(1)

Math. Analysis [II]

1. Functions of several variables
2. Level curves
3. Limits and continuity
4. Partial derivatives
5. Tangent planes and normal lines
6. Distance from a point to a surface.
7. Higher Order Derivatives
8. The Chain Rule
9. Homogeneous functions
10. Linear Approximations
    Differentiability, differentials
11. Functions from n-space to m-space.
12. Gradients, Directional derivatives
13. Implicit Functions
    Systems of equation
    Choosing dependent and independent variables
14. Taylor's Formula, Taylor Series
15. Extreme Values
    Classifying Critical points
16. Lagrange Multipliers
17. Vector fields
    vector and scalar fields
    conservative fields
    vector calculus

(2) 18. Gradient, Divergence, curl

19. Some identities involving Grad, Div., curl.

Scalar and vector potentials

19. Orthogonal curvilinear coordinates

20. vectors and coordinate geometry in $\mathbb{R}^3$.

The cross product in $\mathbb{R}^3$

Determinants, the dot product

Planes and lines in $\mathbb{R}^3$

References

1. R. A. Adams, Christopher Essex

Calculus

2. R. Silverman

Calculus and analytic geometry

Vectors:

A vector is a quantity that involves both magnitude (length) and direction.

~~$\vec{a} = \vec{AB} =$~~

Standard basis vectors in $\mathbb{R}^3$

$\hat{i}, \hat{j}, k$.

③ The Dot product. (scalar product
inner product)

Definition. Given two vectors $u = u_1\hat{i} + u_2\hat{j} + u_3 k$, $v = v_1\hat{i} + v_2\hat{j} + v_3 k$ in $\mathbb{R}^3$, we define their dot product $u \cdot v$ to be

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Theorem. If θ is the angle between the directions of u and v $(0 \le \theta \le \pi)$, then

$$u \cdot v = |u|\,|v| \cos\theta.$$

u and v are perpendicular

$$\iff \theta = \frac{\pi}{2}.$$

Properties of dot product

Let u, v, w be vectors in $\mathbb{R}^3$, $t \in \mathbb{R}$.

Then

① $u \cdot v = v \cdot u$

② $u \cdot (v + w) = u \cdot v + u \cdot w$

③ $(tu) \cdot v = u \cdot (tv) = t(u \cdot v)$

④ $u \cdot u = |u|^2$.

$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$ - the length of u

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Example. Find the angle θ between the vectors $u = 2i + j - 2k$ and $v = 3i - 2j - k$.

Solution. From $u \cdot v = |u|\,|v| \cos\theta \Rightarrow$

$$\cos\theta = \frac{u \cdot v}{|u|\,|v|} = \frac{2 \cdot 3 + (1)(-2) + (-2)(-1)}{3\sqrt{14}}$$

$$= \frac{2}{\sqrt{14}} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right).$$

<u>Vectors in $\mathbb{R}^n$</u>

$e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 1)$, are unit vectors.

$u = (u_1, u_2, \ldots, u_n) = u_1 e_1 + u_2 e_2 + \cdots + u_n e_n$

$|u| = \sqrt{u_1^2 + u_2^2 + \cdots + u_2^2}$ = the length of u.

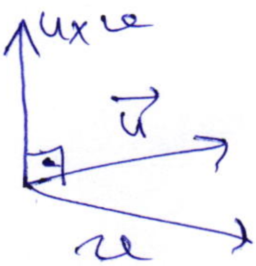
(5) # The cross product in $\mathbb{R}^3$

Definition. For any vectors u, v in $\mathbb{R}^3$, the cross product $u \times v$ is the unique vector satisfying the following three conditions:

1) $(u \times v) \cdot u = 0$ and $(u \times v) \cdot v = 0$

2) $|u \times v| = |u|\,|v| \sin\theta$, θ is the angle between u, v.

3) u, v and $u \times v$ form a right-handed triad.



Theorem. If $u = u_1 i + u_2 j + u_3 k$, $v = v_1 i + v_2 j + v_3 k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Examples

1) $i \times i = 0$ $\quad i \times j = k, \quad j \times i = -k$

$j \times j = 0, \quad j \times k = i, \quad k \times j = -i$

$k \times k = 0 \quad k \times i = j, \quad i \times k = -j.$

2) $(2i + j - 3k) \times (-2j + 5k) = ?$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 0 & -2 & 5 \end{vmatrix} = -i - 10j - 4k.$$

(6) Properties of the cross product

Let u, v, w be vectors in $\mathbb{R}^3$, $t \in \mathbb{R}$. Then

(1) $u \times u = 0$ (2) $u \times v = -v \times u$

(3) $(u+v) \times w = u \times w + v \times w$

(4) $u \times (v+w) = u \times v + u \times w$

(5) $(tu) \times v = u \times (tv) = t(u \times v)$

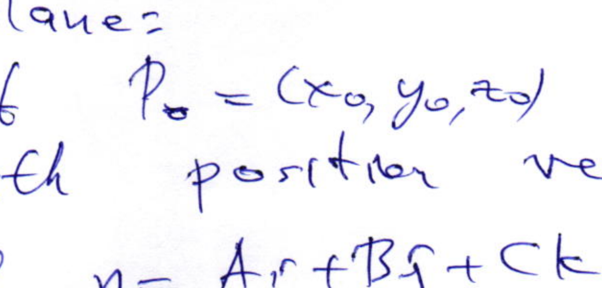
(6) $u \cdot (u \times v) = v \cdot (u \times v) = 0.$

Lines and planes equations in $\mathbb{R}^3$

plane:

Let $P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$ with position vector $r_0 = x_0 i + y_0 j + z_0 k$.

If $n = Ai + Bj + Ck$ is any non-zero vector, then there exists one plane passing through P_0 and perpendicular to n. n is called a normal vector to the plane.



$\vec{P_0P} \perp \vec{n}$

$\Rightarrow \boxed{\vec{P_0P} \cdot \vec{n} = 0}$

plane

$$(x-x_0)i+(y-y_0)j+(z-z_0)k \cdot (Ai+Bj+Ck)$$
$$= A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

$$\Rightarrow Ax+By+Cz\underbrace{-(Ax_0+By_0+Cz_0)}_{D}=0.$$

$$\Rightarrow \boxed{Ax+By+Cz+D=0}$$

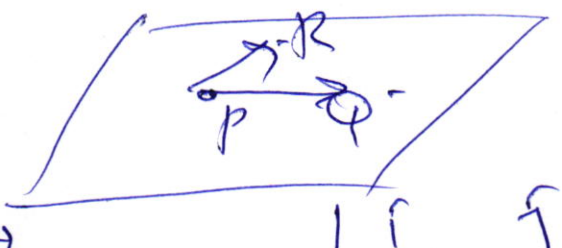
plane equation.

Example. $2x+3y-4z=0$ A plane.

normal $n=2i+3j-4k$, passing through origin $(0,0,0)$.

Example. Find an equation of the plane that passes through the points $P=(1,1,0)$, $Q=(0,2,1)$, $R=(3,2,-1)$.

Solution.



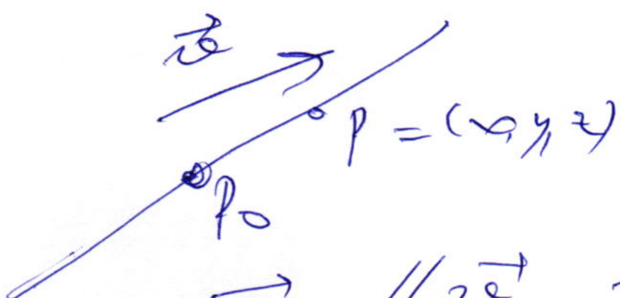
$$\vec{PQ}\times\vec{PR}=n=\begin{vmatrix} i & j & k \\ -1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix}=-2i+j-3k.$$

Using P

$$-2(x-1)+1(y-1)-3(z-0)=0.$$

Lines equations.

Let $r_0 = x_0 i + y_0 j + z_0 k$ be a position vector of point $P_0 = (x_0, y_0, z_0)$ and $v = ai + bj + ck$ be a non-zero vector then, there is a unique line passing through P_0 parallel to v.
v is called direction vector.



$\overrightarrow{P_0P} \parallel \vec{v} \Rightarrow \overrightarrow{P_0P} = t\vec{v}$ for some $t \in \mathbb{R}$.

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(ai + bj + ck)$$

$$\Rightarrow \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

standard eq. of line.

parametric eq. of line:

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$