2. D.Es can be transformed into seperable D.Es.

Consider the equation

$\frac{dy}{dx} = f(ax+by+c)$ where a, b, c are constants.

This equation can be transformed into a seperable D.E by making the transformation

$$\left.\begin{aligned} ax+by+c &= u \\ a+by' &= u' \end{aligned}\right\} \quad \frac{u'-a}{b} = y' = f(u) \Rightarrow u' = a + bf(u)$$

$$\int \frac{du}{a+bf(u)} = \int dx \rightarrow \phi(u) = x + c \rightarrow \phi(ax+by+c) = x + c$$

ex

$$\frac{dy}{dx} = (x+y+1)^2$$

$$\left.\begin{aligned} x+y+1 &= u \\ 1+y' &= u' \end{aligned}\right\} \quad \begin{aligned} &u'-1 = u^2 \rightarrow u' = u^2+1 \\ &\int \frac{du}{u^2+1} = \int dx \rightarrow \arctan u = x + c \\ &\qquad\qquad\qquad\qquad \arctan(x+y+1) = x+c \end{aligned}$$

ex

$$\tan^2(x+y)\,dx - dy = 0$$

$$\left.\begin{aligned} x+y &= u \\ 1+y' &= u' \end{aligned}\right\} \quad \tan^2 u - u' + 1 = 0 \Rightarrow u' = 1 + \tan^2 u$$

$$\int \frac{du}{1+\tan^2 u} = \int dx \rightarrow \int \cos^2 u\, du = \int dx$$

$$\left(\cos 2u = \begin{aligned} &2\cos^2 u - 1 \\ &1 - 2\sin^2 u \end{aligned} \right)$$

$$\int \cos^2 u\, du = \int \left(\frac{1+\cos 2u}{2} \right) du = \frac{u}{2} + \frac{1}{4}\sin 2u$$

$$\frac{u}{2} + \frac{1}{4}\sin 2u = x + c \rightarrow \frac{(x+y)}{2} + \frac{1}{4}\sin 2(x+y) = x + c$$

ex $y' = 2\sqrt{2x+y+1}$

$$\left.\begin{aligned} 2x+y+1 &= u^2 \\ 2+y' &= 2uu' \end{aligned}\right\} \quad \begin{aligned} &2uu' - 2 = 2u \\ &u' = \frac{u+1}{u} \Rightarrow \int \frac{u}{u+1}\,du = \int dx \end{aligned}$$

$$\int \left(1 - \frac{1}{u+1}\right) du = \int dx \Rightarrow u - \ln|u+1| = x + c$$

$$(2x+y+1) - \ln|2x+y+2| = x + c$$

ex

$$y' = (y-4x)^2$$

$$\left.\begin{aligned} y - 4x &= u \\ y' - 4 &= u' \end{aligned}\right\} \quad \begin{aligned} &u' + 4 = u^2 \Rightarrow u' = u^2 - 4 \\ &\int \frac{du}{u^2-4} = \int dx \end{aligned}$$

$$\frac{A}{u-2} + \frac{B}{u+2} = \frac{1}{u^2-4} \Rightarrow (A+B)u + 2A - 2B = 1$$

$$\left.\begin{aligned} A+B &= 0 \\ 2A-2B &= 1 \end{aligned}\right\} 4A = 1 \Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4}$$

$$\frac{1}{4}\left[\int \frac{du}{u-2} - \int \frac{du}{u+2}\right] = x + c$$

$$\frac{1}{4}\left[\ln|u-2| - \ln|u+2|\right] = x + c$$

$$\ln\left|\frac{u-2}{u+2}\right| = 4(x+c) \Rightarrow \ln\left|\frac{y-4x-2}{y-4x+2}\right| = 4(x+c)$$

Homogeneous D.Es

$f(x,y)$ function is said to be homogeneous with degree n, if

$$f(tx,ty)=t^n f(x,y)$$

ex

$$f(x,y)=x^3-2xy^2+y^3$$
$$f(tx,ty)=t^3x^3-2t^3xy^2+t^3y^3=t^3\underbrace{\left[x^3-2xy^2+y^3\right]}_{f(x,y)}$$

ex

$$f(x,y)=e^{x/y}+\ln\frac{y}{x}$$
$$f(tx,ty)=e^{tx/ty}+\ln\frac{ty}{tx}=\underbrace{e^{x/y}+\ln\frac{y}{x}}_{f(x,y)}$$

Let us consider the D.E

$$m(x,y)\,dx+N(x,y)\,dy=0$$

If $\left.\begin{array}{l} m(tx,ty)=t^n m(x,y) \\ N(tx,ty)=t^n N(x,y)\end{array}\right\}$ then, D.E is homogeneous D.E.

To solve the H.D.E, we use the transformation

$\dfrac{y}{x}=u$ $\left(\text{Sometimes } \dfrac{x}{y}=u\right)$

After using this transformation, D.E becomes seperable D.E.

ex

$$(xe^{y/x}+y)\,dx - x\,dy = 0$$

$$\frac{y}{x}=u \Rightarrow \begin{array}{l} y = ux \\ dy = u\,dx + x\,du \end{array}$$

$$(xe^{u}+ux)\,dx - x(u\,dx + x\,du) = 0$$

$$(xe^{u}+\cancel{ux}-\cancel{ux})\,dx - x^2\,du = 0$$

$$xe^{u}\,dx - x^2\,du = 0$$

$$\int\frac{dx}{x} - \int e^{-u}\,du = \int 0 \Rightarrow \ln x + e^{-u} = c$$

$$\ln x + e^{-y/x} = c$$

ex

$$(2x+3y)\,dx + (y-x)\,dy = 0$$

$$\left.\begin{array}{l} y = ux \\ dy = u\,dx + x\,du \end{array}\right\} \begin{array}{l} (2x+3ux)\,dx + (ux-x)(u\,dx+x\,du) = 0 \\ (2x+3ux+u^2x-ux)\,dx + (ux-x)x\,du = 0 \end{array}$$

$$x(u^2+2u+2)\,dx + (u-1)x^2\,du = 0$$

$$\int\frac{dx}{x} + \int\frac{(u-1)\,du}{u^2+2u+2} = \int 0 \Rightarrow \ln x + \frac{1}{2}\int\frac{2(u+1)\,du}{u^2+2u+2} - \int\frac{2\,du}{u^2+2u+2} = 0$$

$$\ln x + \frac{1}{2}\ln|u^2+2u+2| - 2\underbrace{\int\frac{du}{(u+1)^2+1}} = c$$

$$\left.\begin{array}{l} u+1 = t \\ du = dt \end{array}\right\} \int\frac{dt}{1+t^2} = \arctan t = \arctan(u+1)$$

$$\ln x + \frac{1}{2}\ln|u^2+2u+2| - 2\arctan(u+1) = c$$

$$\ln x + \frac{1}{2}\ln\left|\frac{y^2}{x^2}+2\frac{y}{x}+2\right| - 2\arctan\left(\frac{y}{x}+1\right) = c$$

ex $(y+\sqrt{x^2-y^2})dx - x\,dy = 0$

$$\left.\begin{array}{l} \frac{y}{x} = u \Rightarrow y = ux \\ \qquad\qquad dy = u\,dx + x\,du \end{array}\right\} (ux + \sqrt{x^2-u^2x^2})dx - x(u\,dx + x\,du) = 0$$

$$(\cancel{ux} + x\sqrt{1-u^2} - \cancel{ux})dx - x^2du = 0$$

$$x\sqrt{1-u^2}\,dx - x^2du = 0 \rightarrow \int\frac{dx}{x} - \int\frac{du}{\sqrt{1-u^2}} = \int 0$$

$$\ln x - \arcsin u = \ln c \rightarrow \ln x - \ln c = \arcsin u$$

$$\ln\frac{x}{c} = \arcsin u$$

$$\rightarrow x = ce^{\arcsin u} \rightarrow \boxed{x = ce^{\arcsin\frac{y}{x}}}$$

ex

$$\left(x + y\sin\frac{y}{x}\right)dx - x\sin\frac{y}{x}\,dy = 0$$

$$\left.\begin{array}{l} y = ux \\ dy = u\,dx + x\,du \end{array}\right\} \begin{array}{l} (x + ux\sin u)dx - x\sin u\,(u\,dx + x\,du) = 0 \\ (x + \cancel{ux}\sin u - \cancel{ux}\sin u)dx - x^2\sin u\,du = 0 \end{array}$$

$$x\,dx - x^2\sin u\,du = 0 \Rightarrow \int\frac{dx}{x} - \int\sin u\,du = \int 0$$

$$\ln x + \cos u = \ln c \Rightarrow \ln\frac{x}{c} = -\cos u \Rightarrow x = ce^{-\cos u}$$

$$\rightarrow \boxed{x = ce^{-\cos\frac{y}{x}}}$$

ex

$$xy' = y + \frac{x^2}{y}$$

$$\left.\begin{array}{l} y = ux \\ y' = u'x + u \end{array}\right\} \begin{array}{l} x(u'x + u) = ux + \frac{x^2}{ux} \\ u'x^2 + \cancel{ux} = \cancel{ux} + \frac{x}{u} \rightarrow x^2\frac{du}{dx} = \frac{x}{u} \end{array}$$

$$u\,du = \frac{dx}{x} \Rightarrow \frac{u^2}{2} = \ln x + c \rightarrow \frac{y^2}{2x^2} = \ln x + c$$

ex $y(1+e^{x/y})dx + (y-x)e^{x/y}dy = 0$

$$\left.\begin{aligned} \frac{x}{y} = u \Rightarrow x &= uy \\ dx &= u\,dy + y\,du \end{aligned}\right\} \; y(1+e^u)(u\,dy + y\,du) + (y-uy)e^u dy = 0$$

$$(uy + \cancel{uy}e^u + ye^u - \cancel{uy}e^u)dy + y^2(1+e^u)du = 0$$

$$y(u+e^u)dy + y^2(1+e^u)du = 0 \rightarrow \int \frac{dy}{y} + \int \frac{(1+e^u)du}{u+e^u} = \int 0$$

$$\ln y + \ln(u+e^u) = \ln c \Rightarrow y(u+e^u) = c$$

$$\rightarrow y\left(\frac{x}{y} + e^{x/y}\right) = c$$

Homework

1. $(xyy' - y^2)\ln \frac{y}{x} = x^2$

2. Solve the I.V.P $xe^{y/x} + y - xy' = 0$, $y(1) = 0$
3. $y' = \frac{y^2 + 2xy}{x^2}$

D.Es which can be transformed into seperable or homogeneous D.Es

Let us consider the equation

$(a_1 x + b_1 y + c_1)dx + (a_2 x + b_2 y + c_2)dy = 0$ $\quad a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$

Case I

If $\frac{a_2}{a_1} = \frac{b_2}{b_1} = k$, then the transformation

$u = a_1 x + b_1 y$ (or $u = a_2 x + b_2 y$) reduces the D.E to a seperable D.E.

ex

$$(4x+4y+1)dx + (x+y-1)dy = 0$$

$$\left.\begin{array}{l} x+y=u \\ dx+dy=du \end{array}\right\} \begin{array}{l} (4u+1)dx + (u-1)(du-dx) = 0 \\ (4u+1-u+1)dx + (u-1)du = 0 \\ (3u+2)dx + (u-1)du = 0 \end{array}$$

$$\int dx + \int \frac{u-1}{3u+2}\,du = \int 0$$

$$\begin{array}{l|l} u-1 & 3u+2 \\ \hline u+\frac{2}{3} & \frac{1}{3} \\ \hline -\frac{5}{3} & \end{array}$$

$$\int dx + \int \frac{du}{3} - \frac{5}{3}\int \frac{du}{3u+2} = \int 0$$

$$x + \frac{u}{3} - \frac{5}{3}\cdot\frac{1}{3}\ln(3u+2) = c \Rightarrow x + \frac{x+y}{3} - \frac{5}{9}\ln(3x+3y+2) = c$$

ex

$$(3x+3y-1)dx - (x+y+1)dy = 0$$

$$\left.\begin{array}{l} x+y=u \\ dx+dy=du \end{array}\right\} \begin{array}{l} (3u-1)dx - (u+1)(du-dx) = 0 \\ (3u-1+u+1)dx - (u+1)du = 0 \\ 4u\,dx - (u+1)du = 0 \end{array}$$

$$\int 4dx - \int \frac{u+1}{u}\,du = \int 0$$

$$4x - u - \ln u = c \rightarrow 4x - x - y - \ln(x+y) = c$$

$$3x - y - \ln(x+y) = c$$

Homework

$(x+2y+1)dx-(2x+4y-3)dy=0$

Case II

If $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$, then the transformation

$\left.\begin{array}{l} x=x_1+h \\ y=y_1+k \end{array}\right\}$ where (h,k) is the solution of the system

reduce the D.E into the homogeneous D.E.

ex

$(x-y+1)dx+(x+y-1)dy=0$

$\left.\begin{array}{ll} x=x_1+h & , dx=dx_1 \\ y=y_1+k & , dy=dy_1 \end{array}\right\}$ $(x_1+h-y_1-k+1)dx_1+(x_1+h+y_1+k-1)dy_1=0$

$\left.\begin{array}{l} h-k+1=0 \Rightarrow h-k=-1 \\ h+k-1=0 \Rightarrow h+k=1 \end{array}\right\}$ $\begin{array}{l} h=0 \\ k=1 \end{array}$ $\boxed{\begin{array}{l} x=x_1 \\ y=y_1+1 \end{array}}$

$(x_1-y_1)dx_1+(x_1+y_1)dy_1=0$

$\left.\begin{array}{l} \frac{y_1}{x_1}=u \Rightarrow y_1=ux_1 \\ \qquad dy_1=u\,dx_1+x_1\,du \end{array}\right\}$

$(x_1-ux_1)dx_1+(x_1+ux_1)(u\,dx_1+x_1\,du)=0$

$(x_1-\cancel{ux_1}+\cancel{ux_1}+u^2x_1)dx_1+(x_1^2+ux_1^2)du=0$

$x_1(1+u^2)dx_1+x_1^2(1+u)du=0$

$\int\frac{dx_1}{x_1}+\int\frac{(1+u)du}{1+u^2}=\int 0$

$$\ln x_1 + \arctan u + \frac{1}{2}\ln(1+u^2) = c$$

$$\ln x + \arctan\left(\frac{y-1}{x}\right) + \frac{1}{2}\ln\left(1+\left(\frac{y-1}{x}\right)^2\right) = c$$

ex $(2x-y+4)dx + (x+2y+7)dy = 0$

$$\left.\begin{array}{l} x = x_1 + h \,,\; dx = dx_1 \\ y = y_1 + k \,,\; dy = dy_1 \end{array}\right\} (2x_1 - y_1 + 2h - k + 4)dx_1 + (x_1 + 2y_1 + h + 2k + 7)dy_1 = 0$$

$$\left.\begin{array}{l} 2h - k = -4 \\ h + 2k = -7 \end{array}\right\} \left.\begin{array}{l} h = -3 \\ k = -2 \end{array}\right\} \left.\begin{array}{l} x = x_1 - 3 \\ y = y_1 - 2 \end{array}\right\} \begin{array}{l} x_1 = x + 3 \\ y_1 = y + 2 \end{array}$$

$$(2x_1 - y_1)dx_1 + (x_1 + 2y_1)dy_1 = 0$$

$$\left.\begin{array}{l} y_1 = ux_1 \\ dy_1 = u dx_1 + x_1 du \end{array}\right\} (2x_1 - ux_1)dx_1 + (x_1 + 2ux_1)(u dx_1 + x_1 du) = 0$$

$$(2x_1 - \cancel{ux_1} + \cancel{ux_1} + 2u^2x_1)dx_1 + (x_1^2 + 2ux_1^2)du = 0$$

$$2x_1(1+u^2)dx_1 + x_1^2(1+2u)du = 0$$

$$\int \frac{2dx_1}{x_1} + \int \frac{1+2u}{1+u^2}du = \int 0 \rightarrow 2\ln x_1 + \arctan u + \ln(1+u^2) = \ln c$$

$$\frac{x_1^2(1+u^2)}{c} = e^{-\arctan u} \Rightarrow x_1^2\left(1 + \frac{y_1^2}{x_1^2}\right) = c e^{-\arctan \frac{y_1}{x_1}}$$

$$(x_1^2 + y_1^2) = c e^{-\arctan \frac{y_1}{x_1}}$$

$$\left((x+3)^2 + (y+2)^2\right) = c e^{-\arctan\left(\frac{y+2}{x+3}\right)}$$

Homework

$(x-3y)dx + (x+y+4)dy = 0$

Exact D.Es

Consider the D.E

$$m(x,y)\,dx + N(x,y)\,dy = 0$$

where m and N have continuous partial derivatives at all points in domain D.

If $\dfrac{\partial m}{\partial y} = \dfrac{\partial N}{\partial x}$

for all $(x,y)\in D$, then D.E is an "exact D.E".

Solution

Let the solution of an exact D.E be the function

$$u(x,y) = k \quad (k \text{ is a constant})$$

The total differential of u is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 = m(x,y)dx + N(x,y)dy$$

$$\underbrace{\frac{\partial u}{\partial x} = m(x,y)}_{\textcircled{A}} \quad , \quad \underbrace{\frac{\partial u}{\partial y} = N(x,y)}_{AA}$$

Thus, we aim to find a function $u(x,y)$ which satisfies (A) and (AA).

Let us assume that u satisfies (A), then with integrating (A), we obtain

$$u(x,y) = \int m(x,y)\,dx + \phi(y)$$

If we integrates $u(x,y)$ with respect to ∂y,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\int m(x,y)\,dx + \frac{d\phi}{dy} = N(x,y)$$

$$\frac{d\phi}{dy} = N(x,y) - \frac{\partial}{\partial y}\int m(x,y)\,dx$$

By integrating $\frac{d\phi}{dy}$ with respect to y, we find ϕ.

ex

$$\underbrace{(3x^2+4xy)}_{m}dx+\underbrace{(2x^2+2y)}_{N}dy=0$$

$$\frac{\partial m}{\partial y}=4x=\frac{\partial N}{\partial x}=4x \Rightarrow \text{Exact D.E}$$

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=(3x^2+4xy)dx+(2x^2+2y)dy=0$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=3x^2+4xy\\ \frac{\partial u}{\partial y}&=2x^2+2y\end{aligned}\right\} \quad u(x,y)=x^3+2x^2y+\phi(y)$$

$$\frac{\partial u}{\partial y}=2x^2+\frac{d\phi}{dy}$$

$$2x^2+\frac{d\phi}{dy}=2x^2+2y \rightarrow \frac{d\phi}{dy}=2y \Rightarrow \phi(y)=y^2+c$$

$$u(x,y)=x^3+2x^2y+y^2+c=k$$

ex

$$\underbrace{(x+y\cos x)}_{m}dx+\underbrace{(y+\sin x)}_{N}dy=0$$

$$\frac{\partial m}{\partial y}=\cos x=\frac{\partial N}{\partial x}=\cos x$$

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=x+y\cos x\\ \frac{\partial u}{\partial y}&=y+\sin x\end{aligned}\right\} \quad u(x,y)=\frac{x^2}{2}+y\sin x+\phi(y)$$

$$\frac{\partial u}{\partial y}=\sin x+\frac{d\phi}{dy}$$

$$y+\sin x=\sin x+\frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dy}=y \Rightarrow \phi(y)=\frac{y^2}{2}+c$$

$$u(x,y)=\frac{x^2}{2}+y\sin x+\frac{y^2}{2}+c=k$$

ex $\left(x^2 e^y + \frac{x}{y} - 3y^2 + 1\right)dy + \left(2xe^y + \ln y + \frac{1}{\sqrt{1-x^2}}\right)dx = 0$

$$\frac{\partial m}{\partial y} = 2xe^y + \frac{1}{y} = \frac{\partial N}{\partial x} = 2xe^y + \frac{1}{y}$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2xe^y + \ln y + \frac{1}{\sqrt{1-x^2}} \\ \frac{\partial u}{\partial y} &= x^2 e^y + \frac{x}{y} - 3y^2 + 1 \end{aligned}\right\} \begin{aligned} &u(x,y) = x^2 e^y + x\ln y - y^3 + y + \phi(x) \\ &\frac{\partial u}{\partial x} = 2xe^y + \ln y + \frac{d\phi}{dx} \end{aligned}$$

$$\cancel{2xe^y} + \cancel{\ln y} + \frac{1}{\sqrt{1-x^2}} = \cancel{2xe^y} + \cancel{\ln y} + \frac{d\phi}{dx} \Rightarrow \frac{d\phi}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$\phi(x) = \arcsin x + c$

$u(x,y) = x^2 e^y + x\ln y - y^3 + y + \arcsin x + c = k$

Homework

$$\arcsin x\, dy + \frac{y + 2\sqrt{1-x^2}\cos x}{\sqrt{1-x^2}}dx = 0$$

Integrating Factor

Consider the D.E

$$m(x,y)dx + N(x,y)dy = 0$$

Assume that $\frac{\partial m}{\partial y} \neq \frac{\partial N}{\partial x}$

If there is a function $\lambda(x,y)$ which makes the D.E

$$\lambda(x,y)m(x,y)dx + \lambda(x,y)N(x,y)dy = 0$$

as an exact D.E, then $\lambda(x,y)$ is called "integrating factor" of the D.E.

$\lambda = \lambda(x)$

$$\lambda(x)m(x,y)dx + \lambda(x)N(x,y)dy = 0$$

$$\frac{\partial}{\partial y}\left[\lambda(x)m(x,y)\right] = \frac{\partial}{\partial x}\left[\lambda(x)N(x,y)\right]$$

$$\lambda(x)\frac{\partial m}{\partial y} = \frac{d\lambda}{dx}N + \lambda\frac{\partial N}{\partial x}$$

$$\frac{d\lambda}{\partial x}N = \lambda(x)\left[\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right] \Rightarrow \frac{d\lambda}{\lambda} = \frac{1}{N}\left[\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right]dx$$

$$\lambda(x) = e^{\int \frac{\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}}{N}dx}$$

$\lambda = \lambda(y)$

$$\lambda(y)m(x,y)dx + \lambda(y)N(x,y)dy = 0$$

$$\frac{\partial}{\partial y}\left[\lambda(y)m\right] = \frac{\partial}{\partial x}\left[\lambda(y)N\right]$$

$$\frac{d\lambda}{dy}m + \lambda\frac{\partial m}{\partial y} = \lambda\frac{\partial N}{\partial x} \Rightarrow \frac{d\lambda}{dy}m = \lambda\left[\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right]$$

$$\frac{d\lambda}{\lambda} = \frac{1}{m}\left[\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}\right]dy$$

$$\lambda(y) = e^{\int \frac{\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}}{m}dy}$$

ex $(x^2+y^2)dx + xydy = 0$

$$\frac{\partial m}{\partial y} = 2y \neq \frac{\partial N}{\partial x} = y$$

1°) $\lambda = \lambda(x)$

$$\lambda(x) = e^{\int \frac{\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}}{N} dx} = e^{\int \frac{2y-y}{xy} dx} = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$(x^3+xy^2)dx + x^2ydy = 0$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= x^3 + xy^2 \\ \frac{\partial u}{\partial y} &= x^2y \end{aligned}\right\} \quad u(x,y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \phi(y)$$

$$\frac{\partial u}{\partial y} = x^2y + \frac{d\phi}{dy}$$

$$x^2y = x^2y + \frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dy} = 0 \rightarrow \phi(y) = c$$

$$u(x,y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + c = K$$

ex $(xy+1)ydx + (2y-x)dy = 0$

$$\frac{\partial m}{\partial y} = 2xy+1 \neq \frac{\partial N}{\partial x} = -1$$

$$\lambda = \lambda(y) \rightarrow \lambda(y) = e^{\int \frac{\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}}{m} dy} = e^{\int \frac{-1-2xy-1}{(xy+1)y} dy}$$

$$\lambda(y) = e^{\int \frac{-2(xy+1)}{(xy+1)y} dy} = e^{\int -\frac{2dy}{y}} = e^{-2\ln y} \rightarrow \lambda(y) = \frac{1}{y^2}$$

$$\left(x+\frac{1}{y}\right)dx+\left(\frac{2}{y}-\frac{x}{y^2}\right)dy=0$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= x+\frac{1}{y} \\ \frac{\partial u}{\partial y} &= \frac{2}{y}-\frac{x}{y^2} \end{aligned}\right\} \quad u(x,y)=\frac{x^2}{2}+\frac{x}{y}+\phi(y)$$

$$\frac{\partial u}{\partial y}=-\frac{x}{y^2}+\frac{d\phi}{dy}$$

$$\frac{2}{y}-\frac{x}{y^2}=-\frac{x}{y^2}+\frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dy}=\frac{2}{y} \rightarrow \phi(y)=2\ln y+c$$

$$u(x,y)=\frac{x^2}{2}+\frac{x}{y}+2\ln y+c=K$$

ex

$$y'\sin x+\sin^2 x-y\cos x=0$$

$$\sin x\,dy+(\sin^2 x-y\cos x)\,dx=0$$

$$\frac{\partial m}{\partial y}=-\cos x \neq \frac{\partial N}{\partial x}=\cos x$$

$$\lambda=\lambda(x) \qquad \lambda(x)=e^{\int \frac{\frac{\partial m}{\partial y}-\frac{\partial N}{\partial x}}{N}dx}=e^{\int \frac{-\cos x-\cos x}{\sin x}dx}$$

$$\lambda(x)=e^{-2\int \cot x\,dx}=e^{-2\ln(\sin x)}=\frac{1}{\sin^2 x}$$

$$\frac{dy}{\sin x}+\left(1-y\frac{\cos x}{\sin^2 x}\right)dx=0$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 1-y\frac{\cos x}{\sin^2 x} \\ \frac{\partial u}{\partial y} &= \frac{1}{\sin x} \end{aligned}\right\} \quad u(x,y)=\frac{y}{\sin x}+\phi(x)$$

$$\frac{\partial u}{\partial x}=-\frac{y\cos x}{\sin^2 x}+\frac{d\phi}{dx}$$

$$1 - y\cancel{\frac{\cos x}{\sin^2 x}} = -y\cancel{\frac{\cos x}{\sin^2 x}} + \frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dx} = 1 \Rightarrow \phi(x) = x + c$$

$$u(x,y) = \frac{y}{\sin x} + x + c = K$$

ex $y\,dx + (2xy - e^{-2y})\,dy = 0$

$$\frac{\partial m}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2y$$

$$\lambda = \lambda(y) \Rightarrow e^{\int \frac{\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y}}{m} dy} = e^{\int \frac{2y-1}{y} dy} = e^{\int (2 - \frac{1}{y}) dy}$$

$$\lambda(y) = e^{2y} \cdot e^{-\ln y} = \frac{e^{2y}}{y}$$

$$e^{2y}\,dx + \left(2xe^{2y} - \frac{1}{y}\right) dy = 0$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^{2y} \\ \frac{\partial u}{\partial y} &= 2xe^{2y} - \frac{1}{y} \end{aligned}\right\} \quad \begin{aligned} & u(x,y) = xe^{2y} + \phi(y) \\ & \frac{\partial u}{\partial y} = 2xe^{2y} + \frac{d\phi}{dy} \end{aligned}$$

$$\cancel{2xe^{2y}} - \frac{1}{y} = \cancel{2xe^{2y}} + \frac{d\phi}{dy} \Rightarrow \frac{d\phi}{dy} = -\frac{1}{y} \Rightarrow \phi(y) = -\ln y + c$$

$$u(x,y) = xe^{2y} - \ln y + c = K$$

Linear Differential Equations

The form of

$$y' + P(x)\,y = Q(x)$$

is called linear differential equation.

For the solution, let us write the L.D.E as

$$dy + [P(x)\,y - Q(x)]\,dx = 0$$

$$\frac{\partial M}{\partial y} = P(x) \quad , \quad \frac{\partial N}{\partial x} = 0$$

$$\lambda = \lambda(x) \qquad \lambda(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx} = e^{\int \frac{P(x) - 0}{1} dx} = e^{\int P(x)dx}$$

$$\underbrace{e^{\int P(x)dx}\, y' + P(x)\, y\, e^{\int P(x)dx}} = Q(x)\, e^{\int P(x)dx}$$

$$\frac{d}{dx}\left[e^{\int P(x)dx}\, y\right] = Q(x)\, e^{\int P(x)\,dx}$$

By integrating the equation,

$$e^{\int P(x)dx}\, y = \int Q(x)\, e^{\int P(x)dx}\, dx + c$$

$$y = e^{-\int P(x)dx}\left[\int Q(x)\, e^{\int P(x)dx}\, dx + c\right]$$

$$y = \frac{1}{\lambda}\left[\int Q(x)\,\lambda\, dx + c\right]$$

ex $y' = y\cot x + \sin x$

$\left.y' - y\cot x = \sin x\right\} P(x) = -\cot x \ , \ Q(x) = \sin x$

$$\lambda(x) = e^{-\int \cot x\,dx} = e^{-\ln(\sin x)} = \frac{1}{\sin x}$$

$$y = \sin x\left[\int \sin x \cdot \frac{1}{\sin x}\,dx + c\right] = \sin x\left[x + c\right]$$

ex $y' + \frac{y}{x} = \frac{e^{2x} - \sin x}{x}$

$$P(x) = \frac{1}{x} \ , \ Q(x) = \frac{e^{2x} - \sin x}{x}$$

$$\lambda(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$y = \frac{1}{x}\left[\int \frac{e^{2x} - \sin x}{x} \cdot x\,dx + c\right] = \frac{1}{x}\left[\frac{1}{2}e^{2x} + \cos x + c\right]$$

ex $y' + \left(\frac{2x+1}{x}\right)y = e^{-2x}$

$$P(x) = \frac{2x+1}{x} \ , \ Q(x) = e^{-2x}$$

$$\lambda(x) = e^{\int\left(2 + \frac{1}{x}\right)dx} = e^{2x + \ln x} = e^{2x} \cdot e^{\ln x} = e^{2x} \cdot x$$

$$y = \frac{1}{xe^{2x}}\left[\int e^{-2x} \cdot e^{2x} \cdot x\,dx + c\right] = \frac{1}{xe^{2x}}\left(\frac{x^2}{2} + c\right)$$

Variation of Parameters

Let us assume $y' + P(x)y = Q(x)$.

If we take

$$y' + P(x)y = 0 \rightarrow dy + P(x)y\,dx = 0$$

$$\int \frac{dy}{y} + \int P(x)dx = \int 0 \rightarrow \ln y + \int P(x)dx = \ln c$$

$$y = c e^{-\int P(x)dx}$$

Now, we have $Q(x)$, so let us consider $c = c(x)$

So, solution will be

$$\left.\begin{aligned} y &= c(x) e^{-\int P(x)dx} \\ y' &= c' e^{-\int P(x)dx} - cP(x) e^{-\int P(x)dx} \end{aligned}\right\}$$

$$y' + P(x)y = Q(x) \rightarrow c' e^{-\int P(x)dx} \cancel{- cPe^{-\int P(x)dx}} \cancel{+ cPe^{-\int P(x)dx}} = Q(x)$$

$$c' = Q(x) e^{\int P(x)dx} \rightarrow c(x) = \int Q(x) e^{\int P(x)dx} + k$$

$$y = e^{-\int P(x)dx} \left[\int Q(x) e^{\int P(x)dx} + k \right]$$

ex $y' - 2y = x^2 e^{2x}$

$$y' - 2y = 0 \rightarrow dy - 2y\,dx = 0 \rightarrow \int \frac{dy}{y} - \int 2\,dx = \int 0$$

$$\ln y - 2x = \ln c \rightarrow y = c e^{2x}$$

Assume that $c = c(x)$

$$\left.\begin{aligned} y &= c(x)e^{2x} \\ y' &= c'e^{2x} + 2ce^{2x} \end{aligned}\right\}$$

$$c'e^{2x} + \cancel{2c}e^{2x} - \cancel{2c}e^{2x} = x^2e^{2x} \Rightarrow c' = x^2 \rightarrow c(x) = \frac{x^3}{3} + k$$

$$y = \left(\frac{x^3}{3} + k\right)e^{2x}$$

Bernoulli Differential Equation

An equation of the form

$$y' + P(x)y = Q(x)y^n$$

is called "Bernoulli Differential Equation."

If $n = 0$, then $y' + P(x)y = Q(x) \Rightarrow$ Linear D.E

If $n = 1$, then $\underline{y' + P(x)y = Q(x)y}$

$$\frac{dy}{dx} + (P - Q)y = 0 \Rightarrow \frac{dy}{y} + (P - Q)dx = 0$$

seperable D.E

So, let us consider $n \neq 0, 1$

Theorem: Suppose that $n \neq 0, 1$. Then the transformation

$$\left.\begin{aligned} u &= y^{1-n} \\ u' &= (1-n)y^{-n}y' \end{aligned}\right\}$$

reduces the Bernoulli D.E into Linear D.E. But, before the application, we need to multiply each term with y^{-n}. That is

$$y'y^{-n} + P(x)y^{1-n} = Q(x)$$

$$\frac{u'}{1-n} + P(x)u = Q(x) \Rightarrow \text{L.D.E}$$

ex

$y' + y = xy^3$. Solve the D.E.

$$\left.\begin{aligned} y^{1-3} &= y^{-2} = u \\ -2y^{-3}y' &= u' \end{aligned}\right\} \qquad y'y^{-3} + y^{-2} = x$$

$$-\frac{u'}{2} + u = x \Rightarrow u' - 2u = -2x \quad \text{L.D.E}$$

$$\lambda(x) = e^{\int -2dx} = e^{-2x}$$

$$u(x) = \frac{1}{\lambda}\left[\int Q(x)\lambda\, dx + c\right]$$

$$u(x) = e^{2x}\left[\int -2x e^{-2x} dx + c\right]$$

For the integral $I = \int x e^{-2x} dx$

$$\left.\begin{aligned} x &= \alpha & e^{-2x}dx &= d\beta \\ dx &= d\alpha & -\frac{1}{2}e^{-2x} &= \beta \end{aligned}\right\} \quad I = \alpha\beta - \int \beta\, d\alpha$$

$$I = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\left(-\frac{1}{2}\right)e^{-2x}$$

$$u = e^{2x}\left[(-2)\left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right] + c\right]$$

$$u = x + \frac{1}{2} + ce^{2x}$$

$$y^{-2} = \frac{1}{y^2} = u \Rightarrow y = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{x + \frac{1}{2} + ce^{2x}}}$$

ex $xy' = 2x\sqrt{y} + 4y$ Solve the D.E

$$xy' - 4y = 2x\sqrt{y}$$

$$\left.\begin{aligned} y^{1-\frac{1}{2}} &= y^{\frac{1}{2}} = u \\ \frac{1}{2} y^{-\frac{1}{2}} y' &= u' \end{aligned}\right\} \quad x y' y^{-\frac{1}{2}} - 4 y^{1/2} = 2x$$

$$2u'x - 4u = 2x \Rightarrow u' - \frac{2u}{x} = 1$$

$$\lambda(x) = e^{\int -\frac{2dx}{x}} = e^{-2\ln x} = e^{\ln x^{-2}} = \frac{1}{x^2}$$

$$u = x^2\left[\int \frac{1}{x^2} \cdot 1 \cdot dx + c\right] = x^2\left(-\frac{1}{x} + c\right) = -x + cx^2$$

$$\sqrt{y} = u \Rightarrow y = u^2 = (cx^2 - x)^2$$

ex $y'\cos x + y\sin x = -y^3$

$$\left.\begin{aligned} y^{1-3} &= y^{-2} = u \\ (-2) y^{-3} y' &= u' \end{aligned}\right\} \quad y' y^{1-3} \cos x + y^{-2} \sin x = -1$$

$$-\frac{u'}{2}\cos x + u\sin x = -1 \Rightarrow u' - 2u\frac{\sin x}{\cos x} = \frac{2}{\cos x}$$

$$u' - 2u\tan x = 2\sec x$$

$$\lambda(x) = e^{-2\int \tan x\, dx} = e^{2\ln(\cos x)} = \cos^2 x$$

$$u(x) = \frac{1}{\cos^2 x}\left[\int \cos^2 x \cdot 2\sec x\, dx + c\right]$$

$$u(x) = \frac{1}{\cos^2 x}\left[2\sin x + c\right] = 2\tan x \sec x + c\sec^2 x$$

$$y^{-2} = \frac{1}{y^2} = u \Rightarrow y = \frac{1}{\sqrt{u}} = \frac{1}{\sqrt{2\tan x \sec x + c\sec^2 x}}$$

ex Solve the initial value problem

$x y' + y = (xy)^{3/2}$, $y(1) = 4$.

$$x y' + y = x\sqrt{x} \cdot y^{3/2}$$

$$\left.\begin{aligned} & y^{1-\frac{3}{2}} = y^{-\frac{1}{2}} = u \\ & \left(-\frac{1}{2}\right) y^{-3/2} y' = u' \end{aligned}\right\} \quad x y' y^{-3/2} + y^{-1/2} = x\sqrt{x}$$

$$-2u'x + u = x\sqrt{x}$$

$$u' - \frac{u}{2x} = -\frac{\sqrt{x}}{2}$$

$$\lambda(x) = e^{\int -\frac{dx}{2x}} = e^{-\frac{1}{2}\ln x} = \frac{1}{\sqrt{x}}$$

$$u(x) = \sqrt{x}\left[\int \frac{1}{\sqrt{x}} \cdot \left(-\frac{\sqrt{x}}{2}\right) dx + c\right]$$

$$u(x) = \sqrt{x}\left[-\frac{x}{2} + c\right] = -\frac{x\sqrt{x}}{2} + c\sqrt{x}$$

$$\frac{1}{\sqrt{y}} = u \Rightarrow y = \frac{1}{u^2} = \frac{1}{\left(-\frac{x\sqrt{x}}{2} + c\sqrt{x}\right)^2}$$