Vector Calculus

Gradient, Divergence, curl

$$\text{grad } f(x,y,z) = \nabla f(x,y,z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Let $F(x,y,z) = P(x,y,z) i + Q(x,y,z) j + R(x,y,z) k$. be a vector field.

$$\text{div}(F) = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\text{curl}(F) = \nabla \times F$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}.$$

$$\nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) j$$

$$+ \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) k.$$

(2)

the Laplacian operator : $\nabla^2 = \nabla \cdot \nabla$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \text{div grad } \phi$$
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Where $\phi(x,y,z)$ is a scalar field

Let $F(x,y,z) = Pi + Qj + Rk$ be a vector field

$$\nabla^2 F = (\nabla^2 P)i + (\nabla^2 Q)j + (\nabla^2 R)k.$$

Theorem

1. $\nabla(\phi \psi) = \phi \nabla \psi + \psi \nabla \phi$
2. $\nabla \cdot (\phi F) = (\nabla \phi) \cdot F + \phi (\nabla \cdot F)$
3. $\nabla \times (\phi F) = (\nabla \phi) \times F + \phi (\nabla \times F)$
4. $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G - F \cdot (\nabla \times G)$
5. $\nabla \cdot (\nabla \times F) = 0$ (div curl = 0)
6. $\nabla \times (\nabla \phi) = 0$ (curl ~~grad~~ = 0)
7. $\nabla \times (\nabla \times F) = \nabla (\nabla \cdot F) - \nabla^2 F.$

curl curl = grad div − Laplacian.