① Exercises

1) Find and classify the critical points of the following functions:

a) $f(x,y) = x^2 + 2y^2 - 4x + 4y$.

b) $f(x,y) = x^3 + y^3 - 3xy$.

c) $f(x,y) = x^2 y\, e^{-(x^2+y^2)}$.

d) $f(x,y) = x\, e^{-x^3+y^3}$.

Lagrange Multipliers

Extreme Values of functions defined on restricted domains

~~Example.~~ For this, we do the following:

1) Find any critical points or singular points of f on the interior of D (domain).

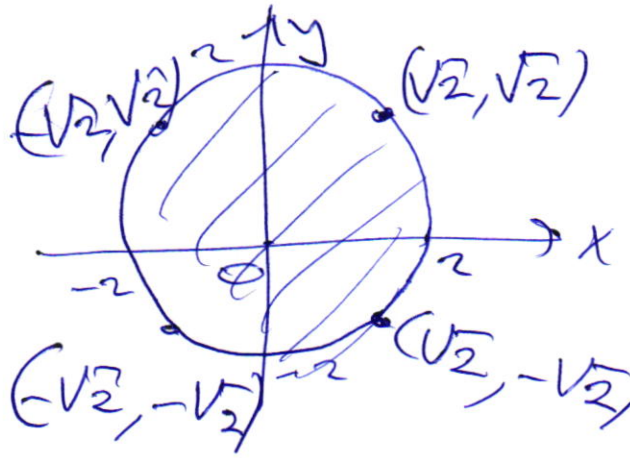
2) Find any points on the boundary of D where f might have extreme values. To do this, you can parametrize the whole boundary, or parts of it, and express f as a function of the parameters. If you break the boundary

into pieces, you must consider the endpoints of those pieces.

3) Evaluate f at all the points found in step 1 and 2.

Example Find the maximum and minimum values of $f(x,y) = 2xy$ on the closed disk $x^2 + y^2 \leq 4$.

Solution



Since f is continuous on the closed disk, f must have absolute maximum and minimum values at some points of the disk.

$f_x = 2y = 0$ $f_y = 2x = 0$ $\Rightarrow$ No singular points. Critical point $(0,0) = P_1$

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On the boundary $x^2+y^2=4$.

Parametrize this circle:

$x=2\cos t,\ y=2\sin t\ ,\ -\pi\le t\le\pi.$

Then,

$f(2\cos t, 2\sin t) = 8\cos t\sin t = g(t)$

$g(t) = 4\sin 2t = 8\cos t\sin t.$

$0 = g'(t) = -8\sin^2 t + 8\cos^2 t. \Leftrightarrow$

$\tan^2 t = 1 \Leftrightarrow t = \mp\frac{\pi}{4}$ or $\mp\frac{3\pi}{4}$.

Maximum value of g is 4,

Minimum value of g is -4

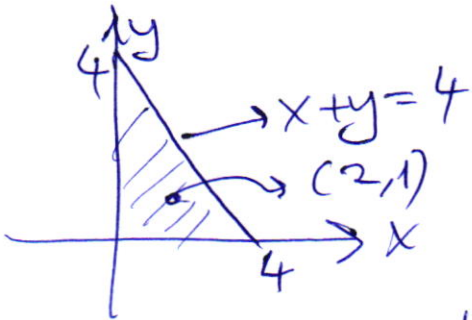
f has max. value 4 at the boundary $(\sqrt{2},\sqrt{2})$ $(-\sqrt{2},-\sqrt{2})$.

minimum value -4 at the boundary $(\sqrt{2},-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$.

Example

Find the extreme values of the function $f(x,y) = x^2y\,e^{-(x+y)}$ on the triangular region T given by $x\geq 0, y\geq 0$ and $x+y\leq 4$.

Solution.



critical points:

$f_1 = xy(2-x)e^{-(x+y)} = 0 \iff x=0, y=0, \text{ or } x=2$

$f_2 = x^2(1-y)e^{-(x+y)} = 0 \iff x=0 \text{ or } y=1.$

So, critical points are $(0,y)$ for any y, and $(2,1)$.

$f(2,1) = 4/e^3$.

on the boundary:

on coordinate axes, f is zero.

on the line $x+y=4$, $y=4-x$, $0\le x\le 4$.

$g(x) = f(x,4-x) = x^2(4-x)e^{-4}$, $0\le x\le 4$

$g(0)=g(4)=0$.

If $0<x<4$, then $g(x)>0$.

$g'(x) = (8x-3x^2)e^{-4} = 0$

$\to x=0, \; x=8/3.$

$f\left(\frac{8}{3},\frac{4}{3}\right) = g(8/3) = \frac{256}{27}e^{-4} < f(2,1)$

Hence, the maximum of f is $4/e^3$

the minimum of f is 0.

The Method of Lagrange Multipliers

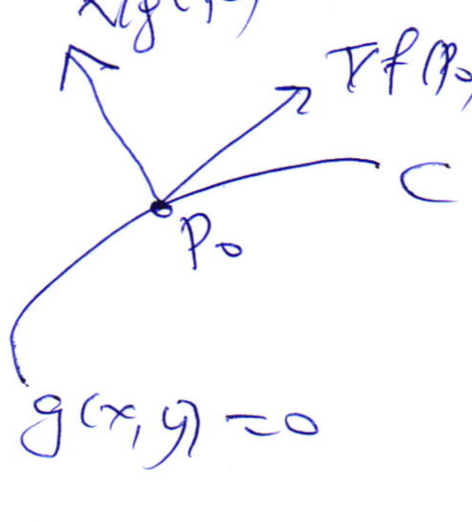
Suppose that f, g have continuous first partial derivatives near the point $P_0 = (x_0, y_0)$ on the curve C with equation $g(x,y)=0$. Suppose that, when restricted to points on C, the function $f(x,y)$ has a local maximum or minimum value at P_0. Suppose that

(i) P_0 is not an endpoint of C,

(ii) $\nabla g(P_0) \neq 0$.

Then, there exists a number λ_0 such that (x_0, y_0, λ_0) is a critical point of the Lagrange function

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y).$$



$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$\left.\begin{aligned} 0 = \frac{\partial L}{\partial x} &= f_1 + \lambda g_1 = 0 \\ 0 = \frac{\partial L}{\partial y} &= f_2 + \lambda g_2 = 0 \end{aligned}\right\} \nabla f \parallel \nabla g.$$

$$\frac{\partial L}{\partial \lambda} = g(x,y) = \text{constraint equation}$$

Example. Find the shortest distance from the origin to the curve $x^2 y = 16$.

Solution. minimize $f(x,y) = x^2 + y^2$
subject to $g(x,y) = x^2 y - 16 = 0$.

$$L(x,y,\lambda) = f(x,y) + \lambda g(x,y)$$

$$L(x,y,\lambda) = x^2 + y^2 + \lambda(x^2 y - 16)$$

(1) $0 = \frac{\partial L}{\partial x} = 2x + 2\lambda xy = 2x(1+\lambda y)$

(2) $0 = \frac{\partial L}{\partial y} = 2y + \lambda x^2$

(3) $0 = \frac{\partial L}{\partial \lambda} = x^2 y - 16$

(1) $\Rightarrow$ $x = 0$ or $\lambda y = -1$.

$x = 0 \Rightarrow$ inconsistent with (3)

So, $\lambda y = -1$. By (2),

$0 = 2y^2 + \lambda y x^2 = 2y^2 - x^2$

$\Rightarrow x = \mp\sqrt{2}\, y$.

By (3), $2y^3 = 16 \rightarrow y = 2$.

$x^2 y = 16 \Rightarrow x = \mp 2\sqrt{2}$.

So, $(\mp 2\sqrt{2}, 2)$.

$f(\mp 2\sqrt{2}, 2) = 8 + 4 = 12 = d^2$

$d = 2\sqrt{3}$ unit.

Example. Find the maximum and minimum values of $f(x,y,z) = xy^2z^3$ on the ball $x^2+y^2+z^2 \le 1$

<u>Solut.</u> $f_1 = y^2z^3 = 0 \Rightarrow y = 0$ or $z = 0$.

On boundary $x^2+y^2+z^2 = 1$

$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z)$

$= xy^2z^3 + \lambda(x^2+y^2+z^2-1)$.

$x \neq 0, y \neq 0, z \neq 0$

$0 = \dfrac{\partial L}{\partial x} = y^2z^3 + 2\lambda x \Leftrightarrow \dfrac{y^2z^3}{x} = -2\lambda$

$0 = \dfrac{\partial L}{\partial y} = 2xyz^3 + 2\lambda y \Leftrightarrow 2xz^3 = -2\lambda$

$0 = \dfrac{\partial L}{\partial z} = 3xy^2z^2 + 2\lambda z \Leftrightarrow 3xy^2z = -2\lambda$

$$0 = \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0$$

$$\frac{y^2 z^3}{x} = 2xz^3 = 3xy^2 z.$$

$$\to \cancel{y^2 z^3}\ y^2 = \cancel{2x}\ 2x^2\ z^2 = \left(\frac{3}{2}\right) y^2 = 3x^2$$

$$x^2 + 2x^2 + 3x^2 = 1 \Rightarrow$$

$$x^2 = 1/6,\ y^2 = 1/3,\ z^2 = \frac{1}{2}.$$

$$f(x,y,z) = \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2\sqrt{2}}\right) = \frac{1}{6\sqrt{3}}.$$

maximum value is $\frac{1}{6\sqrt{3}}$.

minimum value is $-\frac{1}{6\sqrt{3}}$.

Example. Find the distance from the origin to the plane $x + 2y + 2z = 3$., by Lagrange multipliers.

Solution. $f(x,y,z) = x^2 + y^2 + z^2$

constraint. $g(x,y,z) = x + 2y + 2z - 3 = 0$,

$$L(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z).$$

$$= x^2 + y^2 + z^2 + \lambda(x + 2y + 2z - 3).$$

$$\frac{\partial L}{\partial x} = 2x + \lambda = 0,$$

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$$\frac{\partial L}{\partial y} = 2y + 2\lambda = 0$$

$$\frac{\partial L}{\partial z} = 2z + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x + 2y + 2z - 3 = 0.$$

$$\Rightarrow x = -\lambda/2, \; y = -\lambda, \; z = -\lambda.$$

$$\Rightarrow -\frac{\lambda}{2} + 2(-\lambda) + 2(-\lambda) - 3 = 0.$$

$$\left(-\frac{1}{2} - 4\right)\lambda = 3.$$

$$-\frac{9}{2}\lambda = 3 \longrightarrow \lambda = -\frac{6}{9} = -\frac{2}{3}.$$

$$x = \frac{2/3}{2} = 1/3, \; y = 2/3, \; z = 2/3.$$

$$f(x,y,z) = f\left(\frac{1}{3}, 2/3, 2/3\right)$$

$$= \frac{1}{9} + \frac{4}{9} + 4/9 = 1 = d^2.$$

$d = 1$ unit.