≠ SYSTEMS OF FIRST ORDER LINEAR DES ≠

In the previous lectures we have been concerned with one DE in one unknown function. Now we shall consider systems of two DES in two unknown functions.

Definition: The order a DE system is equal to the sum of the orders of the highest derivative of all unknown functions.

Definition: The system which has a term of indep. variable is called nonhomogeneous.

Example: $\frac{dx}{dt} + \frac{d^2y}{dt^2} + x + 2y = t$

$\frac{d^2x}{dt^2} + \frac{dy}{dt} + 2x + y = t^2$

x, y : unknown funct.

$t \to$ ind. variable

The order of this system is : $2+2=4$

This system is non-hom b/c of the terms t, t^2.

Example: $\frac{dy}{dx} = -5y + z$

$\frac{dz}{dx} = -y - 3z$

y, z : unknown functions

x : ind. variable.

The order of the system: $1+1=2$

a homogeneous system.

Definition: If the highest derivative of each equation in a system, could be leave alone in one side of the equation, then the system is called a standart (canonical) form.

Definition: If a standart system is of the form;

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$$\frac{dx}{dt} = a_{11}(t)x + a_{12}(t)y + F_1(t)$$
$$\frac{dy}{dt} = a_{21}(t)x + a_{22}(t)y + F_2(t)$$

then it is called normal form in the case of two linear DEs in two unknown functions. The characteristic feature of such a system is having the first derivatives only.

THE METHOD OF DERIVATION AND ELIMINATION

Consider the normal system;

$$\left.\begin{aligned} \frac{dx}{dt} &= a_{11}x + a_{12}y + F_1(t) \\ \frac{dy}{dt} &= a_{21}x + a_{22}y + F_2(t) \end{aligned}\right\}$$ 2nd order non-hom. linear constant coefficient normal system.

x, y : unknown functions ; t : independent variable.

The main purpose of this method is eliminating one of the first derivatives by taking derivatives. After eliminating, a DE in one unknown function is obtained.

Example: $\frac{dy}{dx} = -5y - z + 1 + x^2$ --- (1)

$\frac{dz}{dx} = y - 3z + e^{2x}$ --- (2)

y, z : unknown functions ; x : ind. variable

The derivative of (1): $\frac{d^2y}{dx^2} = -5\frac{dy}{dx} - \frac{dz}{dx} + 2x$ --- (3)

The " " (2): $\frac{d^2z}{dx^2} = \frac{dy}{dx} - 3\frac{dz}{dx} + 2e^{2x}$ --- (4)

Substituting (2) in (3), we have;

$$\frac{d^2y}{dx^2} = -5\frac{dy}{dx} - y + 3z - e^{2x} + 2x \quad \text{or}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y - 3z = 2x - e^{2x}$$

If the resulting equation does not contain the term $3z$, we can solve it. Thus, if we leave alone the unknown function z from (1), and then substitute in the resulting DE, we have;

$$(1) \rightarrow \quad z = -\frac{dy}{dx} - 5y + 1 + x^2 \; ...(5)$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y - 3\left(-\frac{dy}{dx} - 5y + 1 + x^2\right) = 2x - e^{2x}$$

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y + 3\frac{dy}{dx} + 15y - 3 - 3x^2 = 2x - e^{2x}$$

$$\star \quad \frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = \underbrace{3x^2 + 2x + 3}_{f_1(x)} \underbrace{- e^{2x}}_{f_2(x)}$$

A liear DE with const. coeff.

CE: $r^2 + 8r + 16 = 0$
$(r+4)^2 = 0$
$r_1 = r_2 = -4 \Rightarrow y_h = c_1 e^{-4x} + c_2 x e^{-4x}$

1) For $f_1(x) = 3x^2 + 2x + 3$
$y_{p_1} = ax^2 + bx + c$
$y_{p_1}' = 2ax + b$
$y_{p_1}'' = 2a$

$2a + 16ax + 8b + 16ax^2 + 16bx + 16c = 3x^2 + 2x + 3$

$16a = 3$, $a = 3/16$, $16a + 16b = 2$, $16b = -1$, $b = -1/16$

$2a + 8b + 16c = 3$

$$\frac{6}{16} - \frac{8}{16} + 16c = 3 \Rightarrow c = 25/128$$

2) for $f_2(x) = -e^{2x}$

$$y_{p_2} = Ae^{2x}$$
$$y_{p_2}' = 2Ae^{2x}$$
$$y_{p_2}'' = 4Ae^{2x}$$

$$(4A + 16A + 16A)e^{2x} = -e^{2x}$$
$$36A = -1$$
$$A = -1/36$$

$$y_{p_1} = \frac{3}{16}x^2 - \frac{1}{16}x + \frac{25}{8} \quad , \quad y_{p_2} = -\frac{1}{36}e^{2x}$$

$$y_p = y_{p_1} + y_{p_2} = \frac{3x^2}{16} - \frac{x}{16} + \frac{25}{8} - \frac{1}{36}e^{2x}$$

$$y = y_h + y_p = c_1e^{-4x} + c_2xe^{-4x} + \frac{3x^2}{16} - \frac{x}{16} + \frac{25}{8} - \frac{1}{36}e^{2x}$$

If we substitute y in (5), we obtain z;

$$z = -\frac{dy}{dx} - 5y + 1 + x^2 \text{ , so}$$

$$z = -\left(-4c_1e^{-4x} + c_2e^{-4x} - 4c_2xe^{-4x} + \frac{6}{16}x - \frac{1}{16} - \frac{2}{36}e^{2x}\right)$$
$$-5\left(c_1e^{-4x} + c_2xe^{-4x} + \frac{3x^2}{16} - \frac{x}{16} + \frac{25}{8} - \frac{1}{36}e^{2x}\right) + 1 + x^2$$

$$z = e^{-4x}(-c_1 - c_2 - c_2x) - \frac{1}{16}(15x^2 - x - 249) + \frac{1}{12}e^{2x} + 1 + x^2$$

Example:
$$\left.\begin{aligned} \frac{dx}{dt} &= -y + t \text{ ---(1)} \\ \frac{dy}{dt} &= x - t \text{ ----(2)} \end{aligned}\right\} \text{2nd order non-hom. linear system.}$$

The derivation of (1): $\frac{d^2x}{dt^2} = -\frac{dy}{dt} + 1$

$$\frac{d^2x}{dt^2} = -x + t + 1$$

$$\frac{d^2x}{dt^2} + x = t + 1$$
$$r^2 + 1 = 0$$
$$r = \mp i$$

$$x_h = c_1\cos t + c_2\sin t$$
$$x_p = at + b$$
$$x_p' = a$$
$$x_p'' = 0$$

$$0 + at + b = t + 1$$
$$a = 1 \text{ , } b = 1$$
$$x_p = t + 1$$

$$\boxed{x = c_1\cos t + c_2\sin t + t + 1}$$

$$\frac{dx}{dt} = -y + t \Rightarrow y = t - \frac{dx}{dt}$$

$$y = t - (-c_1 \sin t + c_2 \cancel{\sin t}\,\cos t + 1)$$

$$y = t + c_1 \sin t - c_2 \cancel{\sin t}\,\cos t - 1$$

Example: $\left.\begin{aligned} x' &= x + y + e^t \quad \ldots (1) \\ y' &= 9x + y + \sin t \quad \ldots (2) \end{aligned}\right\}$ Find the solution.

2nd order linear non hom. eq. system.

Derivation of (1): $x'' = x' + y' + e^t$

$$x'' = x' + 9x + y + \sin t + e^t$$
$$x'' = x' + 9x + (x' - x - e^t) + \sin t + e^t$$
$$x'' = x' + 9x + x' - x - \cancel{e^t} + \sin t + \cancel{e^t}$$
$$x'' - 2x' - 8x = \sin t$$ a 2nd order linear eq.

$$\left.\begin{aligned} & r^2 - 2r - 8 = 0 \\ & (r-4)(r+2) = 0 \\ & r_1 = 4, \; r_2 = -2 \end{aligned}\right\} x_h = c_1 e^{4t} + c_2 e^{-2t}$$

$$\left.\begin{aligned} x_p &= A\cos t + B\sin t \\ x_p' &= -A\sin t + B\cos t \\ x_p'' &= -A\cos t - B\sin t \end{aligned}\right\}$$

$$-A\cos t - B\sin t + 2A\sin t - 2B\cos t - 8A\cos t - 8B\sin t = \sin t$$

$$(\underbrace{-A - 2B - 8A}_{0})\cos t + (\underbrace{-B + 2A - 8B}_{1})\sin t = \sin t$$

$$2A - 9B = 1, \quad -2B - 9A = 0 \Rightarrow \begin{aligned} 2B &= -9A \\ B &= -\frac{9}{2}A \end{aligned}$$

$$2A + \frac{81}{2}A = 1$$

$$\frac{85}{2}A = 1$$

$$A = \frac{2}{85}, \quad B = -\frac{9}{85}$$

$$x_p = \frac{2}{85}\cos t - \frac{9}{85}\sin t$$

$$x = x_h + x_p = c_1 e^{4t} + c_2 e^{-2t} + \frac{1}{85}(2\cos t - 9\sin t)$$

$$y = x' - x - e^t$$

$$y = 4c_1 e^{4t} - 2c_2 e^{-2t} + \frac{1}{85}(-2\sin t - 9\cos t) - x - e^t$$

THE METHOD OF DETERMINANT

In this section we shall present a symbolic operator method for solving linear systems with constant coefficients.

Let x be an n-times differentiable function of the independent variable t. We denote the operation of differentiation with respect to t by the symbol D and call D a differential operator. In terms of this differential operator the derivative $\frac{dx}{dt}$ is denoted by Dx. That is

$$Dx = \frac{dx}{dt}.$$

In like manner, we denote the second derivative of x with respect to t by D^2x. That is;

$$D^n x = \frac{d^n x}{dt^n} \qquad (n = 1, 2, \ldots).$$

Further extending this operator notation, we write

$(D+c)x$ to denote $\frac{dx}{dt} + cx$ and

$$(aD^n + bD^m x) = a\frac{d^n x}{dt^n} + b\frac{d^m x}{dt^m}$$

where a, b and c are constants.

In this notation, the general linear differential expression with constant coefficients $a_0, a_1, \dots, a_n$,

$$a_0 \frac{d^n x}{dt^n} + a_1 \frac{d^{n-1}x}{dt^{n-1}} + \dots + a_{n-1}\frac{dx}{dt} + a_n x .$$

is written

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)x .$$

Example: $(3D^2 + 5D - 2)x$ denotes $3\frac{d^2x}{dt^2} + 5\frac{dx}{dt} - 2x$.

If $x = t^3$, we have

$$(3D^2 + 5D - 2)t^3 = 18t + 15t^2 - 2t^3$$

Example: $\left.\begin{aligned} \frac{dx}{dt} &= 3x - y - 1 \\ \frac{dy}{dt} &= x + y + 4e^t \end{aligned}\right\}$ Find the solution?

$$\left.\begin{aligned} (D-3)x + y &= -1 \\ -x + (D-1)y &= 4e^t \end{aligned}\right\} \quad \Delta = \begin{vmatrix} D-3 & 1 \\ -1 & D-1 \end{vmatrix} = (D-3)(D-1) + 1 = D^2 - 4D + 4 = (D-2)^2$$

$$\Delta = (D-2)^2$$

$$x = \frac{\Delta_1}{\Delta} \Rightarrow \underline{\Delta_1 = x . \Delta}$$

$$(D-2)^2 x = \begin{vmatrix} -1 & 1 \\ 4e^t & D-1 \end{vmatrix} = (D-1)(-1) - 4e^t = 1 - 4e^t$$

$$(D-2)^2 x = 1 - 4e^t \qquad \left(\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 1 - 4e^t \right)$$

$$(r-2)^2 = 0$$

$$r_1 = r_2 = 2 \qquad \Rightarrow x_h = c_1 e^{2t} + c_2 t e^{2t}$$

$F(t) = \underset{f_1(t)}{1} - \underset{f_2(t)}{4e^t}$

$\rightarrow$ for $f_1(t) = 1$

$x_{p_1} = A$

$x_{p_1}' = x_{p_1}'' = 0$

$4A = 1$

$A = 1/4$

$x_{p_1} = \frac{1}{4}$

$\rightarrow$ for $f_2(t) = -4e^t$

$x_{p_2} = Ae^t$

$x_{p_2}' = x_{p_2}'' = Ae^t$

$(A - \cancel{4A} + \cancel{4A})e^t = -4e^t$

$A = -4$

$x_{p_2} = -4e^t$

$x = x_h + x_{p_1} + y_{p_2}$

$x = c_1 e^{2t} + c_2 te^{2t} + \frac{1}{4} - 4e^t$

Ⓐ $(D-2)^2 y = \begin{vmatrix} D-3 & -1 \\ -1 & 4e^t \end{vmatrix} = (D-3)4e^t - 1$

$= 4e^t - 12e^t - 1$

$= -8e^t - 1$

$(D-2)^2 y = -8e^t - 1$

$(r-2)^2 = 0$

$r_1 = r_2 = 2$

$y_h = c_3 e^{2t} + c_4 te^{2t}$

$\rightarrow$ $y_{p_1} = A$

$y_{p_1}' = y_{p_1}'' = 0$

$4A = -1$

$A = -1/4$

$\rightarrow$ $y_{p_2} = Be^t$

$y_{p_2}' = y_{p_2}'' = Be^t$

$(B - 4B + 4B)e^t = -8e^t$

$B = -8$

$y = y_h + y_{p_1} + y_{p_2} = c_3 e^{2t} + c_4 te^{2t} - \frac{1}{4} - 8e^t$

We have 4 arbitrary constanst in the g s but we must have only two!

$$x' = 3x - y - 1$$

$$\underline{2c_1e^{2t}} + \underline{c_2e^{2t}} + 2c_2te^{2t} \cancel{-4e^t} = \underline{3c_1e^{2t}} + 3c_2te^{2t} \cancel{+\frac{3}{4}}$$
$$\cancel{-12e^t} - \underline{c_3e^{2t}} - c_4te^{2t} + \cancel{\frac{1}{4}} \cancel{+8e^t} \cancel{-1}$$

$$(2c_1 + c_2 - 3c_1 + c_3)e^{2t} + (2c_2t - 3c_2t + c_4t)e^{2t} = 0$$

$$-c_1 + c_2 + c_3 = 0 \Rightarrow c_1 = c_2 + c_3$$
$$-c_2t + c_4t = 0$$
$$\underline{c_2 = c_4} \qquad \underline{c_1 = c_4 + c_3}$$

$$\left\{ \begin{aligned} x(t) &= (c_4+c_3)e^{2t} + c_4te^{2t} + \frac{1}{4} - 4e^t \\ y(t) &= c_3e^{2t} + c_4te^{2t} - \frac{1}{4} - 8e^t \end{aligned} \right\}$$

Example: $\left.\begin{aligned}(D+1)x + 2Dy &= 6\cos 2t \\ Dx + (D-1)y &= t\end{aligned}\right\}$ Find the G.S.

$$\Delta = \begin{vmatrix} D+1 & 2D \\ D & D-1 \end{vmatrix} = (D+1)(D-1) - 2D^2$$
$$= D^2 - 1 - 2D^2$$
$$= -(D^2+1)$$

$$-(D^2+1)x = \begin{vmatrix} 6\cos 2t & 2D \\ t & D-1 \end{vmatrix} = (D-1)(6\cos 2t) - 2D(t)$$

$$-(D^2+1)x = -12\sin 2t - 6\cos 2t - 2$$
$$(D^2+1)x = 12\sin 2t + 6\cos 2t + 2$$
$$\frac{d^2x}{dt^2} + x = 12\sin 2t + 6\cos 2t + 2$$

$$r^2 + 1 = 0 \Rightarrow x_h = c_1\cos t + c_2\sin t$$
$$r = \mp i$$

$$\left.\begin{aligned} x_{p_1} &= A \\ x_{p_1}' &= x_{p_1}'' = 0 \end{aligned}\right\} \quad \left.\begin{aligned} 0 + A &= 2 \\ A &= 2 \end{aligned}\right\} \quad x_{p_1} = A = 2$$

$x_{p_2} = A\cos 2t + B\sin 2t$

$x_{p_2}' = -2A\sin 2t + 2B\cos 2t$

$x_{p_2}'' = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + A\cos 2t + B\sin 2t = 6\cos 2t + 12\sin 2t$

$-3A\cos 2t - 3B\sin 2t = 6\cos 2t + 12\sin 2t$

$-3A = 6 \Rightarrow A = -2 \quad , \quad -3B = 12 \Rightarrow B = -4$

$x_{p_2} = -2\cos 2t - 4\sin 2t$

$\underline{x = c_1\cos t + c_2\sin t + 2 - 2\cos 2t - 4\sin 2t}$

$$-(D^2+1)y = \begin{vmatrix} D+1 & 6\cos 2t \\ D & t \end{vmatrix} = (D+1)t - D(6\cos 2t)$$

$$= 1 + t + 12\sin 2t$$

$(D^2+1)y = -t - 1 - 12\sin 2t$

$y_h = c_3\cos t + c_4\sin t$

$$\left.\begin{aligned} y_{p_1} &= at+b \\ y_{p_1}' &= a \\ y_{p_1}'' &= 0 \end{aligned}\right\} \begin{aligned} 0 + at + b &= -t-1 \\ a = -1 \ &, \ b = -1 \end{aligned} \Bigg\} \ y_{p_1} = -t-1$$

$y_{p_2} = A\cos 2t + B\sin 2t$

$y_{p_2}' = -2A\sin 2t + 2B\cos 2t$

$y_{p_2}'' = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + A\cos 2t + B\sin 2t = -12\sin 2t$

$-3A = 0 \quad , \quad -4B + B = -12 \Rightarrow B = 4$

$A = 0$

$y_{p_2} = 4\sin 2t$

$\underline{y = c_3\cos t + c_4\sin t - t - 1 + 4\sin 2t}$

$x' + y' - y = t$

$-c_1\sin t + c_2\cos t + 4\sin 2t - 8\cos 2t - c_3\sin t + c_4\cos t - 1$

$+ 8\cos 2t - c_3\cos t - c_4\sin t + t + 1 - 4\sin 2t = t$

$\Rightarrow \underline{c_1 = -c_3 - c_4} \quad , \quad \underline{c_2 = c_3 - c_4}$

Example: $\left.\begin{array}{l} Dx = -y+t \\ Dy = x-t \end{array}\right\}$ Find the solution.

$$\left.\begin{array}{l} Dx + y = t \\ -x + Dy = -t \end{array}\right\} \qquad \Delta = \begin{vmatrix} D & 1 \\ -1 & D \end{vmatrix} = D^2+1$$

$$x = \frac{\Delta_1}{\Delta} \Rightarrow x = \frac{\begin{vmatrix} t & 1 \\ -t & D \end{vmatrix}}{D^2+1} \Rightarrow (D^2+1)x = Dt+t$$

$$(D^2+1)x = 1+t$$

$$\frac{d^2x}{dt^2} + x = 1+t$$

$$\left.\begin{array}{l} r^2+1=0 \\ r = \mp i \end{array}\right\} \quad x_h = c_1\cos t + c_2 \sin t$$

$$x_p = at+b, \quad x_p' = a, \quad x_p'' = 0$$

$$0 + at + b = 1+t \Rightarrow a=1, \ b=1$$

$$x_p = t+1$$

$$x = c_1\cos t + c_2 \sin t + t + 1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} D & t \\ -1 & -t \end{vmatrix}}{D^2+1} \Rightarrow (D^2+1)y = D(-t)+t$$

$$(D^2+1)y = -1+t$$

$$\left.\begin{array}{l} y_h = c_3\cos t + c_4 \sin t \\ y_p = at+b \\ y_p' = a, \quad y_p'' = 0 \end{array}\right\} \quad \left.\begin{array}{l} at+b = t-1 \\ a=1, \ b=-1 \end{array}\right\} \ y_p = t-1$$

$$y = c_3\cos t + c_4 \sin t + t - 1$$

$$\frac{dx}{dt}+y-t=0$$

$$-c_1\sin t+c_2\cos t+\cancel{1}+c_3\cos t+c_4\sin t+\cancel{t}-\cancel{1}-\cancel{t}=0$$

$$(\underbrace{-c_1+c_4}_{0})\sin t+(\underbrace{c_2+c_3}_{0})\cos t=0$$

$$c_1=c_4\quad,\quad c_2=-c_3$$

$$\left.\begin{aligned} x(t)&=c_4\cos t-c_3\sin t+t+1\\ y(t)&=c_3\cos t+c_4\sin t+t-1\end{aligned}\right\}\ \text{G.S.}$$

<u>Example:</u>
$$\left.\begin{aligned}&\frac{dx}{dt}+2x+3y=0\\ &\frac{dy}{dt}+2y+3x=2e^{2t}\end{aligned}\right\}$$

$$\left.\begin{aligned}(D+2)x+3y&=0\\ 3x+(D+2)y&=2e^{2t}\end{aligned}\right\}\quad \Delta=\begin{vmatrix} D+2 & 3\\ 3 & D+2\end{vmatrix}=(D+2)^2-9$$

$$\Delta=D^2+4D-5$$

$$x=\frac{\Delta_1}{\Delta}=\frac{\begin{vmatrix}0 & 3\\ 2e^{2t} & D+2\end{vmatrix}}{D^2+4D-5}=\frac{-6e^{2t}}{D^2+4D-5}$$

$$(D^2+4D-5)x=-6e^{2t}$$

$$\left.\begin{aligned}&\text{C.E: } r^2+4r-5=0\\ &\qquad\quad +5\quad -1\\ &r_1=-5\quad r_2=1\end{aligned}\right\}\ x_h=c_1e^{-5t}+c_2e^{t}$$

$$x_p=Ae^{2t}\ ,\ x_p'=2Ae^{2t}\ ,\ x_p''=4Ae^{2t}$$

$$(4A+8A-5A)e^{2t}=-6e^{2t}$$

$$7A=-6\qquad\Rightarrow\ x_p=-\frac{6}{7}e^{2t}$$
$$A=-6/7$$

$$\boxed{x=x_h+x_p=c_1e^{-5t}+c_2e^{t}-\frac{6}{7}e^{2t}}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{\begin{vmatrix} D+2 & 0 \\ 3 & 2e^{2t} \end{vmatrix}}{D^2+4D-5} = \frac{(D+2)(2e^{2t})-0}{D^2+4D-5}$$

$$(D^2+4D-5)y = 4e^{2t}+4e^{2t} = 8e^{2t}$$

$$y_h = c_3e^{-5t} + c_4e^{t}$$

$$y_p = Be^{2t}, \quad y_p' = 2Be^{2t}, \quad y_p'' = 4Be^{2t}$$

$$(4B+8B-5B)e^{2t} = 8e^{2t}$$

$$7B = 8$$

$$B = 8/7 \quad \Rightarrow y_p = \frac{8}{7}e^{2t}$$

$$\boxed{y = y_h + y_p = c_3e^{-5t} + c_4e^{t} + \frac{8}{7}e^{2t}}$$

$$* \quad \frac{dx}{dt} + 2x + 3y = 0$$

$$-5c_1e^{-5t} + c_2e^{t} - \cancel{\frac{12}{7}e^{2t}} + 2c_1e^{-5t} + 2c_2e^{t} - \cancel{\frac{12}{7}e^{2t}} + 3c_3e^{-5t} +$$

$$3c_4e^{t} + \cancel{\frac{24}{7}e^{2t}} = 0$$

$$\underbrace{(-5c_1+2c_1+3c_3)}_{0}e^{-5t} + \underbrace{(c_2+2c_2+3c_4)}_{0}e^{t} = 0$$

$$-3c_1 + 3c_3 = 0 \qquad\qquad 3c_2 + 3c_4 = 0$$

$$c_1 = c_3 \qquad\qquad c_2 = -c_4$$

$$\left\{\begin{aligned} x &= c_3e^{-5t} + (-c_4)e^{t} - \frac{6}{7}e^{2t} \\ y &= c_3e^{-5t} + c_4e^{t} + \frac{8}{7}e^{2t} \end{aligned}\right\}$$

Example: $\frac{dx}{dt} + \frac{dy}{dt} - 2y = -e^t$

$\frac{dy}{dt} + x + y = 0$

Solve this DE system with the derivation - elimination method.

x, y are unknown functions

t is ind. variable

$$\frac{dx}{dt} = 2y - \frac{dy}{dt} - e^t$$

$$\frac{d^2y}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$$

$$\frac{d^2y}{dt^2} + 2y - \cancel{\frac{dy}{dt}} - e^t + \cancel{\frac{dy}{dt}} = 0$$

$$\frac{d^2y}{dt^2} + 2y = e^t$$

CE: $r^2 + 2 = 0$

$r_{1,2} = \mp\sqrt{2}i$

$y_h = C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t$

$y_p = Ae^t = y_p' = y_p''$

$(A + 2A)e^t = e^t$

$A = 1/3$

$y_p = \frac{1}{3}e^t$

$$y = y_h + y_p = C_1 \cos\sqrt{2}t + C_2 \sin\sqrt{2}t + \frac{1}{3}e^t$$

$$x = -y - \frac{dy}{dt}$$

$$x = -C_1\cos\sqrt{2}t - C_2\sin\sqrt{2}t - \frac{1}{3}e^t - \left(-\sqrt{2}C_1\sin\sqrt{2}t + \sqrt{2}C_2\cos\sqrt{2}t + \frac{1}{3}e^t\right)$$

$$x = (-C_1 - \sqrt{2}C_2)\cos\sqrt{2}t + (-C_2 + \sqrt{2}C_1)\sin\sqrt{2}t - \frac{2}{3}e^t$$

Example: $\dfrac{d^2x}{dt^2} - \dfrac{dy}{dt} = e^t$, $\dfrac{dx}{dt} + \dfrac{dy}{dt} - 4x - y = 2e^t$

$$\left\{\begin{aligned} &D^2x - Dy = e^t \\ &(D-4)x + (D-1)y = 2e^t \end{aligned}\right.$$

$$\Delta = \begin{vmatrix} D^2 & -D \\ D-4 & D-1 \end{vmatrix} = D^2(D-1) + D(D-4)$$

$$\Delta = D^3 - \cancel{D^2} + \cancel{D^2} - 4D = D^3 - 4D$$

$$\Delta_1 = \begin{vmatrix} e^t & -D \\ 2e^t & D-1 \end{vmatrix} = (D-1)e^t + D(2e^t)$$

$$\Delta_1 = \cancel{e^t} - \cancel{e^t} + 2e^t \Rightarrow \Delta_1 = 2e^t$$

$$x = \frac{\Delta_1}{\Delta} \Rightarrow (D^3 - 4D)x = 2e^t$$

$$\frac{d^3x}{dt^3} - 4\frac{dx}{dt} = 2e^t$$

CE: $r^3 - 4r = 0$

$r(r^2 - 4) = 0$

$r_1 = 0 \quad r_2 = 2 \quad r_3 = -2$

$$x_h = C_1 + C_2 e^{2t} + C_3 e^{-2t}$$

$$x_p = Ae^t \Rightarrow x_p' = x_p'' = x_p''' = Ae^t$$

$(A - 4A)e^t = 2e^t$

$-3A = 2$

$A = -2/3$

$$x_p = -\frac{2}{3}e^t$$

$x = x_h + x_p$

$$x = C_1 + C_2 e^{2t} + C_3 e^{-2t} - \frac{2}{3}e^t$$

$$y = \frac{\Delta_2}{\Delta}$$

$$\Delta_2 = \begin{vmatrix} D^2 & e^t \\ D-4 & 2e^t \end{vmatrix} = D^2(2e^t) - (D-4)e^t$$

$$\Delta_2 = 2e^t - (e^t - 4e^t) = 5e^t$$

$$(D^3 - 4D)\,y = 5e^t$$

$$\frac{d^3y}{dt^3} - 4\frac{dy}{dt} = 5e^t$$

$$\left.\begin{aligned} & y_h = c_4 + c_5 e^{2t} + c_6 e^{-2t} \\ & y_p = Be^t \quad , y_p' = y_p'' = y_p''' = Be^t \\ & (B - 4B)\,e^t = 5e^t \\ & \quad -3B = 5 \\ & \quad\quad B = -5/3 \end{aligned}\right\} y_p = -\frac{5}{3}e^t$$

$$\underline{y = c_4 + c_5 e^{2t} + c_6 e^{-2t} - \frac{5}{3}e^t}$$

$*\;\; x' + y' - 4x - y = 2e^t$

$$\underline{2c_2e^{2t}} - \underline{2c_3e^{-2t}} - \cancel{\frac{2}{3}e^t} + \underline{2c_5e^{2t}} - \underline{2c_6e^{-2t}} - \cancel{\frac{5}{3}e^t} - 4c_1 +$$

$$\underline{-4c_2e^{2t}} - \underline{4c_3e^{-2t}} + \cancel{\frac{8}{3}e^t} - c_4 - \underline{c_5e^{2t}} - \underline{c_6e^{-2t}} + \cancel{\frac{5}{3}e^t} = \cancel{2e^t}$$

$$(\underbrace{-2c_2 + c_5}_{0})e^{2t} + (\underbrace{-6c_3 - 3c_6}_{0})e^{-2t} - (\underbrace{4c_1 + c_4}_{0}) = 0$$

$\underline{c_5 = 2c_2}$ $\qquad$ $\underline{c_6 = -2c_3}$ $\qquad$ $\underline{c_4 = -4c_1}$

$$\left\{\begin{aligned} x &= c_1 + c_2e^{2t} + c_3e^{-2t} - \frac{2}{3}e^t \\ y &= -4c_1 + 2c_2e^{2t} - 2c_3e^{-2t} - \frac{5}{3}e^t \end{aligned}\right\}$$

Example: $2\frac{dx}{dt} - 2\frac{dy}{dt} - 3x = t$, $2\frac{dx}{dt} + 2\frac{dy}{dt} + 3x + 8y = 2$ } Find the solution?

$$\left.\begin{aligned}(2D-3)x - 2Dy &= t\\ (2D+3)x + (2D+8)y &= 2\end{aligned}\right\}$$

$$\Delta = \begin{vmatrix} 2D-3 & -2D \\ 2D+3 & 2D+8 \end{vmatrix} = (2D-3)(2D+8) + 2D(2D+3)$$

$$\Delta = 4D^2 + 16D - \cancel{6D} - 24 + 4D^2 + \cancel{6D}$$

$$\Delta = 8D^2 + 16D - 24$$

$$\Delta_1 = \begin{vmatrix} t & -2D \\ 2 & 2D+8 \end{vmatrix} = (2D+8)t + 2D(2) = 2 + 8t$$

$$x = \frac{\Delta_1}{\Delta} \Rightarrow (8D^2+16D-24)x = 2+8t$$

$$(D^2+2D-3)x = t + \frac{1}{4}$$

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = t + \frac{1}{4}$$

CE: $r^2 + 2r - 3 = 0$, $r_1 = 1$, $r_2 = -3$ } $x_h = c_1e^{t} + c_2e^{-3t}$

$$x_p = at + b, \quad x_p' = a, \quad x_p'' = 0$$

$$0 + 2a - 3at - 3b = t + \frac{1}{4}$$

$-3a = 1$, $a = -1/3$

$2a - 3b = \frac{1}{4}$

$-\frac{2}{3} - 3b = \frac{1}{4}$

$b = -11/36$

} $x_p = -\frac{t}{3} - \frac{11}{36}$

$X = X_h + X_p$

$X = c_1 e^{t} + c_2 e^{-3t} - \frac{t}{3} - \frac{11}{36}$

$$\Delta_2 = \begin{vmatrix} 2D-3 & t \\ 2D+3 & 2 \end{vmatrix} = (2D-3)(2) - (2D+3)(t)$$

$\Delta_2 = -6 - (2+3t) = -8-3t$

$y = \frac{\Delta_2}{\Delta} \Rightarrow 8(D^2+2D-3)y = -8-3t$

$(D^2+2D-3)y = -\frac{3}{8}t - 1$

$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = -\frac{3}{8}t - 1$

$y_h = c_3 e^{t} + c_4 e^{-3t}$

$y_p = At+B \;,\; y_p' = A \;,\; y_p'' = 0$

$0 + 2A - 3At - 3B = -\frac{3}{8}t - 1$

$-3A = -\frac{3}{8} \Rightarrow A = \frac{1}{8} \;,\quad 2A - 3B = -1$

$\frac{1}{4} + 1 = 3B \Rightarrow B = \frac{5}{12}$

$y_p = \frac{t}{8} + \frac{5}{12}$

$y = y_h + y_p$

$y = c_3 e^{t} + c_4 e^{-3t} + \frac{t}{8} + \frac{5}{12}$

$* \; 2x' - 2y' - 3x = t$

$$2c_1e^{t} - 6c_2e^{-3t} - \frac{2}{3} - 2\left(c_3e^{t} - 3c_4e^{-3t} + \frac{1}{8}\right) +$$
$$-3\left(c_1e^{t} + c_2e^{-3t} - \frac{t}{3} - \frac{11}{36}\right) = t$$

$$\underline{2c_1e^{t}} - \underline{6c_2e^{-3t}} - \cancel{\frac{2}{3}} - \underline{2c_3e^{t}} + \underline{6c_4e^{-3t}} - \cancel{\frac{1}{4}} - \underline{3c_1e^{t}} - \underline{3c_2e^{-3t}}$$
$$+\cancel{t} + \cancel{\frac{11}{12}} = \cancel{t}$$

$$(-c_1 - 2c_3)e^{t} + (-9c_2 + 6c_4)e^{-3t} = 0$$

$\underline{c_1 = -2c_3}$ and $\underline{c_2 = \frac{2}{3}c_4}$

$$\begin{cases} x = -2c_3e^{t} + \frac{2}{3}c_4e^{-3t} - \frac{t}{3} - \frac{11}{36} \\ y = c_3e^{t} + c_4e^{-3t} + \frac{t}{8} + \frac{5}{12} \end{cases}$$