Differential Equations

Definition: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables is said to be a "differential equation". Generally we consider the one dependent and one independent form

$$F(x, y, y', y'', \dots, y^{n}) = 0 \quad \text{or}$$

$$F\left(x, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0 \qquad n \in \mathbb{Z}^+$$

Classification of Differential Equations

1. By Type

If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable, then it's said to be an "ordinary D.E". Otherwise, it's called "partial D.E".

Examples

$$\left.\begin{array}{ll} \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = te^{t} & \Rightarrow \begin{array}{l} y \to \text{dependent} \\ t \to \text{independent} \end{array} \\[2ex] y^{IV} - 2y' = x + 4 & \to \begin{array}{l} y \to \text{dependent} \\ x \to \text{indepent} \end{array} \\[2ex] \dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + y & \Rightarrow \begin{array}{l} x, y \to \text{dependent} \\ t \to \text{independent} \end{array} \end{array}\right\} \text{Ordinary D.E}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = xy + 1 \rightarrow \begin{array}{l} u \rightarrow \text{dependent} \\ x, y \rightarrow \text{independent} \end{array}$$

$$\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} - 2u = 0 \rightarrow \begin{array}{l} u \rightarrow \text{dependent} \\ x, y \rightarrow \text{independent} \end{array}$$

Partial D.E

2. By Order

The order of a D.E is the order of the highest derivative in the equation.

$y^{IV} - y'' = x$ forth order ODE

$y'' - (y')^2 = y$ second order ODE

$\dfrac{\partial^3 u}{\partial x^2 \partial y} = u$ third order PDE

First order ODEs are mostly written in the forms

$y' = f(x, y)$ (explicit form)

$F(x, y, y') = 0$ (implicit form)

$m(x, y)dx + N(x, y)dy = 0$ (differential form) $\left(y' = \dfrac{dy}{dx}\right)$

3. By Linearity

The degree of a D.E is the degree of the highest derivative appearing in the equation.

$\dfrac{d^2 y}{dx^2} + 4y = 0 \rightarrow$ second order, first degree

$(y''')^2 + (y')^3 = 4y \Rightarrow$ third order, second degree

If the degree of each dependent term is 1, then D.E is linear. Otherwise, it's called nonlinear D.E.

$y'' - 2y' + y = x^2$ → second order linear ODE

$(y')^3 + y = 1$ → first order nonlinear ODE

4. By Coefficients

A linear ODE of order n is in the form

$$a_0(x)\, y^{n} + a_1(x)\, y^{n-1} + \dots + a_n y = f(x) \quad [a_0(x) \neq 0]$$

★ If all $a_i(x)$ $(i = 1, \dots, n)$ are constants, then it's called "ODE with constant coefficients."

★ If at least one coefficient is variable, then it's called "ODE with variable coefficients."

$xy'' + 4y = 0$ D.E with variable coefficients.

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$ D.E with constant coefficients.

Examples

★ $\frac{dy}{dx} + x^2 y = xe^x$ first order linear ODE with variable coefficients

★ $(y'')^3 + 4y = \sin x$ second order nonlinear ODE with constant coefficients.

★ $yy''' + 6y' = 0$ third order nonlinear ODE with constant coefficients.

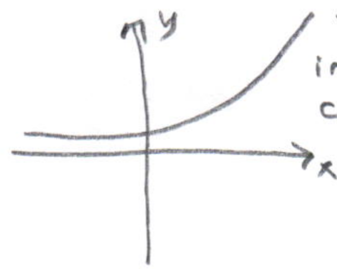
★ $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} = z$ second order linear PDE with constant coefficients.

Solution of an ODE

A solution to an ODE on an interval I is any function $y=f(x)$ which is defined on I and satisfies the ODE. The graph of $y=f(x)$ is called as integral (solution) curves.

For example, $y=e^x$ is the solution of $y''-2y'+y=0$

$$\left.\begin{array}{l} y=e^x \\ y'=e^x \\ y''=e^x \end{array}\right\} e^x-2e^x+e^x=0 \ \checkmark$$



$y=e^x$
integral
curve

Explicit and Implicit Solutions

A relation $G(x,y)=0$ is called implicit solution.

A " $y=f(x)$ " " explicit "

★ $x^2y^2+xy=4$ implicit solution

★ $y=x^2+\sin x+4$ explicit solution

Mainly, we have 3 kinds of solutions:

1. General Solution

Consider the nth order ODE

$F(x,y,y',\dots,y^n)=0$

and the following family of function with n-parameter

$F(x,y,c_1,c_2,\dots,c_n)=0$

where $c_1,c_2,\dots,c_n$ are arbitrary constants.

If each function in this family is a solution of the D.E, then this family is called the "general solution" of ODE. General solutions are obtained from integrating ODEs and contains n arbitrary constants resulting from integrating n times.

2. Particular Solution

"Particular solutions" are the solutions which obtained by assigning specific values to the arbitrary constants in the general solution.

3. Singular Solution

Solutions that can not be expressed by the general solutions are called "singular solutions".

Ex $y'' = 6x \Rightarrow y' = 3x^2 + c_1 \rightarrow \underbrace{y = x^3 + c_1 x + c_2}_{\text{general solution}}$

For $y(0) = 1$, $y'(0) = 1$ specific values

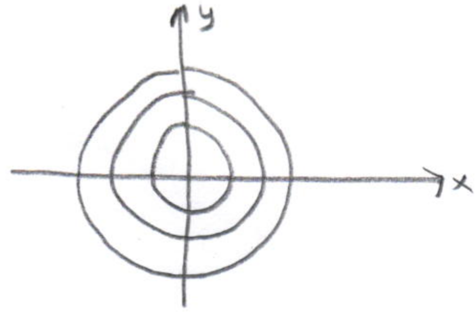
$y(0) = c_2 = 1$, $y'(0) = c_1 = 1 \Rightarrow \underbrace{y = x^3 + x + 1}_{\text{particular solution}}$

Family of curves

A set of curves whose equations are of the same form but which have different values assigned to one or more parameters in the equations.

For example,

$$x^2+y^2=c^2$$



To find the family of curves of the DE, we have to differentiate the curves n times. Then we must find a relation among this derivatives and family of curves.

ex

Find the D.E of the family of curves

$$\left.\begin{aligned} y &= c_1 e^{-x} + c_2 e^{x} \\ y' &= -c_1 e^{-x} + c_2 e^{x} \\ y'' &= c_1 e^{-x} + c_2 e^{x} \end{aligned}\right\} \quad y'' = y \Rightarrow y'' - y = 0$$

ex

$$\left.\begin{aligned} y &= (x+c)e^{-x} \\ y' &= e^{-x} - (x+c)e^{-x} \end{aligned}\right\} \quad y' = e^{-x} - y \Rightarrow y' + y = e^{-x}$$

ex

$$x = \tan(y+c)$$

I. way

$$1 = y'\left[1+\tan^2(y+c)\right]$$

$$1 = y'\,(1+x^2)$$

II. way

$$y+c = \arctan x$$

$$y' = \frac{1}{1+x^2}$$

ex

$$\left.\begin{aligned} y &= \frac{c_1}{x} + c_2 \\ y' &= -\frac{c_1}{x^2} \\ y'' &= \frac{2c_1}{x^3} \end{aligned}\right\} \quad y'' = -\frac{2}{x}\,y'$$

ex $\ln y = c_1 x^2 + c_2$

$\frac{y'}{y} = 2c_1 x \Rightarrow y' = 2c_1 xy$

$y'' = 2c_1 y + 2c_1 x y' = 2c_1 [y + xy']$

$\hookrightarrow c_1 = \frac{y'}{2xy}$ $\Big\}$ $y'' = \frac{y'}{xy}[y + xy']$

$xyy'' - x(y')^2 = yy'$

Initial and Boundary Value Problems

A D.E along with the same conditions on the unknown function and its derivatives, all given at the same value of the independent variable constitutes an IVP.

$y'' + 2y' = e^x \quad y(\pi) = 1, \; y'(\pi) = 2 \quad \Big\}$ I.V.P

$y'' + 2y' = e^x \quad y(0) = 1, \; y'(1) = 1$

$\qquad\qquad\qquad y(0) = 1, \; y(1) = 0 \quad \Big\}$ B.V.Ps

First Order Differential Equations

$$F(x,y,y')=0 \quad \text{or} \quad y'=f(x,y)$$

is the standart form of a first order D.E. But sometimes first order D.E can be written as differential form

$$m(x,y)dx+N(x,y)dy=0$$

ex

$y'=\dfrac{x^2+y^2}{x-y}$ standard form $\Rightarrow \dfrac{dy}{dx}=\dfrac{x^2+y^2}{x-y}$

$\Rightarrow (x^2+y^2)dx-(x-y)dy=0$

1. Seperable D.E

Let us consider the differential form of the first order D.E

$$m(x,y)dx+N(x,y)dy=0$$

If $m(x,y)$ and $N(x,y)$ can be written as

$m(x,y)=m_1(x)m_2(y)$ and $N(x,y)=N_1(x)N_2(y)$

then, D.E is called "seperable D.E"

$$m_1(x)m_2(y)dx+N_1(x)N_2(y)dy=0$$

$$\frac{m_1(x)}{N_1(x)}dx+\frac{N_2(y)}{m_2(y)}dy=0$$

If we integrate each terms,

$$F(x)+G(y)=c$$

ex $x(1-y^2)dx+y(1-x^2)dy=0$

$$\frac{x}{1-x^2}dx+\frac{y}{1-y^2}dy=0$$

$$\left(\int\frac{x\,dx}{1-x^2}=\left(-\frac{1}{2}\right)\int\frac{(-2)x\,dx}{1-x^2}=-\frac{1}{2}\ln|1-x^2|\right)$$

$$-\frac{1}{2}\ln|1-x^2|-\frac{1}{2}\ln|1-y^2|=-\frac{1}{2}\ln c$$

$$\rightarrow \ln|1-x^2|+\ln|1-y^2|=\ln c \rightarrow |1-x^2||1-y^2|=c$$

ex

$$y'+e^{x+y}=0$$

$$\frac{dy}{dx}=-e^x e^y \rightarrow \frac{dy}{e^y}=-e^x dx \rightarrow \int e^{-y}dy=-\int e^x dx$$

$$-e^{-y}=-e^x+c$$

ex

$$y=\ln y'$$

$$y'=e^y \rightarrow \frac{dy}{dx}=e^y \rightarrow \int e^{-y}dy=\int dx$$

$$-e^{-y}=x+c \rightarrow e^{-y}=-(x+c)$$

$$e^y=-\frac{1}{x+c}$$

ex $x^2 y\,dy=\cos y^2 dx$

$$\frac{y}{\cos y^2}dy=\frac{dx}{x^2} \rightarrow \underbrace{\int y\sec y^2 dy}=\int\frac{dx}{x^2}$$

$$\left.\begin{array}{l} y^2=t \\ 2y\,dy=dt \end{array}\right\} \int\frac{\sec t\,dt}{2}=\frac{1}{2}\ln|\sec t+\tan t|$$

$$=\frac{1}{2}\ln|\sec y^2+\tan y^2|$$

$$\frac{1}{2}\ln|\sec y^2+\tan y^2|=-\frac{1}{x}+c$$

ex

$$yy' + \sqrt{x - xy^4} = 0$$

$$y\,dy + \sqrt{x - xy^4}\,dx = 0 \Rightarrow y\,dy + \sqrt{x}\sqrt{1-y^4}\,dx = 0$$

$$\frac{y}{\sqrt{1-y^4}}\,dy + \sqrt{x}\,dx = 0$$

$$\left.\begin{aligned} y^2 &= t \\ 2y\,dy &= dt \end{aligned}\right\} \int \frac{dt}{2\sqrt{1-t^2}} = \frac{1}{2}\arcsin t = \frac{1}{2}\arcsin y^2$$

$$\frac{1}{2}\arcsin y^2 + \frac{2}{3}x\sqrt{x} = c$$

ex

Solve the I.V.P $e^x(y-1)dx + 2(e^x+4)\,dy = 0$, $y(0) = 2$.

$$\int \frac{e^x}{e^x+4}\,dx + \int \frac{2dy}{y-1} = \int 0$$

$$\ln(e^x+4) + 2\ln|y-1| = \ln c \rightarrow (e^x+4)(y-1)^2 = c$$

$$y(0) = 2 \Rightarrow 5(2-1)^2 = c \rightarrow \boxed{c=5} \rightarrow (e^x+4)(y-1)^2 = 5$$

Homework

1 $y' = y^2 - 4$

$y' = (1+x)(1+y^2)$

$y' = x^3 e^{+y}$ $\quad y(2) = 0$ IVP

2. D.Es can be transformed into seperable D.Es.

Consider the equation

$\frac{dy}{dx} = f(ax+by+c)$ where a,b,c are constants.

This equation can be transformed into a seperable D.E by making the transformation

$$\left.\begin{aligned} ax+by+c &= u \\ a+by' &= u' \end{aligned}\right\} \quad \frac{u'-a}{b} = y' = f(u) \Rightarrow u' = a+bf(u)$$

$$\int \frac{du}{a+bf(u)} = \int dx \rightarrow \phi(u) = x+c \rightarrow \phi(ax+by+c) = x+c$$

ex $\frac{dy}{dx} = (x+y+1)^2$

$$\left.\begin{aligned} x+y+1 &= u \\ 1+y' &= u' \end{aligned}\right\} \quad u'-1 = u^2 \rightarrow u' = u^2+1$$

$$\int \frac{du}{u^2+1} = \int dx \rightarrow \arctan u = x+c$$

$$\arctan(x+y+1) = x+c$$

ex

$\tan^2(x+y)\,dx - dy = 0$

$$\left.\begin{aligned} x+y &= u \\ 1+y' &= u' \end{aligned}\right\} \quad \tan^2 u - u' + 1 = 0 \Rightarrow u' = 1+\tan^2 u$$

$$\int \frac{du}{1+\tan^2 u} = \int dx \rightarrow \int \cos^2 u\, du = \int dx$$

$$\left(\cos 2u = \begin{aligned} 2\cos^2 u - 1 \\ 1 - 2\sin^2 u \end{aligned}\right)$$

$$\int \cos^2 u\, du = \int \left(\frac{1+\cos 2u}{2}\right) du = \frac{u}{2} + \frac{1}{4}\sin 2u$$

$$\frac{u}{2} + \frac{1}{4}\sin 2u = x+c \rightarrow \frac{(x+y)}{2} + \frac{1}{4}\sin 2(x+y) = x+c$$

ex $y' = 2\sqrt{2x+y+1}$

$$\left.\begin{aligned} 2x+y+1 &= u^2 \\ 2+y' &= 2uu' \end{aligned}\right\} \quad \begin{aligned} &2uu' - 2 = 2u \\ &u' = \frac{u+1}{u} \Rightarrow \int \frac{u}{u+1}\,du = \int dx \end{aligned}$$

$$\int\left(1-\frac{1}{u+1}\right)du = \int dx \Rightarrow u - \ln|u+1| = x+c$$

$$(2x+y+1) - \ln|2x+y+2| = x+c$$

ex

$$y' = (y-4x)^2$$

$$\left.\begin{aligned} y-4x &= u \\ y'-4 &= u' \end{aligned}\right\} \quad \begin{aligned} &u'+4 = u^2 \Rightarrow u' = u^2-4 \\ &\int \frac{du}{u^2-4} = \int dx \end{aligned}$$

$$\frac{A}{u-2} + \frac{B}{u+2} = \frac{1}{u^2-4} \Rightarrow (A+B)u + 2A - 2B = 1$$

$$\left.\begin{aligned} A+B &= 0 \\ 2A-2B &= 1 \end{aligned}\right\} 4A = 1 \Rightarrow A = \frac{1}{4},\ B = -\frac{1}{4}$$

$$\frac{1}{4}\left[\int \frac{du}{u-2} - \int \frac{du}{u+2}\right] = x+c$$

$$\frac{1}{4}\left[\ln|u-2| - \ln|u+2|\right] = x+c$$

$$\ln\left|\frac{u-2}{u+2}\right| = 4(x+c) \Rightarrow \ln\left|\frac{y-4x-2}{y-4x+2}\right| = 4(x+c)$$