Implicit differentiation

Let $F(x, y, z) = 0$, $z = z(x,y)$.

Let's find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = ?$

By the chain rule,

$$\frac{\partial F}{\partial x} \cdot 1 + \frac{\partial F}{\partial y} \cdot 0 + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0 \implies$$

$$\frac{\partial z}{\partial x} = - \frac{\partial F / \partial x}{\partial F / \partial z} = - \frac{F_x}{F_z}, \quad F_z \neq 0.$$

$$\cancel{\frac{\partial F}{\partial x} \cdot 0} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

0

$$\implies \frac{\partial z}{\partial y} = - \frac{\partial F / \partial y}{\partial F / \partial z} = - \frac{F_y}{F_z}, \quad F_z \neq 0.$$

Example. Let $x^2 y^3 + z^2 y^2 + xyz = 1$.

$z = z(x,y)$. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} = ?$

Solution

$\frac{\partial z}{\partial x} = ?$

$$2xy^3 + 2z \cdot z_x \cdot y^2 + yz + xyz_x = 0$$

$$\implies z_x = ?$$

$$2xy^3 + yz + z_x(2zy^2 + xy) = 0$$

$$z_x = - \frac{(2xy^3 + yz)}{2zy^2 + xy}$$

OR

$$F(x,y,z) = x^2y^3 + z^2y^2 + xyz - 1 = 0.$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy^3 + yz}{2zy^2 + xy}$$

$$\frac{\partial z}{\partial y} = ?$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3x^2y^2 + 2z^2y + xz}{2zy^2 + xy}$$

Gradient and Directional Derivatives

Definition. At any point (x,y) where the first partial derivatives of the function $f(x,y)$ exists, we define the gradient vector $\nabla f(x,y) = \text{grad}\, f(x,y)$ by

$$\nabla f(x,y) = f_1(x,y)\, i + f_2(x,y)\, j.$$

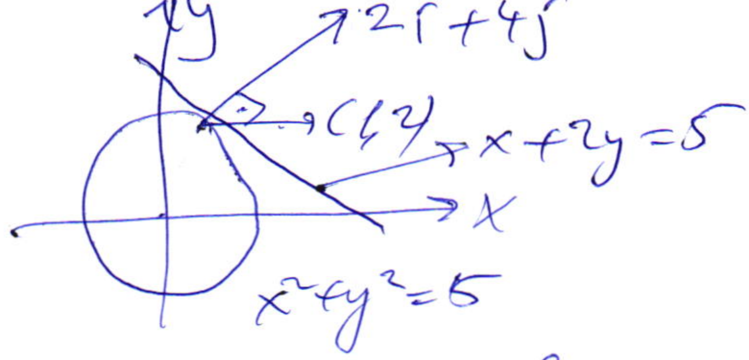
$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}$: del or nabla vector differential operator

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y}.$$

Example. Let $f(x,y) = x^2 + y^2$.

$$\nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} = 2x\, i + 2y\, j.$$

$\nabla f(1,2) = 2i + 4j$. is perpendicular to the tangent line $x+2y=5$ to the circle $x^2+y^2=5$ at $(1,2)$.



Theorem. If $f(x,y)$ is differentiable at the point (a,b) and $\nabla f(a,b) \neq 0$, then $\nabla f(a,b)$ is a normal vector to the level curve off that passes through (a,b).

Definition.
Let $u = ai + bj$ be a unit vector so that $a^2+b^2=1$. The directional derivative of $f(x,y)$ at (x_0,y_0) in the direction of u is the rate of change of $f(x,y)$ with respect to distance measured at (x_0,y_0) along a ray in the direction of u in the xy-plane. The directional derivative is given by

$$D_u f(x_0,y_0) = \lim_{h \to 0^+} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

Also, it is given by

$$D_u f(x_0,y_0) = \frac{d}{dt} f(x_0+ta, y_0+tb)\Big|_{t=0}$$

if the derivative on the right side exists.

Theorem. If f is differentiable at (a,b) and $v = ui + vj$ is a unit vector, then the directional derivative of f at (a,b) in the direction of u is given by

$$D_u f(a,b) = u \cdot \nabla f(a,b).$$

In general,

$$D_v f(a,b) = \frac{v}{|v|} \cdot \nabla f(a,b).$$

In three dimension,

$$D_v f(a,b,c) = \frac{v}{|v|} \cdot \nabla f(a,b,c).$$

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$

Example. Find the rate of change of $f(x,y) = y^4 + 2xy^3 + x^2y^2$ at $(0,1)$ measured in ~~and~~ the direction $u = i + 2j$.

Solution - $|u| = \sqrt{1+4} = \sqrt{5}$.

$$\frac{u}{|u|} = \frac{1}{\sqrt{5}}(i + 2j)$$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} = i(2y^3 + 2xy^2) + j(4y^3 + 6xy^2 + 2x^2y)$$

$$\nabla f(0,1) = 2i + 4j.$$

$$D_u f(0,1) = \frac{u}{|u|} \cdot \nabla f(0,1)$$

$$= \frac{(i + 2j)}{\sqrt{5}} \cdot (2i + 4j) = \frac{2+8}{\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5},$$

Properties of the gradient vector

1) At (a,b), $f(x,y)$ increases rapidly in the direction of the gradient vector $\nabla f(a,b)$ the maximum rate of increase is

$$|\nabla f(a,b)|$$

ii) At (a,b), $f(x,y)$ decreases rapidly in the direction of $-\nabla f(a,b)$.

The maximum rate of decrease is $|\nabla f(a,b)|$

iii) The rate of change of $f(x,y)$ at (a,b) is zero in the direction tangent to the level curve of f passing through (a,b).

The gradient in n dimension.

Let $f(x_1, x_2, \dots, x_n)$.

$$\nabla f(x_1, \dots, x_n) = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \dots + \frac{\partial f}{\partial x_n} e_n.$$

where $e_1 = (1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1)$.

Example Let $f(x,y,z) = x^2 + y^2 + z^2$.

a) Find $\nabla f(x,y,z)$ and $\nabla f(1,-1,2)$.

b) Find an equation of the tangent plane to the sphere $x^2 + y^2 + z^2 = 6$ at the point $(1,-1,2)$.

c) What is the maximum rate of increase of f at $(1,-1,2)$

d) What is the rate of change with

respect to distance of f at $(1,-1,2)$ measured in the direction from that point toward the point $(3,1,1)$.

Solution-

a) $\nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$= 2x i + 2y j + 2z k.$

and $\nabla f(1,-1,2) = 2i - 2j + 4k$.

b) $\nabla f(1,-1,2) = 2i - 2j + 4k$ is normal vector

So, tangent plane,

$2(x-1) - 2(y+1) + 4(z-2) = 0.$

or $x - y + 2z = 6.$

c) the maximum rate of increase

$|\nabla f(1,-1,2)| = 2\sqrt{6}.$

in the direction $i - 2j + 2k.$

$|2i - 2j + 4k| = \sqrt{4+4+16} = \sqrt{24}$

$= \sqrt{4 \cdot 6} = 2\sqrt{6}.$

d) the direction vector is $2i + 2j - k = u$ from $(1,-1,2)$ to $(3,1,1)$

$D_u f(1,-1,2) = \frac{u}{|u|} \cdot \nabla f(1,-1,2)$

$$= \frac{2i+2j-k}{\sqrt{4+4+1}} \cdot (2i-2j+4k) = 4/3.$$

Example.

Find a vector tangent to the curve of intersection of the two surfaces $z = x^2 - y^2$, $xyz + 30 = 0$ at the point $(-3,2,5)$.

$$n_1 = \nabla(x^2-y^2-z)\Big|_{(-3,2,5)} = 2xi - 2yj - k\Big|_{(-3,2,5)}$$

$$= -6i - 4j - k.$$

$$n_2 = \nabla(xyz+30)\Big|_{(-3,2,5)} = (yzi + xzj + xyk)\Big|_{(-3,2,5)}$$

$$= 10i - 15j - 6k$$

The tangen vector $= n_1 \times n_2$

$$= \begin{vmatrix} i & j & k \\ -6 & -4 & -1 \\ 10 & -15 & -6 \end{vmatrix} = 9i - 46j + 130k.$$