(1)

The Chain Rule

1) If $z = f(x,y)$ is a function with continuous first partial derivatives, and if $x = x(t)$, $y = y(t)$ are differentiable functions at t, ~~then~~

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$

2) If $z = f(u,v)$ is a function with continuous first partial derivatives with respect to u and v, and if $x = x(s,t)$, $y = y(s,t)$ are function with continuous~~ly~~ ~~df~~ first partial derivatives w.r.t. s and t, ~~then~~ z has first partial derivatives given by

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial s} \quad \text{and}$$

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial t}.$$

We can write it by matrix,

$$\begin{pmatrix} \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{pmatrix}.$$

<u>Example.</u> If $z=\sin(x^2y)$, where $x=st^2$ and $y=s^2+\frac{1}{t}$, find

$$\frac{\partial z}{\partial s}, \quad \frac{\partial z}{\partial t} \, ?$$

a) by direct substitution and the single-variable from the Chain Rule, and

b) by using the (two-variable) Chain Rule.

Solution.

a) $$z=\sin\left((st^2)^2\left(s^2+\frac{1}{t}\right)\right)=\sin(s^4t^4+s^2t^3)$$

$$\frac{\partial z}{\partial s}=(4s^3t^4+2st^3)\cos(s^4t^4+s^2t^3)$$

$$\frac{\partial z}{\partial t}=(4s^4t^3+3s^2t^2)\cos(s^4t^4+s^2t^3).$$

b) By the Chain Rule,

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$=(2xy\cos(x^2y))\,t^2+(x^2\cos(x^2y))\,2s$$

$$=\left(2st^2\left(s^2+\frac{1}{t}\right)+2s^3t^4\right)\cos(s^4t^4+s^2t^3)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$=(2xy\cos(x^2y))\cdot 2st+(x^2\cos(x^2y))\left(-\frac{1}{t^2}\right)$$

$$=\left(2st^2\left(s^2+\frac{1}{t}\right)2st+s^2t^4\left(-\frac{1}{t^2}\right)\right)\cos(s^4t^4+s^2t^3)$$

Example. Find $\frac{\partial}{\partial x} f(x^2y, x+2y)$ and $\frac{\partial}{\partial x} f(x^2y, x+2y)$ in terms of the partial derivatives of f, assuming that these partial derivatives are continuous.

Solution.

$$\frac{\partial}{\partial x} f(x^2y, x+2y) = f_1(x^2y, x+2y) \cdot \frac{\partial}{\partial x}(x^2y) + f_2(x^2y, x+2y) \cdot \frac{\partial}{\partial x}(x+2y)$$

$$= 2xy \cdot f_1 + 1 \cdot f_2.$$

$$\frac{\partial}{\partial y} f(x^2y, x+2y) = f_1(x^2y, x+2y) \frac{\partial}{\partial y}(x^2y) + f_2(x^2y, x+2y) \cdot \frac{\partial}{\partial y}(x+2y)$$

$$= x^2 \cdot f_1 + 2 f_2.$$

Example. Express the partial derivatives of $z = h(s,t) = f(g(s,t))$ in terms of the derivative f' of f and the partial derivatives of g.

Solution.

$$h_1(s,t) = \frac{\partial z}{\partial s} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial s} = f'(g(s,t)) \cdot g_1(s,t).$$

$$h_2(s,t) = \frac{\partial z}{\partial t} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial t} = f'(g(s,t)) \cdot g_2(s,t)$$

Example. Find $\frac{dz}{dt}$, where $z = f(x,y,t)$, $x = g(t)$, $y = h(t)$. (Assume that f, g and h all have continuous derivatives).

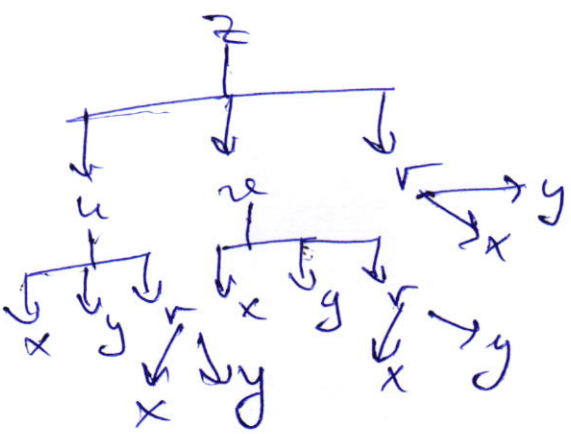
Solution. By Chain Rule,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial z}{\partial t}\cdot\frac{dt}{dt}$$

$$= f_1(x,y,t)\cdot g'(t) + f_2(x,y,t)\cdot h'(t) + f_3(x,y,t)\cdot 1$$

Example. Let $z = z(u,v,r)$, $u = u(x,y,r)$, $v = v(x,y,r)$, $r = r(x,y)$.

Find $\frac{\partial z}{\partial x} = ?$



$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x}.$$

Definition. (Homogeneous functions).
A function $f(x_1, x_2, \dots, x_n)$ is said to be positively homogeneous of degree k if for every point $(x_1, \dots, x_n)$ in its domain and every real number $t>0$, we have
$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, \dots, x_n).$$

Example. 1) $f(x,y) = x^2 + xy - y^2$ is positively homog. of degree 2.

2) $f(x,y) = x^2 + y$ not positively hom.

Theorem (Euler's theorem).
If $f(x_1, \dots, x_n)$ has continuous first partial derivatives and is positively homogeneous of degree k, then
$$\sum_{i=1}^{n} x_i f_i(x_1, \dots, x_n) = k f(x_1, \dots, x_n).$$

Proof: Differentiate
$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, \dots, x_n)$$
w.r.t. t,
$$x_1 \cdot f_1(tx_1, \dots, tx_n) + x_2 f_2(tx_1, \dots, tx_n) + \dots + x_n f_n(tx_1, \dots, tx_n) = k t^{k-1} f(x_1, \dots, x_n)$$
Substitute $t=1$, $x_1 f_1 + \dots + x_n f_n = k f$.

Example. Calculate $\frac{\partial^2}{\partial x \partial y} f(x^2-y^2, xy)$ in terms of partial derivatives of the function f. Assume that the second order partial derivatives of f are continuous.

Solution.

Let $u = x^2-y^2$, $v = xy$.

$$\frac{\partial^2 f(u,v)}{\partial x \partial y} = ?$$

$$\frac{\partial f(u,v)}{\partial y} = -2y f_1 + x f_2$$

$$\frac{\partial^2}{\partial x \partial y} f(u,v) = -2y \left[2x f_{11} + y f_{12}\right] + f_2 + x\left[2x f_{21} + y f_{22}\right]$$

~~Example.~~ Linear Approximation and Differentiability and differentials

Definition. Linearization of $f(x,y)$ at (a,b)

$$f(x,y) \approx L(x,y) = f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b).$$

Example Find an approximate value for $f(x,y)=\sqrt{2x^2+e^{2y}}$ at $(2.2, -0.2)$.

Solution

$$f_1 = \frac{2y}{\sqrt{2x^2+e^{2y}}} \longrightarrow f_1(2,0)=4/3$$

$$f_2 = \frac{e^{2y}}{\sqrt{2x^2+e^{2y}}} \longrightarrow f_2(2,0)=\frac{1}{5}$$

$$f(2,0)=3.$$

So
$$L(x,y)=f(2,0)+f_1(2,0)\cdot(x-2) + f_2(2,0)\cdot(y-0)$$

$$= 3+\frac{4}{3}(x-2)+\frac{1}{3}y$$ and

$$f(2.2,-0.2)\simeq L(2.2,-0.2)$$
$$=3+\frac{4}{3}(2.2-2)+\frac{1}{3}\cdot(-0.2)\simeq 3.2$$

Definition We say that the function $f(x,y)$ is differentiable at the point (a,b) if

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k)-f(a,b)-hf_1(a,b)-kf_2(a,b)}{\sqrt{h^2+k^2}} = 0.$$