Metric Spaces

Definition: Let X be a non-empty set. A function $d: X \times X \longrightarrow \mathbb{R}$ is called a metric if the following conditions are satisfied:

1) $d(x,y) \geq 0$ for all $x, y \in X$,
2) $d(x,y) = 0 \Leftrightarrow x = y$.
3) $d(x,y) = d(y,x)$ for all $x, y \in X$,
4) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$. (triangle ineq.)

Then, X is called a metric space and we denote it by (X, d).

Proposition. Let (X, d) be a metric space. Then,

$$|d(x,y) - d(z,t)| \leq d(x,z) + d(y,t)$$

for every $x, y, z, t \in X$.

Proof:

$$d(x,y) \leq d(x,z) + d(z,y)$$
$$d(z,y) \leq d(z,t) + d(t,y)$$
$$\Longrightarrow d(x,y) - d(z,t) \leq d(x,z) + d(y,t)$$

$d(z,t) \leq d(z,x) + d(x,t)$ and
$d(x,t) \leq d(x,y) + d(y,t)$

$\Rightarrow -(d(x,y) - d(z,t)) \leq d(x,z) + d(y,t)$

So,
$|d(x,y) - d(z,t)| \leq d(x,z) + d(y,t)$.

Let (X,d) be a metric space and $x_1, x_2, \ldots, x_n \in X$.
By induction, we have
$d(x_1,x_n) \leq d(x_1,x_2) + \cdots + d(x_{n-1},x_n)$.

Examples.
1) Let $X = \mathbb{R}$.
Define $d(x,y) = |x-y|$ for all $x,y \in X$. d is a metric on $\mathbb{R}$.

a) $d(x,y) = |x-y| \geq 0$ for all $x,y \in \mathbb{R}$ by the definition of absolute value.

b) $d(x,y) = |x-y| = 0 \Leftrightarrow x-y=0$
$\Leftrightarrow x=y.$

c) $d(x,y) = |x-y| = |-(y-x)|$
$= |y-x| = d(y,x)$
for all $x,y \in \mathbb{R}$.

d) For any $x,y,z \in X_e = \mathbb{R}$,
$|x-y| \leq |x-z| + |z-y|$
$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

Hence, d is a metric on $\mathbb{R}$.

2) Let $X = \mathbb{R}^n$ $n \geq 1$.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$

i) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

ii) $d_p(x,y) = \left[\sum_{i=1}^{n} |x_i - y_i|^p\right]^{1/p}$

for $1 \leq p < \infty$

iii) $d_\infty(x,y) = \max\{|x_i - y_i| : i = 1, \ldots, n\}$

i, ii, iii define metrics on $\mathbb{R}^n$.

3) Let $X \neq \emptyset$ a set.

$B(X) = \{ f \mid f : X \to \mathbb{R}$ is a bounded function $\}$.

Define

$$d(f,g) = \sup_{t \in X} |f(t) - g(t)|$$

for all $f, g \in B(X)$.

$|f(t) - g(t)| \leq |f(t)| + |g(t)| \Rightarrow$

d is meaning full.

d is a metric

1) $d(f,g) \geq 0$ for all $f, g \in B(X)$ by the definition.

2) $0 \leq |f(t) - g(t)| \leq \sup_{t \in X} |f(t) - g(t)| = 0$

$\Rightarrow f = g$.

3)

$$d(f,g) = \sup_{t \in X} |f(t) - g(t)| = \sup_{t \in X} |g(t) - f(t)|$$

$$= d(g,f)$$

4) Let $f, g, h \in B(X)$.

$|f(t) - g(t)| \leq |f(t) - h(t)| + |h(t) - g(t)|$

$\Rightarrow d(f,g) \leq d(f,h) + d(h,g)$.

4) Let $C[0,1] = \{ f \mid f : [0,1] \to \mathbb{R}$ is a continuous funct.$\}$.

Define

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

for all $f, g \in C[0,1]$.

d is a metric on $C[0,1]$.

5) $c_0 = \{ x = (x_i) \mid \lim_{n \to \infty} x_n = 0 \}$

Define

$$d(x,y) = \sup_n |x_n - y_n|$$

for all $x = (x_n), y = (y_n) \in c_0$.

d is a metric.

6. $c = \{x = (x_n) : (x_n) \text{ converges}\}$.

Define
$$d(x,y) = \sup_n |x_n - y_n|$$
for every $x = (x_n), y = (y_n) \in c$.

c is a metric space

7. $l_\infty = \{x = (x_n) : (x_n) \text{ is a bounded sequence}\}$.

Define
$$d(x,y) = \sup_n |x_n - y_n|$$ for every $x = (x_n), y = (y_n) \in l_\infty$.

l_∞ is a metric space