Higher Derivatives

A function $f(x)$ is said to be differentiable on an interval I if it has a derivative $f'(x)$ at every point of I.

Let $f(x)$ be differentiable on an open interval I, and suppose the derivative $f'(x)$ is differentiable on I, with derivative

$$D_x f'(x).$$

Then the function $D_x f'(x)$ is called the second order derivative of $f(x)$ and is denoted by $f''(x) = \frac{d^2 f(x)}{dx^2} = f^{(2)}(x)$

If the second derivative $f''(x)$ is in turn dif-ble on I, with derivative

$$D_x f''(x)$$

we call $D_x f''(x)$ the third (order) derivative of $f(x)$, denoted by $f'''(x) = f^{(3)}(x)$.

The nth derivative of $f(x)$ denoted by $f^{(n)}(x)$ and defined by

$$f^{(n)}(x) = D_x f^{(n-1)}(x), \quad n = 1, 2, \ldots$$

(2)

The zeroth derivative of $f(x)$ is itself. $f^{(0)}(x) = f(x)$.

Example. Let $f(x) = \frac{1}{x}$, $x \neq 0$.

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}$$

$$f'''(x) = -\frac{6}{x^4}.$$

Let $y = f(x)$.

$$y' = f'(x) = \frac{dy}{dx}, \quad y'' = f''(x) = \frac{d^2y}{dx^2}, \dots$$

$$y^{(n)} = \frac{d^ny}{dx^n} = f^{(n)}(x).$$

$$\boxed{\frac{d}{dx} = D}$$

Implicit differentiation

Let $F(x,y) = 0$, $y = y(x)$. y is called an implicit function.

Example. Given that $x^2 - xy + y^3 = 1$ evaluate the derivative $x = 1$.

Solution. $2x - y - xy' + 3y^2 \cdot y' = 0$

$$y' = \frac{2x - y}{x - 3y^2}$$

For $x=1$, $1-y+y^3=1 \Rightarrow y^3=y \Rightarrow$
$y=0, 1, -1$.

$\cancel{y'(\#)=}$ $y'\big|_{x=1,y=0} = 2 = \frac{2\cdot 1-0}{1-3\cdot 0}$

$y'\big|_{x=1,y=1} = -1/2$, $y'\big|_{x=1,y=-1} = -3/2$.

<u>Ex</u>. Let $y=x^r$, $r\in\mathbb{Q}$.

$r=\frac{m}{n}$, $n\neq 0$.

$y=x^{m/n} \rightarrow y^n=x^m$.

$n\cdot y^{n-1}\cdot y' = m x^{m-1} \Rightarrow y' = \frac{m}{n}\cdot\frac{x^{m-1}}{y^{n-1}}$

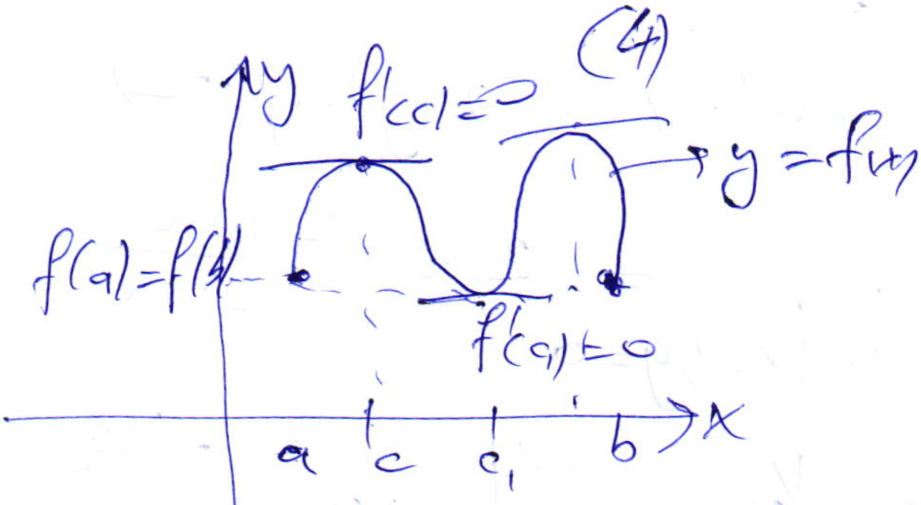
$\rightarrow y' = \frac{m}{n}\cdot\frac{x^{m-1}}{(x^{m/n})^{n-1}} = \frac{m}{n}\cdot x^{m/n-1} = r x^{r-1}$.

<u>Rolle Theorem.</u>

Let f be a continuous function on the closed interval $I=[a,b]$ and differentiable with derivative $f'(x)$ on the open interval (a,b). Suppose that $f(a)=f(b)=k$. Then, there is a point $c\in(a,b)$ such that

$f'(c)=0$.

(4)



<u>Example</u>. Let $f(x)=|x-2|$, $x\in[1,3]$.
Can we apply the Rolle's th.?

1) f is continuous on $[1,3]$
$\Leftrightarrow$ a) f is cont. $\forall x\in(1,3)$
b) f is cont. at $x=1$ from right side, $\lim_{x\to 1^+} f(x)=f(1)$
c) $\lim_{x\to 3^-} f(x)=f(3)$. Yes.

2) f is not dif.ble on $[1,3]$
At $x=2$ f' is not dif.ble

3) $f(1)=f(3)=1$.

So, we can not apply.

<u>Ex.</u> Let $f(x)=\sqrt{x(4-x)}$, $x\in[0,4]$
Can we apply Rolle's th.?
Yes $c=2$.