MATRICES

DEFINITION 1 A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

Some examples of matrices are

$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \\ -1 & 4 \end{bmatrix}, \quad [2 \quad 1 \quad 0 \quad -3], \quad \begin{bmatrix} e & \pi & -\sqrt{2} \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad [4]$$

A matrix with only one row, such as the second in Example 1, is called a ***row vector*** (or a ***row matrix***), and a matrix with only one column, such as the fourth in that example, is called a ***column vector*** (or a ***column matrix***). The fifth matrix in that example is both a row vector and a column vector.

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij}. Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$[a_{ij}]_{m \times n} \quad \text{or} \quad [a_{ij}]$$

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have

$$(A)_{ij} = a_{ij}$$

and for the matrix

$$A = \begin{bmatrix} 2 & -3 \\ 7 & 0 \end{bmatrix}$$

we have $(A)_{11} = 2$, $(A)_{12} = -3$, $(A)_{21} = 7$, and $(A)_{22} = 0$.

A matrix A with n rows and n columns is called a ***square matrix of order n***, and the shaded entries $a_{11}, a_{22}, \ldots, a_{nn}$ in (2) are said to be on the ***main diagonal*** of A.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{2}$$

DEFINITION 3 If A and B are matrices of the same size, then the ***sum*** $A + B$ is the matrix obtained by adding the entries of B to the corresponding entries of A, and the ***difference*** $A - B$ is the matrix obtained by subtracting the entries of B from the corresponding entries of A. Matrices of different sizes cannot be added or subtracted.

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij} = a_{ij} + b_{ij} \quad \text{and} \quad (A-B)_{ij} = (A)_{ij} - (B)_{ij} = a_{ij} - b_{ij}$$

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix},$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

▶ EXAMPLE 4 Scalar Multiples

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \tfrac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$. ◀

DEFINITION 5 If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the ***product*** AB is the $m \times n$ matrix whose entries are determined as follows: To find the entry in row i and column j of AB, single out row i from the matrix A and column j from the matrix B. Multiply the corresponding entries from the row and column together, and then add up the resulting products.

$$\begin{array}{ccccc} A & & B & & AB \\ m \times r & & r \times n & = & m \times n \end{array}$$

Inside

Outside

▶ EXAMPLE 5 Multiplying Matrices

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, then, as illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix} \tag{4}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj} \tag{5}$$

Formula (5) is called the ***row-column rule*** for matrix multiplication.

DEFINITION 6 If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a ***linear combination*** of $A_1, A_2, \ldots, A_r$ with ***coefficients*** $c_1, c_2, \ldots, c_r$.

To see how matrix products can be viewed as linear combinations, let A be an $m \times n$ matrix and $\mathbf{x}$ an $n \times 1$ column vector, say

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

THEOREM 1.3.1 *If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.*

▶ EXAMPLE 8 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

> **DEFINITION 7** If A is any $m \times n$ matrix, then the ***transpose of A***, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

▶ EXAMPLE 11 Some Transposes

The following are some examples of matrices and their transposes.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4] \blacktriangleleft$$

Properties of the Transpose The following theorem lists the main properties of the transpose.

> **THEOREM 1.4.8** *If the sizes of the matrices are such that the stated operations can be performed, then:*
>
> (a) $(A^T)^T = A$
>
> (b) $(A + B)^T = A^T + B^T$
>
> (c) $(A - B)^T = A^T - B^T$
>
> (d) $(kA)^T = kA^T$
>
> (e) $(AB)^T = B^T A^T$

$$(A^T)_{ij} = (A)_{ji}$$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

DEFINITION 8 If A is a square matrix, then the ***trace of A***, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

EXAMPLE 12 Trace

The following are examples of matrices and their traces.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} \qquad \operatorname{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Exercise

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

3. (a) $D + E$ (b) $D - E$ (c) $5A$
(d) $-7C$ (e) $2B - C$ (f) $4E - 2D$
(g) $-3(D + 2E)$ (h) $A - A$ (i) $\operatorname{tr}(D)$
(j) $\operatorname{tr}(D - 3E)$ (k) $4\operatorname{tr}(7B)$ (l) $\operatorname{tr}(A)$

5. (a) AB (b) BA (c) $(3E)D$
(d) $(AB)C$ (e) $A(BC)$ (f) CC^T
(g) $(DA)^T$ (h) $(C^TB)A^T$ (i) $\operatorname{tr}(DD^T)$

What is the value of k?

15. $$\begin{bmatrix} k & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} k \\ 1 \\ 1 \end{bmatrix} = 0$$

16. $$\begin{bmatrix} 2 & 2 & k \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ k \end{bmatrix} = 0$$

32. Find a 4×4 matrix $A = [a_{ij}]$ whose entries satisfy the stated condition.

(a) $a_{ij} = i + j$ (b) $a_{ij} = i^{j-1}$

(c) $a_{ij} = \begin{cases} 1 & \text{if} \quad |i - j| > 1 \\ -1 & \text{if} \quad |i - j| \leq 1 \end{cases}$

True-False Exercises

TF. In parts (a)–(o) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal.

(b) An $m \times n$ matrix has m column vectors and n row vectors.

(c) If A and B are 2×2 matrices, then $AB = BA$.

(d) The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B.

Inverses; Algebraic Properties of Matrices

THEOREM 1.4.1 Properties of Matrix Arithmetic

Assuming that the sizes of the matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

(*a*) $A + B = B + A$ **[Commutative law for matrix addition]**

(*b*) $A + (B + C) = (A + B) + C$ **[Associative law for matrix addition]**

(*c*) $A(BC) = (AB)C$ **[Associative law for matrix multiplication]**

(*d*) $A(B + C) = AB + AC$ **[Left distributive law]**

(*e*) $(B + C)A = BA + CA$ **[Right distributive law]**

(*f*) $A(B - C) = AB - AC$

(*g*) $(B - C)A = BA - CA$

(*h*) $a(B + C) = aB + aC$

(*i*) $a(B - C) = aB - aC$

(*j*) $(a + b)C = aC + bC$

(*k*) $(a - b)C = aC - bC$

(*l*) $a(bC) = (ab)C$

(*m*) $a(BC) = (aB)C = B(aC)$

A matrix whose entries are all zero is called a ***zero matrix***. Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

THEOREM 1.4.2 Properties of Zero Matrices

If c is a scalar, and if the sizes of the matrices are such that the operations can be perfomed, then:

(*a*) $A + 0 = 0 + A = A$

(*b*) $A - 0 = A$

(*c*) $A - A = A + (-A) = 0$

(*d*) $0A = 0$

(*e*) *If* $cA = 0$, *then* $c = 0$ *or* $A = 0$.

A square matrix with 1's on the main diagonal and zeros elsewhere is called an ***identity matrix***. Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An identity matrix is denoted by the letter I. If it is important to emphasize the size, we will write I_n for the $n \times n$ identity matrix.

DEFINITION 1 If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B can be found, then A is said to be ***singular***.

Remark The relationship $AB = BA = I$ is not changed by interchanging A and B, so if A is invertible and B is an inverse of A, then it is also true that B is invertible, and A is an inverse of B. Thus, when

$$AB = BA = I$$

we say that A and B are *inverses of one another*.

EXAMPLE 5 An Invertible Matrix

Let

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

$$AA^{-1} = I \quad \text{and} \quad A^{-1}A = I$$

EXAMPLE 6 A Class of Singular Matrices

A square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

THEOREM 1.4.4 *If B and C are both inverses of the matrix A, then $B = C$.*

THEOREM 1.4.5 *The matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

EXAMPLE 7 Calculating the Inverse of a 2 × 2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$\text{(a) } A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

THEOREM 1.4.6 *If A and B are invertible matrices with the same size, then AB is invertible and*

$$(AB)^{-1} = B^{-1}A^{-1}$$

Powers of a Matrix If A is a *square* matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA \cdots A \quad [n \text{ factors}]$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \cdots A^{-1} \quad [n \text{ factors}]$$

THEOREM 1.4.7 *If A is invertible and n is a nonnegative integer, then:*

(*a*) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(*b*) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(*c*) kA *is invertible for any nonzero scalar* k, *and* $(kA)^{-1} = k^{-1}A^{-1}$.

THEOREM 1.4.9 *If A is an invertible matrix, then A^T is also invertible and*

$$(A^T)^{-1} = (A^{-1})^T$$

Matrix Polynomials If A is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_mA^m \tag{3}$$

where I is the $n \times n$ identity matrix; that is, $p(A)$ is obtained by substituting A for x and replacing the constant term a_0 by the matrix a_0I. An expression of form (3) is called a ***matrix polynomial in A***.

EXAMPLE 12 A Matrix Polynomial

Find $p(A)$ for

$$p(x) = x^2 - 2x - 3 \quad \text{and} \quad A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} p(A) &= A^2 - 2A - 3I \\ &= \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 4 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$. ◀

Exercise

9. Find the inverse of

$$\begin{bmatrix} \frac{1}{2}(e^x + e^{-x}) & \frac{1}{2}(e^x - e^{-x}) \\ \frac{1}{2}(e^x - e^{-x}) & \frac{1}{2}(e^x + e^{-x}) \end{bmatrix}$$

10. Find the inverse of

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

▶ In Exercises 15–18, use the given information to find A. ◀

15. $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

16. $(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

17. $(I + 2A)^{-1} = \begin{bmatrix} -1 & 2 \\ 4 & 5 \end{bmatrix}$

18. $A^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

▶ In Exercises 21–22, compute $p(A)$ for the given matrix A and the following polynomials.

(a) $p(x) = x - 2$

(b) $p(x) = 2x^2 - x + 1$

(c) $p(x) = x^3 - 2x + 1$ ◀

21. $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

22. $A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$

True-False Exercises

TF. In parts (a)–(k) determine whether the statement is true or false, and justify your answer.

(a) Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

(b) For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

(c) For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

(d) If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(e) If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^TB^T$.

(f) The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

(g) If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$.

(h) If A is an invertible matrix, then so is A^T.

(i) If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

(j) A square matrix containing a row or column of zeros cannot be invertible.

(k) The sum of two invertible matrices of the same size must be invertible.

DEFINITION 1 Matrices A and B are said to be ***row equivalent*** if either (hence each) can be obtained from the other by a sequence of elementary row operations.

DEFINITION 2 A matrix E is called an ***elementary matrix*** if it can be obtained from an identity matrix by performing a *single* elementary row operation.

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1.

THEOREM 1.5.3 Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(*a*) *A is invertible.*

(*b*) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(*c*) *The reduced row echelon form of A is I_n.*

(*d*) *A is expressible as a product of elementary matrices.*

▶ EXAMPLE 4 Using Row Operations to Find A^{-1}

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we will adjoin the identity matrix to the right side of A, thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$[I \mid A^{-1}]$$

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← We added −2 times the first row to the second and −1 times the first row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← We added 2 times the second row to the third.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We multiplied the third row by −1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added 3 times the third row to the second and −3 times the third row to the first.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added −2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} ◀$$

▶ **EXAMPLE 5 Showing That a Matrix Is Not Invertible**

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \longleftarrow$$ We added −2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \longleftarrow$$ We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

Exercise

▶ In Exercises 11–12, use the inversion algorithm to find the inverse of the matrix (if the inverse exists). ◀

11. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{bmatrix}$

12. (a) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & -\frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & -\frac{3}{10} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{10} \end{bmatrix}$

In Exercises 13–18, use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

13. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

14. $\begin{bmatrix} \sqrt{2} & 3\sqrt{2} & 0 \\ -4\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

15. $\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 5 & 0 \\ 1 & 3 & 5 & 7 \end{bmatrix}$

17. $\begin{bmatrix} 2 & -4 & 0 & 0 \\ 1 & 2 & 12 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & -4 & -5 \end{bmatrix}$

18. $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & 3 & 0 \\ 2 & 1 & 5 & -3 \end{bmatrix}$

True-False Exercises

TF. In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) The product of two elementary matrices of the same size must be an elementary matrix.

(b) Every elementary matrix is invertible.

(c) If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent.

(d) If A is an $n \times n$ matrix that is not invertible, then the linear system $A\mathbf{x} = 0$ has infinitely many solutions.

(e) If A is an $n \times n$ matrix that is not invertible, then the matrix obtained by interchanging two rows of A cannot be invertible.

(f) If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

(g) An expression of an invertible matrix A as a product of elementary matrices is unique.

1.7 Diagonal, Triangular, and Symmetric Matrices

In this section we will discuss matrices that have various special forms. These matrices arise in a wide variety of applications and will play an important role in our subsequent work.

Diagonal Matrices

A square matrix in which all the entries off the main diagonal are zero is called a ***diagonal matrix***. Here are some examples:

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

A general $n \times n$ diagonal matrix D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \qquad D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular Matrices

A square matrix in which all the entries above the main diagonal are zero is called ***lower triangular***, and a square matrix in which all the entries below the main diagonal are zero is called ***upper triangular***. A matrix that is either upper triangular or lower triangular is called ***triangular***.

EXAMPLE 2 Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

THEOREM 1.7.1

(*a*) *The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.*

(*b*) *The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.*

(*c*) *A triangular matrix is invertible if and only if its diagonal entries are all nonzero.*

(*d*) *The inverse of an invertible lower triangular matrix is lower triangular, and the inverse of an invertible upper triangular matrix is upper triangular.*

EXAMPLE 3 Computations with Triangular Matrices

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows from part (*c*) of Theorem 1.7.1 that the matrix A is invertible but the matrix B is not. Moreover, the theorem also tells us that A^{-1}, AB, and BA must be upper triangular. We leave it for you to confirm these three statements by showing that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

DEFINITION 1 A square matrix A is said to be ***symmetric*** if $A = A^T$.

EXAMPLE 4 Symmetric Matrices

The following matrices are symmetric, since each is equal to its own transpose (verify).

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}, \quad \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & d_2 & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix}$$

$$(A)_{ij} = (A)_{ji}$$

THEOREM 1.7.2 *If A and B are symmetric matrices with the same size, and if k is any scalar, then:*

(*a*) *A^T is symmetric.*

(*b*) *$A + B$ and $A - B$ are symmetric.*

(*c*) *kA is symmetric.*

THEOREM 1.7.4 *If A is an invertible symmetric matrix, then A^{-1} is symmetric.*

THEOREM 1.7.5 *If A is an invertible matrix, then AA^T and A^TA are also invertible.*

Exercise

▶ In Exercises 7–10, find A^2, A^{-2}, and A^{-k} (where k is any integer) by inspection. ◀

7. $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

8. $A = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

9. $A = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

10. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

True-False Exercises

TF. In parts (a)–(m) determine whether the statement is true or false, and justify your answer.

(a) The transpose of a diagonal matrix is a diagonal matrix.

(b) The transpose of an upper triangular matrix is an upper triangular matrix.

(c) The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix.

(d) All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal.

(e) All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal.

(f) The inverse of an invertible lower triangular matrix is an upper triangular matrix.

(g) A diagonal matrix is invertible if and only if all of its diagonal entries are positive.

(h) The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix.

(i) A matrix that is both symmetric and upper triangular must be a diagonal matrix.

(j) If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric.

(k) If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular.

(l) If A^2 is a symmetric matrix, then A is a symmetric matrix.

(m) If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix.

DEFINITION 1 If A is a complex matrix, then the ***conjugate transpose*** of A, denoted by A^*, is defined by

$$A^* = \overline{A}^T \tag{1}$$

DEFINITION 2 A square matrix A is said to be ***unitary*** if

$$AA^* = A^*A = I \tag{2}$$

or, equivalently, if

$$A^* = A^{-1} \tag{3}$$

and it is said to be ***Hermitian****[*] if

$$A^* = A \tag{4}$$

THEOREM 7.5.1 *If k is a complex scalar, and if A and B are complex matrices whose sizes are such that the stated operations can be performed, then:*

(a) $(A^*)^* = A$

(b) $(A + B)^* = A^* + B^*$

(c) $(A - B)^* = A^* - B^*$

(d) $(kA)^* = \overline{k}A^*$

(e) $(AB)^* = B^*A^*$

for example, we can tell by inspection that

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{bmatrix}$$

is Hermitian.

Skew-Symmetric and Skew-Hermitian Matrices

We will now consider two more classes of matrices that play a role in the analysis of the diagonalization problem. A square *real* matrix A is said to be ***skew-symmetric*** if $A^T = -A$, and a square *complex* matrix A is said to be ***skew-Hermitian*** if $A^* = -A$. We leave it as an exercise to show that a skew-symmetric matrix must have zeros on the main diagonal, and a skew-Hermitian matrix must have zeros or pure imaginary numbers on the main diagonal. Here are two examples:

$$A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} i & 1-i & 5 \\ -1-i & 2i & i \\ -5 & i & 0 \end{bmatrix}$$

[skew-symmetric] [skew-Hermitian]

Idempotent Matrix

An idempotent matrix, $\mathbf{P}$, is one that is equal to its square, that is, $\mathbf{P}^2 = \mathbf{PP} = \mathbf{P}$.

An **involutory matrix** is a square and invertible matrix whose inverse matrix is the matrix itself.

Exercise

In Exercises 1–2, find A^*.

1. $A = \begin{bmatrix} 2i & 1-i \\ 4 & 3+i \\ 5+i & 0 \end{bmatrix}$

2. $A = \begin{bmatrix} 2i & 1-i & -1+i \\ 4 & 5-7i & -i \end{bmatrix}$

In Exercises 3–4, substitute numbers for the $\times$'s so that A is Hermitian.

3. $A = \begin{bmatrix} 1 & i & 2-3i \\ \times & -3 & 1 \\ \times & \times & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} 2 & 0 & 3+5i \\ \times & -4 & -i \\ \times & \times & 6 \end{bmatrix}$

In Exercises 19–20, substitute numbers for the $\times$'s so that A is skew-Hermitian.

19. $A = \begin{bmatrix} 0 & i & 2-3i \\ \times & 0 & 1 \\ \times & \times & 4i \end{bmatrix}$

20. $A = \begin{bmatrix} 0 & 0 & 3-5i \\ \times & 0 & -i \\ \times & \times & 0 \end{bmatrix}$

Echelon Forms

In Example 6 of the last section, we solved a linear system in the unknowns x, y, and z by reducing the augmented matrix to the form

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

from which the solution $x = 1$, $y = 2$, $z = 3$ became evident. This is an example of a matrix that is in ***reduced row echelon form***. To be of this form, a matrix must have the following properties:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. We call this a ***leading 1***.
2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column that contains a leading 1 has zeros everywhere else in that column.

A matrix that has the first three properties is said to be in ***row echelon form***. (Thus, a matrix in reduced row echelon form is of necessity in row echelon form, but not conversely.)

▶ EXAMPLE 1 Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Some Facts About Echelon Forms

There are three facts about row echelon forms and reduced row echelon forms that are important to know but we will not prove:

1. Every matrix has a unique reduced row echelon form; that is, regardless of whether you use Gauss–Jordan elimination or some other sequence of elementary row operations, the same reduced row echelon form will result in the end.*
2. Row echelon forms are not unique; that is, different sequences of elementary row operations can result in different row echelon forms.
3. Although row echelon forms are not unique, the reduced row echelon form and all row echelon forms of a matrix A have the same number of zero rows, and the leading 1's always occur in the same positions. Those are called the ***pivot positions*** of A. A column that contains a pivot position is called a ***pivot column*** of A.

▶ EXAMPLE 9 Pivot Positions and Columns

Earlier in this section (immediately after Definition 1) we found a row echelon form of

$$A = \begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

to be

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -\frac{7}{2} & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The leading 1's occur in positions (row 1, column 1), (row 2, column 3), and (row 3, column 5). These are the pivot positions. The pivot columns are columns 1, 3, and 5.