# Exercises (I)

1) Write parametric equation for the line through the point $A=(1,-2,4)$ parallel to the vector $m=2i+3j-k$

2) Write the line equation through the point $A=(-3,2,0)$ parallel to the line $\frac{x-6}{-2}=\frac{y+4}{5}=\frac{z+7}{8}$.

3) Write the line equation through the points $A=(4,1,4)$ and $B=(-1,5,3)$.

4) Find the plane equation through the point $A=(2,1,-1)$ with normal vector $n=i-2j+3k$.

5) Write an equation of the plane through the points $A=(3,-1,2)$, $B=(4,-1,-1)$, and $C=(2,0,2)$.

6) Write an equation of the plane through the point $A=(2,-1,1)$ perpendicular to both planes $y=0$, and $2x-z+1=0$.

7) Find an equation of the plane through the points $A=(1,-1,-2)$, $B=(3,1,1)$ perpendicular to the plane $x-2y-3z-5=0$.

8) Find the angle between the planes $6x+3y-2z=0$, $x+2y+2z=0$.

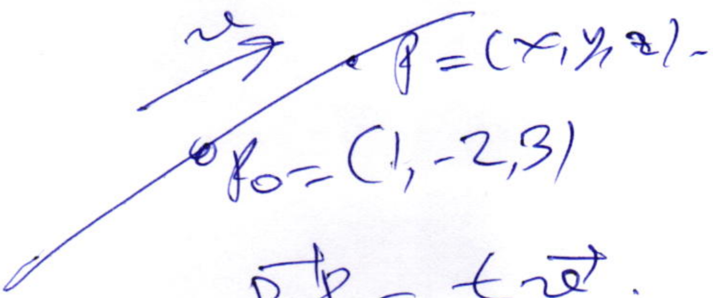
9) Find the point in which the line $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{3}$ intersects the plane $2x+3y+z-11=0$.

Example. Write the line equation through $(1,-2,3)$ perpendicular to the plane $x-2y+4z=5$.

Solution. normal vector $n$ of the plane is $n = i - 2j + 4k$

Take $n$ as direction vector of the line. So,

$n = v = i - 2j + 4k$.




$\overrightarrow{P_0P} = t\vec{v}$. for some $t \in \mathbb{R}$.

$(x-1)i + (y+2)j + (z-3)k = t(i - 2j + 4k)$.

$$\Rightarrow \left.\begin{aligned} x-1 &= t \\ y+2 &= -2t \\ z-3 &= 4t \end{aligned}\right\} \text{ line eq.}$$

Example. Find the direction vector for the line of intersection of the planes $x+y-z=0$ and $y+2z=0$.

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Solution.

$\pi_1: x+y-z=0 \Rightarrow \vec{n_1} = i+j-k$ (normal)

$\pi_2: y+2z=0 \Rightarrow \vec{n_2} = j+2k.$ ( " )

direction vector $= v = \vec{n_1} \times \vec{n_2}$

$$v = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 3i-2j+k.$$

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Function of several variables

$f: \mathbb{R}^2 \to \mathbb{R}$, a function of two variables
$(x,y) \to f(x,y) = z$

$f: \mathbb{R}^3 \to \mathbb{R}$, a function of 3 variables
$(x,y,z) \to f(x,y,z) = u.$

$\vdots$

$f: \mathbb{R}^n \to \mathbb{R}$, a function of $n$-variables
$(x_1,\dots,x_n) \to f(x_1,\dots,x_n) = w.$

Definition. A function of $n$ variables is a rule that assigns a unique real number $f(x_1,\dots,x_n)$ to each point $(x_1,\dots,x_n)$ in some subset $D(f)$ of $\mathbb{R}^n$.

$D(f)$ is called the domain of $f$.

The set of all real numbers $f(x_1,\dots,x_n)$ obtained from points in $D(f)$ is called the range of $f$.

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Definition. A level curve of a function $f(x,y)$ is the projection onto the $xy$-plane of the curve in which the graph of $f$ intersects the horizontal plane $z=c$, where $c$ is any constant in the range of $f$. Thus, the level curve

$f(x,y)=c$ a curve in the $xy$-plane.

Example. The graph of the function $z=\sqrt{1-x^2-y^2}$ is the hemisphere.

For $z=0$, $1-x^2-y=0 \Rightarrow 1=x^2+y^2$

circle is a level curve.

Definition. A level surface of a function $f(x,y,z)$ is the graph of the equation $f(x,y,z)=c$, where $c$ is any constant in the range of $f$.

Example. Level surface of the function $f(x,y,z)=x-2y+3z$ for $c=2$.

It is $x-2y+3z=2$ a plane.

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Example. Let $f(x,y) = \sqrt{4-x^2-y^2}$.

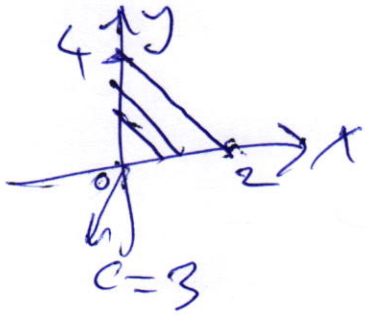
Domain of $f$ is $\{(x,y) : x^2+y^2 \leq 4\}$.

Example. The level curve of the function $f(x,y) = 3(1 - x/2 - y/4)$.

Take $c \in \mathbb{R}$, $0 \leq c \leq 3$. $\Rightarrow$

$$3(1 - x/2 - y/4) = c \Rightarrow$$

$$\frac{x}{2} + y/4 = 1 - c/3$$




Examples. Specify the domains of the functions

1) $f(x,y) = \sqrt{xy}$,

2) $f(x,y) = \sqrt{4x^2+9y^2-36}$

3) $f(x,y) = \dfrac{xy}{x^2-y^2}$

# Limits and continuity

Definition- $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L \iff$

i) every neighborhood of $(a,b)$ contains points of the domain of $f$ different from $(a,b)$,

ii) For every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon) > 0$ such that $|f(x,y) - L| < \varepsilon$ holds whenever $(x,y)$ is in the domain of $f$ and satisfies

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

## Properties of limits

1) If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) \mp g(x,y) = L \mp M.$$

2) If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M$,

then $\lim\limits_{(x,y)\to(a,b)} f(x,y) \cdot g(x,y) = L \cdot M.$

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3) If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = M \neq 0$,

then $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$.

Examples Evaluate

1) $\lim\limits_{(x,y)\to(2,1)} 2x - y^2 \overset{?}{=} 4-1=3.$

2) $\lim\limits_{(x,y)\to(\pi/3,2)} y \cdot \sin(x/y) \overset{?}{=} 2 \cdot \sin\left(\frac{\pi}{2\cdot3}\right) = 1.$

Example

Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2} = ?$

Solution

$\lim\limits_{\substack{x\to0\\y=0}} \dfrac{0}{x^2} = 0$, $\lim\limits_{\substack{y\to0\\x=0}} \dfrac{0}{y^2} = 0.$

$\lim\limits_{\substack{x\to0\\y=x}} \dfrac{2x^2}{x^2+x^2} = 1$

So, limit does not exist.

Definition. The function $f(x,y)$ is continuous at the point $(a,b)$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

## Partial Derivatives

Definition.

The first partial derivatives of the function $f(x,y)$ w.r.t. the variables $x$ and $y$ are the functions $f_1(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

and $f_2(x,y) = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$

if limits exist.

Example Let $f(x,y) = x^2 \sin y$.

Then $f_1 = 2x \sin y$

$f_2 = x^2 \cos y$.

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Notation

Let $z = f(x,y)$. Then,

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = f_1 = D_1 f(x,y).$$

$$z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = f_2 = D_2 f(x,y).$$

Values of partial derivatives

$$\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = f_1(a,b) = D_1 f(a,b).$$

$$\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = f_2(a,b) = D_2 f(a,b).$$

**Example** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = x^3y^2 + x^4y + y^4$.

Solution.

$$\frac{\partial z}{\partial x} = 3x^2y^2 + 4x^3y.$$

$$\frac{\partial z}{\partial y} = 2x^3y + x^4 + 4y^3.$$

**Example** Find $f_1(0,\pi)$ if $f(x,y) = e^{xy}\cos(x+y)$

Solution

$$f_1(x,y) = y\,e^{xy}\cos(x+y) - e^{xy}\sin(x+y).$$

$$f_1(0,\pi) = \pi e^0 \cos(\pi) - \cancel{e^0 \sin(\pi)}$$

$$= \boxed{-\pi}$$

<u>Ex.</u> If $f$ is an everywhere differentiable function of one variable, show that $z = f(x/y)$ satisfies the partial differential equation

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

Solution,

$$\frac{\partial z}{\partial x} = \frac{1}{y} f'(x/y), \quad \frac{\partial z}{\partial y} = \frac{-x}{y^2} f'(x/y).$$

$$\Rightarrow x \cdot \frac{\partial z}{\partial x} + y \cdot \frac{\partial z}{\partial y} = x \cdot \frac{1}{y} f'(x/y) + y \cdot \frac{(-x)}{y^2} f'(\tfrac{x}{y})$$

$$= 0$$

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## Higher Order Derivatives

Let $z = f(x,y)$. The second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{11} = f_{xx}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = z_{xy} = f_{12} = f_{xy}.$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{22} = f_{yy} = z_{yy}.$$

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$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{21} = f_{yx}.$$

## Examples

1) Find $f_{11}$, $f_{12}$ for $f(x,y) = x^3 y^4$.

Solution:

$f_1 = f_x = 3x^2 y^4$

$f_{11} = f_{xx} = 6x y^4.$

$f_{12} = 3x^2 \cdot 4y^3.$

2) Calculate $f_{223}$, $f_{232}$ for

$f(x, y, z) = e^{x-2y+3z}$,

Solution

$f_2 = -2\, e^{x-2y+3z}$

$f_{22} = (+4)\, e^{x-2y+3z}$

$f_{223} = 4 \cdot 3\, e^{x-2y+3z}$

$f_{23} = -6\, e^{x-2y+3z}$

$f_{232} = 12\, e^{x-2y+3z}.$