(1) Exercises

1) If $x = u^3 + v^3$, $y = uv - v^2$ are solved for u and v in terms of x and y, evaluate

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \frac{\partial(u,v)}{\partial(x,y)}.$$

2) Find $\dfrac{\partial z}{\partial x}$ if $F(x^2 - z^2, y^2 + xz) = 0$.

3) Find $\dfrac{du}{dx}$ if $x^2 y + y^2 u - u^3 = 0$, $x^2 + yu = 1$,

4) Show that the equation

$$xy^2 + zu + v^2 = 3$$
$$x^3 z + 2y - uv = 2$$
$$xu + yv - xyz = 1$$

can be solved for x, y, z as functions of u, v near the point $P = (x,y,z,u,v) = (1,1,1,1,1)$ and find $\left(\dfrac{\partial y}{\partial u}\right)_v$ at $(u,v) = (1,1)$.

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5) Find $\dfrac{\partial x}{\partial y}$ if $xy^3 = y - z$.

6) Find $\dfrac{\partial y}{\partial z}$ if $e^{yz} - x^2 z \ln y = x$.

7) Find $\dfrac{dy}{dx}$ if $F(x, y, x^2 - y^2) = 0$.

Taylor's Formula for two-variables functions

$z = f(x,y)$ has continuous partial derivatives for all order. at (a,b).

$$f(x,y) = f(a,b) + (x-a) f_1(a,b) + (y-b) f_2(a,b)$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{11}(a,b) + 2(x-a)(y-b) f_{12}(a,b) \right.$$
$$\left. + (y-b)^2 f_{22}(a,b) \right] + \dots + R_n.$$

remainder

Example

Find a second degree polynomial approximation to the function $f(x,y) = \sqrt{x^2+y^3}$ near the point $(1,2)$ and use it to estimate the value of $\sqrt{(1.02)^2 + (0.97)^3}$.

Solution.

$$f(x,y) = \sqrt{x^2+y^3} \longrightarrow f(1,2) = 3$$

$$f_1(x,y) = \frac{x}{\sqrt{x^2+y^3}} \longrightarrow f_1(1,2) = \frac{1}{3}$$

$$f_2(x,y) = \frac{3y^2}{2\sqrt{x^2+y^3}} \longrightarrow f_2(1,2) = 2$$

(3)

$$f_{11}(x,y) = \frac{y^3}{(x^2+y^3)^{3/2}} \to f_{11}(1,2) = \frac{8}{27}$$

$$f_{12}(x,y) = \frac{-3xy^2}{2(x^2+y^3)^{3/2}} \to f_{12}(1,2) = -\frac{2}{9}$$

$$f_{22}(x,y) = \frac{12x^2y + 3y^4}{4(x^2+y^3)^{3/2}} \to f_{22}(1,2) = \frac{2}{3}$$

thus,

$$f(x,y) \cong 3 + \frac{1}{3}(x-1) + 2(y-2) +$$
$$+ \frac{1}{2!}\left[\frac{8}{27}(x-1)^2 + 2(-2/9)(x-1)(y-2) + \frac{2}{3}(y-2)^2\right]$$

$$\sqrt{(1.02)^2 + (1.97)^3} =$$
$$= f(1+0.02, 2-0.03)$$
$$\cong 3 + \frac{1}{3}(0.02) + 2(-0.03) +$$
$$+ \frac{4}{27}(0.02)^2 - \frac{2}{9}(0.02)\cdot(-0.03)$$
$$+ \frac{1}{3}(-0.03)^2$$

Example.
Find the Taylor polynomial of degree 3 in powers of x and y for the function $f(x,y) = e^{x-2y}$.

<u>Solution</u>

Let $t = x - 2y$.

$$e^{x-2y} = e^t = 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3.$$

$$= 1 + (x-2y) + \frac{1}{2!}(x-2y)^2 + \frac{1}{3!}(x-2y)^3.$$

$$= 1 + x - 2y + \frac{1}{2}x^2 - 2xy + 2y^2$$
$$+ \frac{1}{6}x^3 - x^2y + 2xy^2 - \frac{4}{3}y^3.$$

<u>Extreme Values</u>

A function $f(x)$ has a local maximum value (or a local minimum value) at a point a in its domain if $f(x) \le f(a)$ for all x in ~~the~~ the domain of f that are sufficiently close to a.

If the appropriate inequality holds for all x in the domain of f, we say that f has an absolute maximum value at a.

Such local or absolute extreme values can occur only at points of one of the following three types:

a) critical points, $f'(x)=0$.

b) singular points, where $f'(x)$ does not exist.

c) endpoints of the domain of f.

Similar situation exist for functions of several variables.

We say that a function of two variables has a local maximum or relative maximum value at the point (a,b) in its domain if $f(x,y) \leq f(a,b)$ for all points (x,y) in the domain of f that are sufficiently close to the point (a,b).

If the inequality holds for all (x,y) in the domain of f, then we say that f has a global maximum or absolute maximum value at (a,b)

Similar definitions hold for local and absolute minimum values.

<u>Theorem.</u>

A function $f(x,y)$ can have a local or absolute extreme value at a point (a,b) in its domain only if (a,b) is one of the following

(a) a critical point of f, that is, a point satisfying $\nabla f(a,b) = 0 \iff f_x(a,b) = f_y(a,b) = 0$,

(b) A singular point of f, that is, a point where $\nabla f(a,b)$ does not exist

(c) a boundary point of the domain of f.

Theorem If f is a continuous function of n variables whose domain is a closed and bounded set in $\mathbb{R}^n$, then the range of f is a bounded set of real numbers and there are points in its domain where f takes on absolute maximum and minimum values.

Example. Let $f(x,y) = x^2 + y^2$.

$f_x = 2x = 0, \quad f_y = 2y = 0 \Rightarrow$

$x = 0, \; y = 0.$

So, $(0,0)$ is a critical point.

Since $f(x,y) > f(0,0) = 0$ if $(x,y) \neq (0,0)$

f has an absolute minimum value 0 at $(0,0)$.

Example. $f(x,y) = 1 - x^2 - y^2$ has absolute maximum value, 1 at the critical point $(0,0)$.

<u>Definition.</u> We say that any interior critical point of the domain of a function f of several variables is a saddle point if f does not have a local maximum or minimum value there.

Classifying Critical points

A classification can be made by the difference

$$\Delta f = f(a+h, b+k) - f(a,b)$$

for small values of h and k where (a,b) is the critical point.

If the difference is non-negative (non-positive) for small h and k, then f must have a local maximum (minimum) at (a,b).
If the difference is negative for some points (h,k) arbitrarily near $(0,0)$ an positive for others, then f must have a saddle point at (a,b).

Theorem.
Suppose that (a,b) is a critical point of the function $f(x,y)$ that is interior to the domain of f. Suppose that the second partial derivatives of f are continuous in a neighborhood of (a,b) and have at that point the values

$$A = f_{11}(a,b), \quad B = f_{12}(a,b) = f_{21}(a,b),$$
$$C = f_{22}(a,b).$$

$$\Delta(a,b) = \begin{vmatrix} f_{11}(a,b) & f_{12}(a,b) \\ f_{21}(a,b) & f_{22}(a,b) \end{vmatrix}$$

i) $B^2 - AC < 0$, $A > 0 \Longrightarrow$
f has a local minimum ~~at~~ value at (a,b).

ii) $B^2 - AC < 0$, $A > 0 \Rightarrow$
f has a local max. value at (a,b).

iii) $B^2 - AC > 0 \Rightarrow$ f has a saddle point at (a,b)

iv) $B^2 - AC = 0 \Rightarrow$ No information

Example Find all critical points of $f(x,y) = 2x^3 - 6xy + 3y^2$ and classify them

Solution $f_1 = f_x = 6x^2 - 6y = 0$
$f_2 = f_y = -6x + 6y = 0$
$\Rightarrow$ $x = y = 0$, $x = y = 1$.
$P_1 = (0,0)$, $P_2 = (1,1)$ crit. pts
$f_{11} = 12x$, $f_{12} = -6$, $f_{22} = 6$
At $(0,0) = P_1$, $A = 0$, $B = -6$, $C = 6$.
$B^2 - AC = 36 > 0 \Rightarrow (0,0)$ is a saddle pt

At $(1,1)$,
$A = 12$, $B = -6$, $C = 6$,
$B^2 - AC = -36 < 0 \Rightarrow$
f must have a local minimum at (1,1).