Example. Let $X \neq \emptyset$ be a set.
$B(X) = \{ f \mid f : X \to \mathbb{R}$ is a bounded function$\}$.

Define $d(f,g) = \sup_{x \in X} |f(x) - g(x)|$ for every $f, g \in B(X)$.

d is a metric on $B(X)$.

Let's show this.

1) $d(f,g) \geq 0$? for all $f,g \in B(X)$.

$0 \leq |\cdot| \Rightarrow d(f,g) \geq 0$ by the definition of d.

2) $d(f,g) = 0 \Leftrightarrow f = g$?

$0 \leq |f(x) - g(x)| \leq \sup_{x \in X} |f(x) - g(x)| = 0$

$\Rightarrow |f(x) - g(x)| = 0 \Rightarrow f(x) - g(x) = 0$ for all $x \in X$. $\Rightarrow f(x) = g(x) \ \forall x \in X$.

$\Rightarrow f = g$

If $f = g$, then $f(x) = g(x) \ \forall x \in X$.

$\Rightarrow f(x) - g(x) = 0 \ \forall x \in X$.

$\Rightarrow |f(x) - g(x)| = 0 \Rightarrow$

$d(f,g) = \sup_{x \in X} |f(x) - g(x)| = 0$

3) $d(f,g) = d(g,f)$ for all $f,g \in B(X)$.

$$d(f,g) = \sup_{x \in X} |f(x) - g(x)|$$

$$= \sup_{x \in X} |-(g(x) - f(x))|$$

$$= \sup_{x \in X} |g(x) - f(x)| = d(g,f).$$

4) Let $f, g, h \in B(X)$.

$$|f(x) - g(x)| = |(f(x) - h(x)) + (h(x) - g(x))|$$

$$\leq |f(x) - h(x)| + |h(x) - g(x)|$$

$$\leq \sup_{x \in X} |f(x) - h(x)| + \sup_{x \in X} |h(x) - g(x)|$$

$$\Rightarrow \sup_{x \in X} |f(x) - g(x)| \leq d(f,h) + d(h,g)$$

$$\Rightarrow d(f,g) \leq d(f,h) + d(h,g)$$

Lemma Let $0 \le a$, $0 \le b$.
If $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$, then

$$a^{1/p} b^{1/q} \le \frac{a}{p} + \frac{b}{q}.$$

Proof: If $a = 0$ or $b = 0$, then inequality holds.

So, let $0 < a$, $0 < b$.

Define $f(x) = \alpha(x-1) - (x^{a} - 1)$
for $\alpha \in [0,1]$.

Consider $f(x) \ge 0$ for all $x \in [1, \infty)$.

Hence, $x^\alpha - 1 \le \alpha(x-1)$.

Suppose $a \le b$. Then choose
$x = \frac{b}{a}$, $\alpha = \frac{1}{q}$.

If ~~we choose~~ $b < a$, then we
choose $x = \frac{a}{b}$, $\alpha = \frac{1}{p}$.

Then, we see that inequality holds.

Let $1 \leq p < \infty$.

$$l_p = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

Holders Inequality.

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

If $x = (x_n) \in l_p$ and $y = (y_n) \in l_q$,

then

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{1/p} \left[\sum_{n=1}^{\infty} |y_n|^q \right]^{1/q}.$$

holds.

Proof. Suppose $\sum_{n=1}^{\infty} |x_n|^p \neq 0$ and $\sum_{n=1}^{\infty} |y_n|^q \neq 0$. For $n \in \mathbb{N}$, by the previous lemma,

$$\frac{|x_n|\,|y_n|}{\left[\sum_{n=1}^{\infty} |x_m|^p \right]^{1/p} \left[\sum_{n=1}^{\infty} |y_m|^q \right]^{1/q}} \leq \frac{1}{p} \frac{|x_n|^p}{\sum_{m=1}^{\infty} |x_m|^p} + \frac{1}{q} \frac{|y_n|^q}{\sum_{m=1}^{\infty} |y_m|^q}$$

Here, $a = \dfrac{|x_n|^p}{\sum_{m=1}^{\infty} |x_m|^p}$, $b = \dfrac{|y_n|^q}{\sum_{m=1}^{\infty} |y_m|^q}$.

(5)

In inequality, sum $\sum_{n=1}^{\infty} \Rightarrow$

$$\frac{\sum_{n=1}^{\infty} |x_n y_n|}{\left[\sum_{m=1}^{\infty} |x_m|^p\right]^{1/p} \left[\sum_{m=1}^{\infty} |y_m|^q\right]^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

This implies

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left[\sum_{m=1}^{\infty} |x_m|^p\right]^{1/p} \left[\sum_{m=1}^{\infty} |y_m|^q\right]^{1/q}$$

Write $m = n$.

Therefore,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{1/q}$$

Minkowski's Inequality:

Let $1 \leq p < \infty$ and $x = (x_n) \in l_p$ $y = (y_n) \in l_p$. Then

$$\left[\sum_{n=1}^{\infty} |x_n + y_n|^p\right]^{1/p} \leq \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p} + \left[\sum_{n=1}^{\infty} |y_n|^p\right]^{1/p}$$

holds.