SERIES SOLUTIONS OF SECOND ORDER LINEAR HOMOGENEOUS DEs

We consider the second-order linear homogeneous dE for $y=y(x)$:

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \ldots\ (1)$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials or convergent power series (around $x=x_0$), with no common polynomial factor (that could be divided out). The value $x=x_0$ is called an ordinary point of (1) if $P(x_0)\neq 0$ and is called a singular point if $P(x_0)=0$. Our goal is to find two independent solutions of (1), valid in a neighborhood about $x=x_0$.

Taylor's Theorem: Any function that is analytic at a point $x=x_0$ (that means differentiable) can be expanded in a power series;

$$y(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots = \sum_{n=0}^{\infty} a_n (x-x_0)^n .$$

Because of Taylor's Theorem, the term 'analytic' is sometimes used to mean that a function can be expanded in a power series at a point.
(analytic $\Leftrightarrow$ infinitely differentiable $\Leftrightarrow$ power series exist)

Thus we can say that; if the point $x=x_0$ is an ordinary point of the given DE;

$$P(x)y'' + Q(x)y' + R(x)y = 0 \quad \left(\text{or: } y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0\right)$$

so we can find a solution of this DE in the type: $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ around $x=x_0$.

The method of power series solutions will work at any ordinary point of a DE. A point $x = x_0$ is called an ordinary point of the diff. operator;

$$L_n y = a_n(x) y^{(n)} + a_{n-1}(x) y^{(n-1)} + \dots + a_0(x) y$$ if all the functions;

$$P_k(x) = \frac{a_k(x)}{a_n(x)}$$ are analytic at $t = t_0$.

Example: $y'' - (x-2)y' + 2y = 0$ Find the solution of this DE around $x = 2$.

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-2)^n \\ y' &= \sum_{n=0}^{\infty} a_n . n (x-2)^{n-1} \\ y'' &= \sum_{n=0}^{\infty} a_n n(n-1)(x-2)^{n-2} \end{aligned}\right\}$$

$$\sum_{n=0}^{\infty} a_n n(n-1)(x-2)^{n-2} - (x-2)\sum_{n=0}^{\infty} a_n n (x-2)^{n-1} + 2\sum_{n=0}^{\infty} a_n (x-2)^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1)(x-2)^{n-2} - \sum_{n=2}^{\infty} a_{n-2}(n-2)(x-2)^{n-2} + 2\sum_{n=2}^{\infty} a_{n-2}(x-2)^{n-2} = 0$$

$$\sum_{n=2}^{\infty} [\underbrace{n(n-1)a_n - (n-2)a_{n-2} + 2a_{n-2}}_{0}](x-2)^{n-2} = 0$$

$$n(n-1)a_n - (n-4)a_{n-2} = 0$$

$$n(n-1)a_n = (n-4)a_{n-2}$$

$$\boxed{a_n = \frac{n-4}{n(n-1)} a_{n-2}} \quad n \geq 2$$

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$$a_2 = \frac{2-4}{2.1} a_0 \Rightarrow a_2 = -a_0$$

$$a_3 = -\frac{1}{6} a_1$$

$$a_4 = 0$$

$$a_5 = -\frac{1}{20} a_3 = -\frac{1}{120} a_1$$

$$y = a_0 + a_1(x-2) - a_0(x-2)^2 - \frac{1}{6} a_1 (x-2)^3 - \frac{1}{120} a_1 (x-2)^5 + \ldots$$

$$y = a_0(1-(x-2)^2 + \ldots) + a_1\left((x-2) - \frac{1}{6}(x-2)^3 - \frac{1}{120}(x-2)^5 \ldots\right)$$

Example: $y'' - xy' - y = 0$ Find the solution around $x=0$.

$$y = \sum_{n=0}^{\infty} a_n (x - \overset{0}{x_0})^n, \quad y' = \sum_{n=0}^{\infty} a_n n (x - \overset{0}{x_0})^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n n(n-1)(x - \overset{0}{x_0})^{n-2}$$

$$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} - x . \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=0}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} - \sum_{n=2}^{\infty} a_{n-2}(n-2) x^{n-2} - \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\sum_{n=2}^{\infty} \underbrace{[a_n n(n-1) - a_{n-2}(n-2) - a_{n-2}]}_{0} x^{n-2} = 0$$

$$a_n n(n-1) - a_{n-2}(n-1) = 0$$

$$a_n n = a_{n-2} \Rightarrow a_n = \frac{a_{n-2}}{n}$$

$$a_{2k} = \frac{a_0}{2.4 \ldots 2k} \quad a_{2k+1} = \frac{a_1}{1.3 \ldots (2k+1)} \quad n \geq 2$$

$$a_2 = \frac{a_0}{2} \quad a_3 = \frac{a_1}{3} \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2.4}$$

$$a_5 = \frac{a_3}{5} = \frac{a_1}{3.5} \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2.4.6}$$