Riccati Differential Equation

An equation in the form

$$y' + P(x)y^2 + Q(x)y + R(x) = 0$$

is called Riccati D.E.

For the solution of Riccati D.Es, we have to know a particular solution of $y(x)$.

If we use the transformation

1. with $y = y_1(x) + u$, new D.E will be a Bernoulli D.E.
2. with $y = y_1(x) + \frac{1}{u}$, new D.E will be a Linear D.E.

ex $\quad y' + 2xy = x^2 + y^2 + 1 \qquad y_1(x) = x$

$$\left.\begin{aligned} y &= x + \frac{1}{u} \\ y' &= 1 - \frac{u'}{u^2} \end{aligned}\right\} \quad \cancel{1} - \frac{u'}{u^2} + \cancel{2x^2} + \cancel{\frac{2x}{u}} = \cancel{x^2} + \cancel{1} + \cancel{x^2} + \frac{1}{u^2} + \cancel{\frac{2x}{u}}$$

$$-\frac{u'}{u^2} = \frac{1}{u^2} \Rightarrow u' = -1 \Rightarrow \frac{du}{dx} = -1 \Rightarrow \int du = -\int dx$$

$$u = -x + c$$

$$y = x + \frac{1}{c - x}$$

ex $\quad y' + y^2 - 3y\tan x + \tan^2 x - 1 = 0 \qquad y_1(x) = \tan x$

$$\left.\begin{aligned} y &= \tan x + \frac{1}{u} \\ y' &= 1 + \tan^2 x - \frac{u'}{u^2} \end{aligned}\right\} \quad \cancel{1} + \cancel{\tan^2 x} - \frac{u'}{u^2} + \cancel{\tan^2 x} + \frac{2\tan x}{u} + \frac{1}{u^2}$$

$$- \cancel{3\tan^2 x} - \frac{3\tan x}{u} + \cancel{\tan^2 x} - \cancel{1} = 0$$

$$-\frac{u'}{u^2} - \frac{\tan x}{u} + \frac{1}{u^2} = 0 \Rightarrow u' + u\tan x = 1 \qquad \text{L.D.E}$$

$$\lambda(x) = e^{\int \tan x\, dx} = e^{-\ln(\cos x)} = \frac{1}{\cos x}$$

$$u = \cos x \left[\int \frac{1}{\cos x} dx + c \right] = \cos x \left[\ln|\sec x + \tan x| + c \right]$$

ex $\quad y' = -\frac{1}{x^2} - \frac{y}{x} + y^2 \qquad y_1(x) = \frac{1}{x}$

$$\left.\begin{aligned} y &= \frac{1}{x} + \frac{1}{u} \\ y' &= -\frac{1}{x^2} - \frac{u'}{u^2} \end{aligned}\right\} \cancel{-\frac{1}{x^2}} - \frac{u'}{u^2} = \cancel{-\frac{1}{x^2}} \cancel{-\frac{1}{x^2}} - \frac{1}{ux} \cancel{+\frac{1}{x^2}} + \frac{1}{u^2} + \frac{2}{ux}$$

$$-\frac{u'}{u^2} = \frac{1}{ux} + \frac{1}{u^2} \Rightarrow u' + \frac{u}{x} = -1$$

$$\lambda(x) = e^{\int \frac{dx}{x}} = e^{\ln x} = x$$

$$u = \frac{1}{x}\left[\int x(-1)dx + c \right] = \frac{1}{x}\left(-\frac{x^2}{2} + c \right) = -\frac{x}{2} + \frac{c}{x} = \frac{-x^2+2c}{2x}$$

$$y = \frac{1}{x} + \frac{2x}{-x^2+2c}$$

ex $\quad y' = xy^2 + (1-2x)y + x - 1 \quad , \quad y_1(x) = 1$

$$\left.\begin{aligned} y &= 1 + \frac{1}{u} \\ y' &= -\frac{u'}{u^2} \end{aligned}\right\} -\frac{u'}{u^2} = \cancel{x} + \cancel{\frac{2x}{u}} + \frac{x}{u^2} + \cancel{1} + \frac{1}{u} - \cancel{2x} - \cancel{\frac{2x}{u}} + \cancel{x} - \cancel{1}$$

$$-\frac{u'}{u^2} = \frac{x}{u^2} + \frac{1}{u} \Rightarrow u' - u = -x$$

$$\lambda(x) = e^{\int -dx} = e^{-x}$$

$$u = e^{x}\left[\int e^{-x}(-x)dx + c \right]$$

$$\left.\begin{array}{ll} -x = \alpha & e^{-x}dx = d\beta \\ -dx = d\alpha & -e^{-x} = \beta \end{array}\right\} \begin{array}{l} I = xe^{-x} - \int e^{-x}dx \\ I = xe^{-x} + e^{-x} \end{array}$$

$$u = e^{x}\left[(x+1)e^{-x} + c\right] = (x+1) + ce^{x}$$

$$y = 1 + \frac{1}{(x+1) + ce^{x}}$$

First Order Differential Equations of Higher Degrees

Clairaut Differential Equation

An equation of the form

$$y = xy' + g(y')$$

is called Clairaut D.E. where $g(y')$ is a non-linear differentiable function.

By taking $y' = p(x)$, we will have

$$y = xp + g(p)$$

Here we have three variables. To cancel y, let us derivate each term,

$$\underbrace{y'}_{\cancel{p}} = \cancel{p} + xp' + p'g'(p)$$

$$p'\left[x + g'(p)\right] = 0$$

1°) $p' = 0 \Rightarrow p = c \Rightarrow y = cx + g(c)$ General solution

2°) $\left.\begin{array}{l} x = -g'(p) \\ y = -pg'(p) + g(p) \end{array}\right\}$ parametric form of singular solution ex $y = xy' - (y')^2$ Find the general and singular solution

$y' = p \Rightarrow y = px - p^2$

$y' = p'x + p - 2pp' \Rightarrow p'[x - 2p] = 0$

p

$1^\circ)$ $p' = 0 \Rightarrow p = c \rightarrow y = cx - c^2$ G.S

$2^\circ)$ $\left.\begin{array}{l} x = 2p \\ y = 2p^2 - p^2 = p^2 \end{array}\right\}$ $x^2 = 4y$ S.S

ex

$y = xy' + \ln(y')$

$y' = p \Rightarrow y = xp + \ln p$

$y' = p'x + p + \dfrac{p'}{p} \Rightarrow p'\left[x + \dfrac{1}{p}\right] = 0$

p

$1^\circ)$ $p' = 0 \rightarrow p = c \rightarrow y = cx + \ln c$ G.S

$2^\circ)$ $\left.\begin{array}{l} x = -\dfrac{1}{p} \\ y = -1 + \ln p \end{array}\right\}$ $y = -1 + \ln\left(-\dfrac{1}{x}\right)$

ex

$y = xy' + \sqrt{y'}$

$y' = p \rightarrow y = px + \sqrt{p}$

$y' = p'x + p + \dfrac{p'}{2\sqrt{p}} \rightarrow p'\left[x + \dfrac{1}{2\sqrt{p}}\right] = 0$

p

$1^\circ)$ $p' = 0 \rightarrow p = c \rightarrow y = cx + \sqrt{c}$ G.S

$2^\circ)$ $x = -\dfrac{1}{2\sqrt{p}}$

$y = -\dfrac{p}{2\sqrt{p}} + \sqrt{p} = \dfrac{-p + 2p}{2\sqrt{p}} = \dfrac{\sqrt{p}}{2}$

$\left.\begin{array}{l} \sqrt{p} = -\dfrac{1}{2x} \\ \sqrt{p} = 2y \end{array}\right\}$ $-\dfrac{1}{2x} = 2y$

$4xy + 1 = 0$

S.S

Homework

1. $y = xy' + y' - (y')^2$

2. $y = xy' + \dfrac{1}{(y')^2}$

3. $y' - (y')^3 = xy'$

Lagrange Differential Equation

An equation of the form

$$y = x f(y') + g(y')$$

is called Lagrange D.E where $f(y')$ and $g(y')$ are differentiable functions.

$$y = x f(p) + g(p)$$

$$\underbrace{y'}_{p} = f(p) + x p' f'(p) + p' g'(p)$$

$$p - f(p) = \underbrace{p'}_{\frac{dp}{dx}} \left[x f'(p) + g'(p) \right]$$

$$\frac{dx}{dp} = \frac{x f'(p) + g'(p)}{p - f(p)} \Rightarrow \frac{dx}{dp} - x \frac{f'(p)}{p - f(p)} = \frac{g'(p)}{p - f(p)} \quad \text{L.D.E}$$

ex

$$y = -xy' + (y')^3$$

$$y' = p \Rightarrow y = -xp + p^3$$

$$\underbrace{y'}_{p} = -p - xp' + 3p^2 p' \Rightarrow 2p = p' \left[-x + 3p^2 \right]$$

$$\frac{dx}{dp} = \frac{-x + 3p^2}{2p} \rightarrow \frac{dx}{dp} + \frac{x}{2p} = \frac{3p}{2} \quad \text{L.D.E}$$

$x \rightarrow$ dep.
$p \rightarrow$ indep.

$$\lambda(p) = e^{\int \frac{dp}{2p}} = e^{\frac{1}{2}\ln p} = \sqrt{p}$$

$$x = \frac{1}{\sqrt{p}} \left[\int \sqrt{p} \cdot \frac{3p}{2} dp + c \right] = \frac{1}{\sqrt{p}} \left[\frac{3}{2} \cdot \frac{2}{5} p^{5/2} + c \right] = \frac{3}{5} p^2 + \frac{c}{\sqrt{p}}$$

$$y = -\frac{3}{5}p^3 - c\sqrt{p} + p^3 = \frac{2}{5}p^3 - c\sqrt{p}$$

x and y are the parametric equations of the general solution.

ex

$$y = 2xy' + (y')^2$$

$$y' = p \Rightarrow y = 2px + p^2$$

$$\underset{p}{y'} = 2p + 2xp' + 2pp'$$

$$-p = 2p'(x+p) \Rightarrow -\frac{dx}{dp} = \frac{2(x+p)}{p}$$

$$\frac{dx}{dp} + 2\frac{x}{p} = -2$$

$$\lambda(p) = e^{\int 2\frac{dp}{p}} = e^{2\ln p} = p^2$$

$$x = \frac{1}{p^2}\left[\int p^2(-2)\,dp + c\right] = \frac{1}{p^2}\left(-2\frac{p^3}{3} + c\right) = -\frac{2}{3}p + \frac{c}{p^2}$$

$$y = -\frac{4}{3}p^2 + \frac{2c}{p} + p^2 = -\frac{p^2}{3} + \frac{2c}{p}$$

ex $y = 2xy' + \sqrt{1-(y')^2}$

$$y' = p \Rightarrow y = 2xp + \sqrt{1-p^2}$$

$$\underset{p}{y'} = 2p + 2xp' - \frac{\cancel{2}pp'}{\cancel{2}\sqrt{1-p^2}}$$

$$-p = 2p'\left(2x - \frac{p}{\sqrt{1-p^2}}\right) \rightarrow -\frac{dx}{dp} = \frac{2\left(2x - \frac{p}{\sqrt{1-p^2}}\right)}{p}$$

$$\frac{dx}{dp} + \frac{2x}{p} = \frac{1}{\sqrt{1-p^2}} \qquad \text{L.D.E}$$

$$\lambda(p) = e^{\int \frac{2dp}{p}} = e^{2\ln p} = p^2$$

$$x = \frac{1}{p^2}\left[\int p^2 \frac{dp}{\sqrt{1-p^2}} + c\right]$$

$$x = -\frac{1}{2p}\arcsin\left(\sqrt{1-p^2}\right) - \frac{1}{2}\sqrt{1-p^2} + \frac{c}{p}$$

$$y = -\frac{1}{2}\arcsin\left(\sqrt{1-p^2}\right) + \left(\frac{2-p}{2}\right)\sqrt{1-p^2} + c$$

ex $\quad y + xy' = (y')^4$

$$y' = p \Rightarrow y + xp = p^4$$

$$\underbrace{y'}_{p} + p + xp' = 4p^3p'$$

$$2p = 4p^3p' - xp' \Rightarrow 2p = p'(4p^3 - x)$$

$$\frac{dx}{dp} = \frac{4p^3 - x}{2p} \Rightarrow \frac{dx}{dp} + \frac{x}{2p} = 2p^2 \quad \text{L.D.E}$$

$$\lambda(p) = e^{\int \frac{dp}{2p}} = e^{\frac{1}{2}\ln p} = \sqrt{p}$$

$$x = \frac{1}{\sqrt{p}}\left[\int \sqrt{p} \cdot 2p^2\,dp + c\right] = \frac{1}{\sqrt{p}}\left[2p^{7/2} \cdot \frac{2}{7} + c\right]$$

$$\left.\begin{aligned} x &= \frac{4}{7}p^3 + \frac{c}{\sqrt{p}} \\ y &= p^4 - \frac{4}{7}p^4 - c\sqrt{p} = \frac{3}{7}p^4 - c\sqrt{p} \end{aligned}\right\} \text{parametric equations of general solution}$$

Homework

1. $y = x + (y')^2 - \frac{2}{3}(y')^3$

2. $y = x(2 + y') + (y')^2$