1) Find div F, curl F for

a) $F = yz\hat{i} + xz\hat{j} + xyk$.

Solution

$\operatorname{div} F = \nabla \cdot F =$

$$= \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} k\right) \cdot \left(yz\hat{i} + xz\hat{j} + xyk\right)$$

$$= \frac{\partial}{\partial x} yz + \frac{\partial}{\partial y} xz + \frac{\partial}{\partial z} xy$$

$$= 0 + 0 + 0 = 0.$$

$\operatorname{curl} F = \nabla \times F$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right) i$$

$$- \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right)\hat{j} + \left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)k$$

$$= (x - x)i - (y - y)\hat{j} + (z - z)k$$

$$= 0i - 0\hat{j} + 0k = 0.$$

2) Let $\phi = xy + z$, $F = x^2 i - y^2 j + z^2 k$.

ii) Calculate $\nabla \cdot (\phi F) = ?$

Solution.

$$\phi F = (xy+z)x^2 i - (xy+z)y^2 j + (xy+z)z^2 k.$$

$$\nabla \cdot (\phi F) = \frac{\partial}{\partial x}\left((xy+z)x^2\right) + \frac{\partial}{\partial y}\left(-(xy+z)y^2\right) + \frac{\partial}{\partial z}\left((xy+z)z^2\right)$$

$$= y \cdot x^2 + (xy+z) \cdot 2x - \left((3xy^2 + 2yz)\right) + 2zxy + 3z^2.$$

3) Find the largest value of the product of three positive numbers whose sum is 60.

Solution. Let $x > 0$, $y > 0$, $z > 0$.

$f(x,y,z) = xyz$, $g(x,y,z) = x+y+z-60 = 0$

$$L(x,y,z,\lambda) = f + \lambda g$$

$$= xyz + \lambda(x+y+z-60)$$

$0 = \frac{\partial L}{\partial x} = yz + \lambda$, $0 = \frac{\partial L}{\partial y} = xz + \lambda$

$0 = \frac{\partial L}{\partial z} = xy + \lambda$, $\frac{\partial L}{\partial \lambda} = x + y + z - 60 = 0$

(3)

3) Find the directional derivative of $f(x,y,z) = \cosh(x+y+z)$ at $(\ln 2, 1, -1)$ in the direction $u = i - 4j + 8k$.

Solution $|u| = \sqrt{1+16+64} = 9$.

$$\frac{u}{|u|} = \frac{1}{9}[i - 4j + 8k].$$

$$D_u f(P) = \frac{u}{|u|} \cdot \nabla f(P)$$

$$\nabla f = \frac{\partial}{\partial x}\cosh(x+y+z)\, i + \frac{\partial}{\partial y}\cosh(x+y+z)\, j$$

$$+ \frac{\partial}{\partial z}\cosh(x+y+z)\, k.$$

$$= \sinh(x+y+z)\, i + \cancel{\cosh}\sinh(x+y+z)\, j$$

$$+ \sinh(x+y+z)\, k$$

$$\nabla f(\ln 2, 1, -1) = \sinh(\ln 2)\, i + \sinh(\ln 2)\, j$$

$$+ \sinh(\ln 2)\, k$$

(4)

$$D_u f(P) = \frac{1}{9}\left[i - 4j + 8k\right] \circ \left[\sinh(\ln 2) i + \sinh(\ln 2) j + \sinh(\ln 2) k\right]$$

$$= \frac{1}{9}\left[\sinh(\ln 2) - 4\sinh(\ln 2) + 8\sinh(\ln 2)\right]$$

$$= \frac{1}{9}\left[5\sinh(\ln 2)\right]$$

$$= \frac{5}{9}\,\frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{5}{9}\,\frac{(2 - 1/2)}{2}$$

$$= \frac{5}{\cancel{9}_{3}}\,\frac{\cancel{3}}{4} = \frac{5}{12}$$