Mean Value Theorem:
Let the function f be continuous on the closed interval $[a,b]$ and differentiable, with derivative f', on the open interval (a,b). Then there is a point c in (a,b) such that

$$f(b) - f(a) = f'(c).(b-a).$$

Example. Find a point c satisfying the mean value theorem for the function $f(x) = x^2$.
Solution. $f'(x) = 2x$, by the mean value formula, $b^2 - a^2 = 2c(b-a)$.
So, we get $c = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$

Theorem. Let the function f be differentiable on the interval I, with derivative f' and suppose f' equals zero at every point of I. Then, f is constant on I.

(2)

Ex. Let $f(x) = \begin{cases} 1, & \text{if } x>0 \\ -1, & \text{if } x<0 \end{cases}$

f is not constant, but $f'(x)=0$

Why? because the domain is not interval.

Theorem. Let f be continuous on an interval, and suppose the derivative f' exists and has the same sign at every interior point of I. Then,

(i) f is increasing on I if f' is positive at every interior point of I

(ii) f is decreasing on I if f' is negative at every interior point of I

Proof: (i) Let $f'(x)>0$ for all $x \in I$

Let $a, b \in I$, $a<b$. By the mean value th,

$$f(b)-f(a) = f'(c)(b-a)$$

for some $c \in (a,b)$. So, $f'(c)>0$

$f(b)-f(a)>0 \Rightarrow f$ is increa