e^x+1} = -\int \frac{e^{-x}\,dx}{1+e^{-x}} = \ln(1+e^{-x}) + k_1$$

$$c_2' = -c_1' e^{2x} = +\frac{e^{2x}}{e^x+1} \Rightarrow c_2 = \int \frac{e^{2x}\,dx}{e^x+1} \qquad \begin{aligned} e^x+1 &= t \\ e^x dx &= dt \end{aligned}$$

$$c_2 = \int \frac{(t-1)\,dt}{t} = t - \ln t \Rightarrow c_2 = (e^x+1) - \ln(e^x+1) + k_2$$

$$y = k_1 e^x + k_2 e^{-x} + e^x \ln(1+e^{-x}) + (1+e^{-x}) - e^{-x}\ln(1+e^x)$$

7. $x^2 y'' + 4xy' + 2y = 2\ln x$

$$\left.\begin{aligned} x &= e^t \\ y' &= e^{-t} Dy \\ y'' &= e^{-2t} D(D-1)y \end{aligned}\right\} \quad \begin{aligned} D(D-1)y + 4Dy + 2y &= 2t \\ (D^2+3D+2)y &= 2t \end{aligned}$$

$r^2 + 3r + 2 = 0 \Rightarrow r_1 = -2,\ r_2 = -1 \qquad y_h = c_1 e^{-2t} + c_2 e^{-t}$

$+2\ \ +1$

$$\left.\begin{aligned} y_p &= at + b \\ y_p' &= a \\ y_p'' &= 0 \end{aligned}\right\} \quad \left.\begin{aligned} &3a + 2at + 2b = 2 \\ &a = 1,\quad 3a + 2b = 0 \\ &\qquad\quad b = -\tfrac{3}{2} \end{aligned}\right\} \quad y_p = t - \frac{3}{2}$$

$$\boxed{y = \frac{c_1}{x^2} + \frac{c_2}{x} + \ln x - \frac{3}{2}}$$

8. $y'' - 2y' + y = \frac{e^x}{x^3}$

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \Big\} \; y_h = c_1 e^x + c_2 x e^x$

$c_1' e^x + c_2' x e^x = 0$

$c_1' e^x + c_2' e^x + c_2' x e^x = \frac{e^x}{x^3}$

$-$

$c_2' = \frac{1}{x^3} \Rightarrow c_2 = -\frac{1}{2x^2} + k_2$

$c_1' = -c_2' x = -\frac{1}{x^2} \Rightarrow c_1 = \frac{1}{x} + k_1$

$y = k_1 e^x + k_2 x e^x + \frac{1}{x} e^x - \frac{1}{2x}$

9. $x^3 y'' + x^2 y' + 4xy = 4x \cos(2\ln x)$

$\hookrightarrow x^2 y'' + xy' + 4y = 4\cos(2\ln x)$

$$\left.\begin{array}{l} x = e^t \\ y' = e^{-t} Dy \\ y'' = e^{-2t} D(D-1)y \end{array}\right\} \begin{array}{l} D(D-1)y + Dy + 4y = 4\cos 2t \\ (D^2+4)y = 4\cos 2t \end{array}$$

$r^2 + 4 = 0 \Rightarrow r_{1,2} = \mp 2i$

$y_h = c_1 \cos 2t + c_2 \sin 2t$

$2\sin 2t /$ $c_1' \cos 2t + c_2' \sin 2t = 0$

$\cos 2t /$ $-2c_1' \sin 2t + 2c_2' \cos 2t = 4\cos 2t$

$2c_2' = 4\cos^2 2t \Rightarrow c_2' = 2\cos^2 2t \Rightarrow c_2' = (1 + \cos 4t)$

$\boxed{c_2 = t + \frac{1}{4}\sin 4t + k_2}$

$$c_1' = -c_2' \tan 2t = -2\cos^{\cancel{2}} 2t \cdot \frac{\sin 2t}{\cancel{\cos 2t}} = -\sin 4t$$

$$c_1 = +\frac{1}{4}\cos 4t + k_1$$

$$y = k_1 \cos 2t + k_2 \sin 2t + \frac{1}{4}\cos 4t \cdot \cos 2t + t \sin 2t + \frac{1}{4}\sin 4t \cdot \sin 2t$$

10. $y'' + 2y' + y = e^{-x}\ln x$

$$\left.\begin{array}{l} (r+1)^2 = 0 \\ r_1 = r_2 = -1 \end{array}\right\} \quad y_h = c_1 e^{-x} + c_2 x e^{-x}$$

$$\cancel{c_1' e^{-x}} + \cancel{c_2' x e^{-x}} = 0$$

$$-\cancel{c_1' e^{-x}} + c_2' e^{-x} - \cancel{c_2' x e^{-x}} = e^{-x}\ln x$$

$+$

$$c_2' = \ln x \quad \Rightarrow \quad c_2 = x\ln x - x + k_2$$

$$c_1' = -c_2' x = -x\ln x$$

$$\left.\begin{array}{ll} \ln x = u & -x\,dx = dv \\ \frac{dx}{x} = du & -\frac{x^2}{2} = v \end{array}\right\} \quad c_1 = -\frac{x^2}{2}\ln x + \int \frac{x}{2}\,dx$$

$$c_1 = -\frac{x^2}{2}\ln x + \frac{x^2}{4} + k_1$$

$$y = k_1 e^{-x} + k_2 x e^{-x} + \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^{-x} + x^2(\ln x - 1)e^{-x}$$

11. $x^2y'' - xy' + y = x\ln x$

$$x = e^t \qquad D(D-1)y - Dy + y = te^t$$
$$y' = e^{-t}Dy \qquad (D^2 - 2D + 1)y = te^t$$
$$y'' = e^{-2t}D(D-1)y$$

$$(r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \qquad y_h = c_1e^t + c_2te^t$$

$$y_p = (at+b)t^2e^t = (at^3+bt^2)e^t$$
$$y_p' = (3at^2+2bt)e^t + (at^3+bt^2)e^t$$
$$y_p'' = (6at+2b)e^t + 2(3at^2+2bt)e^t + (at^3+bt^2)e^t$$

$$(6at+2b) + \cancel{2(3at^2+2bt)} + \cancel{(at^3+bt^2)} - \cancel{2(3at^2+2bt)} - \cancel{2(at^3+bt^2)}$$
$$+ \cancel{(at^3+bt^2)} = t$$

$$6a = 1 \quad 2b = 0 \Rightarrow a = \frac{1}{6},\ b = 0$$

$$y_p = \frac{t^3}{6}e^t$$

$$y = c_1x + c_2x\ln x + \frac{1}{6}(\ln x)^3 x$$

12. $y'' + y = \tan x \sec x$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \mp i$

$y_h = c_1 \cos x + c_2 \sin x$

$\sin x$ / $c_1' \cos x + c_2' \sin x = 0$

$\cos x$ / $-c_1' \sin x + c_2' \cos x = \tan x \sec x$

$c_2' = \tan x \Rightarrow c_2 = -\ln|\cos x| + k_2$

$c_1' = -c_2' \tan x = -\tan^2 x \Rightarrow c_1 = \int(-\tan^2 x - 1 + 1)\,dx$

$c_1 = -\tan x + x + k_1$

$y = k_1 \cos x + k_2 \sin x + x\cos x - \tan x \cos x - \sin x \ln|\cos x|$

13. $y'' - 3y' + 2y = \cos e^{-x} + 3xe^{x}$

$r^2 - 3r + 2 = 0 \quad \Rightarrow r_1 = 2,\ r_2 = 1$

$-2 \quad -1$

$y_h = c_1 e^{2x} + c_2 e^{x}$

$c_1' e^{2x} + c_2' e^{x} = 0$

$2c_1' e^{2x} + c_2' e^{x} = \cos e^{-x} + 3xe^{x}$

$-$

$c_1' = e^{-2x} \cos e^{-x} + 3x e^{-x} \Rightarrow \left.\begin{array}{l} e^{-x} = t \\ -e^{-x}dx = dt \end{array}\right\} \int -t\cos t\,dt$

$\left.\begin{array}{ll} t = u & -\cos t\,dt = dv \\ dt = du & -\sin t = v \end{array}\right\} -t\sin t + \int \sin t\,dt = -t\sin t - \cos t$

$= -e^{-x}\sin(e^{-x}) - \cos(e^{-x})$

$$\int x e^{-x}\,dx \qquad \begin{matrix} x = u & e^{-x}dx = dv \\ dx = du & -e^{-x} = v \end{matrix} \Bigg\} -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$$

$$c_1 = -e^{-x}\sin(e^{-x}) - \cos(e^{-x}) - (x+1)e^{-x}$$

$$c_2' = -c_1' e^{x} = -e^{-x}\cos e^{-x} - 3x$$

$$c_2 = -\int e^{-x}\cos(e^{-x})\,dx - \frac{3}{2}x^2 + k_2 \qquad \begin{matrix} e^{-x} = t \\ -e^{-x}dx = dt \end{matrix}$$

$$\int \cos t\,dt = \sin t = \sin(e^{-x})$$

$$c_2 = \sin(e^{-x}) - \frac{3}{2}x^2 + k_2$$

$$y = k_1 e^{2x} + k_2 e^{x} - \cancel{e^{x}\sin(e^{-x})} - e^{2x}\cos(e^{-x}) - (x+1)e^{x} + \cancel{e^{x}\sin(e^{-x})} - \frac{3}{2}x^2 e^{x}$$

15. $2yy'' = 4 + (y')^2$

$$y' = p,\quad y'' = p\frac{dp}{dy} \quad \Big\} \quad 2yp\frac{dp}{dy} = 4 + p^2$$

$$\frac{2p\,dp}{4+p^2} = \frac{dy}{y} \Rightarrow \ln(4+p^2) = \ln y + \ln c_1$$

$$4 + p^2 = c_1 y \Rightarrow p = \sqrt{c_1 y - 4} = \frac{dy}{dx}$$

$$\frac{dy}{\sqrt{c_1 y - 4}} = dx \qquad \begin{aligned} c_1 y - 4 &= t^2 \\ c_1 dy &= 2t\,dt \end{aligned}$$

$$\int \frac{2\cancel{t}\,dt}{c_1 \cdot \cancel{t}} = \frac{2}{c_1} t = \frac{2}{c_1}\sqrt{c_1 y - 4}$$

$$x = \frac{2}{c_1}\sqrt{c_1 y - 4} + c_2$$

3. $2yy'' = (y')^2 + 1$ (x missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= p\frac{dp}{dy} \end{aligned}\right\} \; 2yp\frac{dp}{dy} = p^2+1 \Rightarrow 2yp\,dp = (p^2+1)dy$$

$$\frac{2p\,dp}{p^2+1} = \frac{dy}{y} \Rightarrow \ln(p^2+1) = \ln y + \ln c_1$$

$$p^2+1 = c_1 y$$

$$p = \sqrt{c_1 y - 1} \Rightarrow \frac{dy}{dx} = \sqrt{c_1 y - 1} \Rightarrow \int \frac{dy}{\sqrt{c_1 y - 1}} = \int dx$$

$$\left.\begin{aligned} c_1 y - 1 &= t^2 \\ c_1\,dy &= 2t\,dt \end{aligned}\right\} \int \frac{2t\,dt}{c_1 t} = \frac{2}{c_1}t = \frac{2}{c_1}\sqrt{c_1 y - 1}$$

$$\frac{2}{c_1}\sqrt{c_1 y - 1} = x + c_2$$

4. $y'' + 4y = \tan 2x$

$$r^2 + 4 = 0 \Rightarrow r_{1,2} = \mp 2i \Rightarrow y_h = c_1\cos 2x + c_2 \sin 2x$$

$2\sin 2x$ / $c_1'\cos 2x + c_2'\sin 2x = 0$

$\cos 2x$ / $-2c_1'\sin 2x + 2c_2'\cos 2x = \tan 2x$

+

$$c_2'\underbrace{(2\sin^2 2x + 2\cos^2 2x)}_{2} = \sin 2x \Rightarrow c_2' = \frac{1}{2}\sin 2x \Rightarrow c_2 = \frac{-1}{4}\cos 2x + k_2$$

$$c_1' = -c_2'\frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\sin 2x \cdot \frac{\sin 2x}{\cos 2x} \Rightarrow c_1 = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}dx$$

$$c_1 = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}dx = -\frac{1}{2}\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$c_1 = -\frac{1}{2}\left[\boxed{\tfrac{1}{2}}\ln|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right] + k_1$$

$$y = -\frac{1}{4}\ln|\sec 2x + \tan 2x| \cdot \cos 2x + \cancel{\frac{1}{4}\sin 2x \cdot \cos 2x} + k_1\cos 2x - \cancel{\frac{1}{4}\sin 2x\cos 2x}$$
$$+ k_2 \sin 2x$$

5. $y'' + 4y' + 4y = \dfrac{e^{-2x}}{x^2}$

$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2 \Rightarrow y_h = c_1 e^{-2x} + c_2 x e^{-2x}$

$2/ \quad c_1' e^{-2x} + c_2' x e^{-2x} = 0$

$-2\cancel{c_1' e^{-2x}} + c_2' e^{-2x} - 2c_2' x e^{-2x} = \dfrac{e^{-2x}}{x^2}$

$+$

$c_2' e^{-2x} = \dfrac{e^{-2x}}{x^2} \Rightarrow c_2' = \dfrac{1}{x^2} \Rightarrow c_2 = -\dfrac{1}{x} + k_2$

$c_1' = -c_2' x = -\dfrac{1}{x^2} \cdot x = \dfrac{-1}{x} \Rightarrow c_1 = -\ln x + k_1$

$y = -\ln x \cdot e^{-2x} + k_1 e^{-2x} - e^{-2x} + k_2 x e^{-2x}$

6. $xy'' - 2y' = \dfrac{1}{x^2}$ (Euler or y missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= \frac{dp}{dx} \end{aligned}\right\} \quad x\frac{dp}{dx} - 2p = \frac{1}{x^2} \Rightarrow \frac{dp}{dx} - \frac{2p}{x} = \frac{1}{x^3} \quad \text{Linear}$$

$$\lambda(x) = e^{\int -\frac{2dx}{x}} = e^{-2\ln x} = \frac{1}{x^2}$$

$$p = x^2\left[\int \frac{1}{x^2} \cdot \frac{1}{x^3}\,dx + c_1\right] = x^2\left[\frac{1}{4x^4} + c_1\right] = \frac{1}{4x^2} + \frac{c_1}{x^2}$$

$$p = \frac{dy}{dx} = \frac{1}{4x^2} + \frac{c_1}{x^2} \Rightarrow \int dy = \int\left(\frac{1}{4x^2} + \frac{c_1}{x^2}\right)dx + c_2$$

$$y = \frac{-1}{4x} - \frac{c_1}{x} + c_2$$

7. $x^2 y'' + x y' + 4y = 4\cos(2\ln x)$

$$\left.\begin{aligned} x &= e^t \\ y' &= e^{-t} D y \\ y'' &= e^{-2t} D(D-1) y \end{aligned}\right\} \quad \begin{aligned} &\underbrace{e^{2t}\cdot e^{-2t}}_{1} D(D-1)y + \underbrace{e^{t}\cdot e^{-t}}_{1} Dy + 4y = 4\cos 2t \\ &[D(D-1)+D+4]y = 4\cos 2t \end{aligned}$$

$$(D^2+4)y = 4\cos 2t$$

$$r^2+4=0 \Rightarrow r_{1,2} = \mp 2i \Rightarrow y_h = c_1\cos 2t + c_2\sin 2t$$

$2\sin 2t$ / $\cos 2t$

$$c_1'\cos 2t + c_2'\sin 2t = 0$$

$$-2c_1'\sin 2t + 2c_2'\cos 2t = 4\cos 2t$$

$+$

$$c_2'\underbrace{(2\sin^2 2t + 2\cos^2 2t)}_{2} = 4\cos^2 2t \Rightarrow c_2' = 2\cos^2 2t$$

$$c_2' = (1+\cos 4t) \Rightarrow c_2 = t + \frac{1}{4}\sin 4t + k_2$$

$$c_1' = -c_2'\frac{\sin 2t}{\cos 2t} = -2\cdot\cos^{\cancel{2}} 2t\cdot\frac{\sin 2t}{\cancel{\cos 2t}} = -\sin 4t$$

$$c_1 = \frac{1}{4}\cos 4t + k_1$$

$$y = \underbrace{\frac{1}{4}\cos 4t\cdot\cos 2t} + k_1\cos 2t + t\sin 2t + \underbrace{\frac{1}{4}\sin 4t\sin 2t} + k_2\sin 2t$$

$$y = \frac{1}{4}\cos 2t + t\sin 2t + k_1\cos 2t + k_2\sin 2t$$

$$y = \frac{1}{4}\cos 2(\ln x) + \ln x\sin 2(\ln x) + k_1\cos 2(\ln x) + k_2\sin 2(\ln x)$$

8. $2yy'' - (y')^2 + 9 = 0$ (x missing)

$$\left.\begin{aligned} y' &= p \\ y'' &= p\frac{dp}{dy} \end{aligned}\right\} \quad 2yp\frac{dp}{dy} - p^2 + 9 = 0 \Rightarrow 2yp\frac{dp}{dy} = p^2 - 9$$

$$\frac{2p\,dp}{p^2-9} = \frac{dy}{y} \Rightarrow \ln(p^2-9) = \ln y + \ln c_1$$

$$p^2 - 9 = c_1 y \Rightarrow p = \sqrt{c_1 y + 9}$$

$$\frac{dy}{dx} = \sqrt{c_1 y + 9} \Rightarrow \int \frac{dy}{\sqrt{c_1 y + 9}} = \int dx$$

$$\left.\begin{aligned} c_1 y + 9 &= t^2 \\ c_1 dy &= 2t\,dt \end{aligned}\right\} \int \frac{2t\,dt}{c_1 \cdot t} = \frac{2}{c_1} t = \frac{2}{c_1}\sqrt{c_1 y + 9}$$

$$\frac{2}{c_1}\sqrt{c_1 y + 9} = x + c_2$$

9. $y^{IV} + y''' = \sin 2t$

$$r^4 + r^3 = 0 \Rightarrow r^3(r+1) = 0 \rightarrow r_1 = r_2 = r_3 = 0,\ r_4 = -1$$

$$y_h = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t}$$

$$\left.\begin{aligned} y_p &= A\cos 2t + B\sin 2t \\ y_p' &= -2A\sin 2t + 2B\cos 2t \\ y_p'' &= -4A\cos 2t - 4B\sin 2t \\ y_p''' &= 8A\sin 2t - 8B\cos 2t \\ y_p^{IV} &= 16A\cos 2t + 16B\sin 2t \end{aligned}\right\}$$

$$8A\sin 2t - 8B\cos 2t + 16A\cos 2t + 16B\sin 2t = \sin 2t$$

$$\left.\begin{aligned} 8A + 16B &= 1 \\ 2/\ 16A - 8B &= 0 \\ +\ \hline 40A &= 1 \\ A = \frac{1}{40},\ B &= \frac{1}{20} \end{aligned}\right\} y_p = \frac{1}{40}\cos 2t + \frac{1}{20}\sin 2t$$

$$y = c_1 + c_2 t + c_3 t^2 + c_4 e^{-t} + \frac{1}{40}\cos 2t + \frac{1}{20}\sin 2t$$

6. $y'' - y = \frac{2}{1+e^x}$

$r^2 - 1 = 0 \Rightarrow r_1 = 1, \; r_2 = -1 \qquad y_h = c_1 e^x + c_2 e^{-x}$

$c_1' e^x + c_2' e^{-x} = 0$

$c_1' e^x - c_2' e^{-x} = \frac{2}{1+e^x}$

$+$

$2c_1' e^x = \frac{2}{1+e^x} \Rightarrow c_1 = \int \frac{dx}{e^x(1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{e^x+1} \right) dx$

$\frac{e^{-x}}{1+e^{-x}}$

$c_1 = -e^{-x} - \ln(1+e^{-x}) + k_1$

$c_2' = c_1' e^{2x} = \frac{e^{2x}}{e^x(1+e^x)} = \frac{e^x}{1+e^x} \Rightarrow c_2 = \ln(1+e^x) + k_2$

$y = -1 - e^x \ln(1+e^{-x}) + k_1 e^x + e^{-x} \ln(1+e^x) + k_2 e^{-x}$

13) $y''' + 9y' = -36 \sin 3x + 9$

$r^3 + 9r = 0 \Rightarrow r(r^2+9) = 0 \qquad r_1 = 0, \; r_{2,3} = \mp 3i$

$y_h = c_1 + c_2 \cos 3x + c_3 \sin 3x$

$y_{p1} = (A \cos 3x + B \sin 3x) x$

$y_{p1}' = (-3A \sin 3x + 3B \cos 3x) x + (A \cos 3x + B \sin 3x)$

$y_{p1}'' = (-9A \cos 3x - 9B \sin 3x) x + 2(-3A \sin 3x + 3B \cos 3x)$

$y_{p1}''' = (27A \sin 3x - 27B \cos 3x) x + 3(-9A \cos 3x - 9B \sin 3x)$

$\cancel{(27A \sin 3x - 27B \cos 3x) x} + 3(-9A \cos 3x - 9B \sin 3x) + \cancel{9(-3A \sin 3x + 3B \cos 3x) x}$

$+ 9(A \cos 3x + B \sin 3x) = -36 \sin 3x$

$-18A \cos 3x - 18B \sin 3x = -36 \sin 3x \Rightarrow A = 0, \; B = 2$

22) $yy'' - (y')^3 = 0$

$$\left.y' = p \,,\quad y'' = p\frac{dp}{dy}\right\} \; yp\frac{dp}{dy} - p^3 = 0 \Rightarrow \frac{dp}{p^2} = \frac{dy}{y}$$

$$-\frac{1}{p} = \ln y + \ln c_1 \Rightarrow p = -\frac{1}{\ln(c_1 y)}$$

$$\frac{dy}{dx} = -\frac{1}{\ln(c_1 y)} \Rightarrow \int \ln(c_1 y)\,dy = \int -dx$$

$$\left.\begin{array}{ll} \ln(c_1 y) = u & dy = dv \\ \frac{c_1}{c_1 y}dy = du & y = v \end{array}\right\} y\ln(c_1 y) - \underbrace{\int y\frac{c_1}{c_1 y}dy}_{y}$$

$$y\ln(c_1 y) - y = -x + c_2$$

24. $yy'' + (y')^2 = 1$

$$\left.\begin{array}{l} y' = p \\ y'' = p\frac{dp}{dy} \end{array}\right\} yp\frac{dp}{dy} + p^2 = 1 \Rightarrow yp\frac{dp}{dy} = (1-p^2)$$

$$\int\frac{p\,dp}{1-p^2} = \int\frac{dy}{y} \Rightarrow -\frac{1}{2}\ln(1-p^2) = \ln y + \ln c_1$$

$$\frac{1}{\sqrt{1-p^2}} = c_1 y \Rightarrow \frac{1}{c_1^2 y^2} = 1 - p^2 \Rightarrow \underset{\frac{dy}{dx}}{p} = \sqrt{1 - \frac{1}{c_1^2 y^2}} = \frac{\sqrt{c_1^2 y^2 - 1}}{c_1 y}$$

$$\int\frac{c_1 y\,dy}{\sqrt{c_1^2 y^2 - 1}} = \int dx \qquad \left.\begin{array}{l} c_1^2 y^2 - 1 = t^2 \\ \cancel{2}c_1^2 y\,dy = \cancel{2}t\,dt \end{array}\right\} \int\frac{t\,dt}{c_1 t} = \frac{t}{c_1} = \frac{\sqrt{c_1^2 y^2 - 1}}{c_1}$$

$$\frac{\sqrt{c_1^2 y^2 - 1}}{c_1} = x + c_2$$

25. $2yy'' = 3yy' + (y')^2 \quad y \geq 0$

$$\left.\begin{array}{l} y' = p \\ y'' = p\dfrac{dp}{dy} \end{array}\right\} \quad 2yp\frac{dp}{dy} = 3yp + p^2$$

$$p\left[2y\frac{dp}{dy} - 3y - p\right] = 0$$

$1^{\circ})$ $p = 0 \Rightarrow p = c$ it's not a solution

$2^{\circ})$ $2y\dfrac{dp}{dy} - p = 3y \Rightarrow \dfrac{dp}{dy} - \dfrac{p}{2y} = \dfrac{3}{2}$ LDE

$$\lambda = e^{\int -\frac{dy}{2y}} = e^{-\frac{1}{2}\ln y} = \frac{1}{\sqrt{y}}$$

$$p = \sqrt{y}\left[\int \frac{1}{\sqrt{y}} \cdot \frac{3}{2}\,dy + c_1\right] = \sqrt{y}\left(3\sqrt{y} + c_1\right) = \left(3y + c_1\sqrt{y}\right)$$

$$\frac{dy}{dx} = 3y + c_1\sqrt{y} \Rightarrow \int \frac{dy}{3y + c_1\sqrt{y}} = \int dx$$

$$\left.\begin{array}{l} y = t^2 \\ dy = 2t\,dt \end{array}\right\} \quad \int \frac{2t\,dt}{3t^2 + c_1 t} = \int \frac{2\,dt}{3t + c_1} = \frac{2}{3}\ln(3t + c_1) = \frac{2}{3}\ln(3\sqrt{y} + c_1)$$

$$\frac{2}{3}\ln(3\sqrt{y} + c_1) = x + c_2$$

$y_{p_1} = 2x \sin 3x$

$y_{p_2} = a.x \qquad y_{p_2}' = a \qquad y_{p_2}'' = 0 = y_{p_2}'''$

$9a = 9 \Rightarrow a = 1 \qquad y_{p_2} = x$

$y = c_1 + c_2 \cos 3x + c_3 \sin 3x + 2x \sin 3x + x$

20. $x^2y'' - xy' + y = x \ln x$

$$\left.\begin{array}{l} x = e^t \\ y' = e^{-t} Dy \\ y'' = e^{-2t} D(D-1)y \end{array}\right\} \begin{array}{l} e^{2t}.e^{-2t}D(D-1)y - e^{t}.e^{-t}Dy + y = te^t \\ [\underbrace{D^2 - D - D + 1}_{(D^2-2D+1)}]y = te^t \end{array}$$

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_1 = r_2 = 1 \qquad y_h = c_1 e^t + c_2 te^t$

$y_p = (at+b)e^t.t^2 = (at^3+bt^2)e^t$

$y_p' = (3at^2+2bt)e^t + (at^3+bt^2)e^t$

$y_p'' = (6at+2b)e^t + 2(3at^2+2bt)e^t + (at^3+bt^2)e^t$

$\Big[(6at+2b) + \cancel{2(3at^2+2bt)} + \cancel{(at^3+bt^2)} - \cancel{2(3at^2+2bt)} - \cancel{2(at^3+bt^2)}$

$+ \cancel{(at^3+bt^2)}\Big] e^t = te^t$

$$\left.\begin{array}{l} 6at + 2b = t \\ a = \frac{1}{6} \quad b = 0 \end{array}\right\} y_p = \frac{t^3}{6} e^t$$

$y = c_1 x + c_2 x \ln x + \frac{(\ln x)^3}{6} x$

7) $y'' y^3 = 1$

$$\left.\begin{array}{l} y' = p \\ y'' = p \dfrac{dp}{dy} \end{array}\right\} \quad p \frac{dp}{dy} y^3 = 1 \Rightarrow p\,dp = \frac{dy}{y^3}$$

$$\frac{p^2}{2} = -\frac{1}{2y^2} + c_1 \Rightarrow p^2 = -\frac{1}{y^2} + 2c_1$$

$$\underset{\frac{dy}{dx}}{p} = \frac{\sqrt{-1 + 2c_1 y^2}}{y} \Rightarrow \int \frac{y\,dy}{\sqrt{-1+2c_1 y^2}} = \int dx$$

$$\left.\begin{array}{l} -1 + 2c_1 y^2 = t^2 \\ 4c_1 y\,dy = 2t\,dt \end{array}\right\} \quad \int \frac{t\,dt}{2c_1 t} = \frac{1}{2c_1} t = \frac{1}{2c_1}\sqrt{-1+2c_1 y^2}$$

$$\frac{1}{2c_1}\sqrt{-1+2c_1 y^2} = x + c_2$$