Continuity

Definition. Let f be a function defined in a neighborhood of a.
f is called continuous at $x=a$ if given any $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$.

In the definition:

1) $f(a)$ is defined.
2) $\lim_{x \to a} f(x) = L$ exists.
3) $L = f(a)$

if and only if $\lim_{x \to a} f(x) = f(a)$.

Theorem. If the functions $f(x)$ and $g(x)$ are both continuous at $x=a$, then so are the sum $f(x)+g(x)$, the difference $f(x)-g(x)$, the product $f(x) \cdot g(x)$ and the quotient $\frac{f(x)}{g(x)}$ provided that $g(a) \neq 0$ in the last case.

Example ① The polynomial function $f(x) = P(x) = a_0 + a_1 x + \dots + a_n x^n$, $(a_n \neq 0)$ is continuous for all $x \in \mathbb{R}$

② Any rational function

$$f(x) = R(x) = \frac{P(x)}{Q(x)}$$

is continuous at every point of its domain of definition. That is, f is continuous at every point where the denominator is non-zero.

$$\{x \in \mathbb{R} : Q(x) \neq 0\}.$$

③ Each of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, $\csc x$ is continuous at every point of its domain of definition

Theorem. (Limit of a composite function) If $\lim_{x \to a} f(x) = L$ and g is continuous at L, then $\lim_{x \to a} g(f(x)) = g(L)$.

Example. Evaluate $\lim_{x\to 0} \cos(\sin x)$?

Solution. Since $\sin x$ and $\cos x$ are continuous for all x, so is the composite function $\cos(\sin x)$.

$$\lim_{x\to 0} \sin x = \sin 0 = 0 = L.$$

$$\lim_{x\to 0} \cos(\sin x) = \cos 0 = 1.$$

Example. Evaluate

$$\lim_{x\to 0} \frac{1-\sqrt{x+1}}{x}.$$

$$\lim_{x\to 0} \frac{1-\sqrt{x+1}}{x} \cdot \frac{(1+\sqrt{x+1})}{1+\sqrt{x+1}} = \lim_{x\to 0} \frac{1-(x+1)}{(1+\sqrt{x+1})x}$$

$$= \lim_{x\to 0} \frac{-\cancel{x}}{(1+\sqrt{x+1})\cancel{x}} = \lim_{x\to 0} \frac{-1}{1+\sqrt{x+1}} = -\frac{1}{2}$$

Example. $\lim_{x\to 0} \frac{1-\cos x}{x} = ?$

$$\lim_{x\to 0} \frac{(1-\cos x)}{x} \cdot \frac{(1+\cos x)}{(1+\cos x)} = \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$$

$$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

$$= 1 \cdot \frac{0}{2} = 0.$$

One-sided limits

Definition ① Let f be defined in an open interval with left endpoint a.

f is said to have the right hand limit L at a if for any given $\varepsilon > 0$ we can find $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $a < x < a + \delta$. And we writ $\lim_{x \to a^+} f(x) = L$

~~f is said to have the left~~

② Let f be defined in an open interval with right endpoint a.

f is said to have the left hand limit L at a if for any $\varepsilon > 0$ we can find $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever

$a - \delta < x < a$.

And we write $\lim_{x \to a^-} f(x) = L$.

Theorem.

The limit $\lim_{x \to a} f(x)$ exists if and only if the one-sided limits

$$\lim_{x \to a^+} f(x) \text{ and } \lim_{x \to a^-} f(x)$$

both exist and are equal.

Example.

Evaluate $\lim_{x \to 0^+} \frac{|x|}{x} = ?$

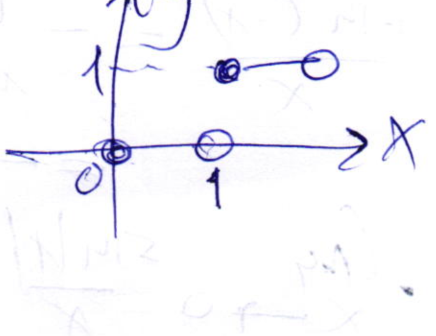
$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

Evaluate $\lim_{x \to 0^-} \frac{|x|}{x} = ?$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = -1.$$

$$\lim_{x \to 0} \frac{|x|}{x} \text{ does not exist.}$$

Example. Evaluate $\lim_{x \to 1^+} [\![x]\!] = ?$



$$\lim_{x \to 1^+} [\![x]\!] = 1.$$

$$\lim_{x \to 1^-} [\![x]\!] = 0.$$

$$\lim_{x \to n^+} [\![x]\!] = n \quad , n \in \mathbb{Z}.$$

$$\lim_{x \to n^-} [\![x]\!] = n-1$$

Example. Let $f(x) = \begin{cases} x, & x < 2 \\ 3x, & x \geq 2 \end{cases}$

Evaluate $\lim_{x \to 2} f(x) = ?$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x = 6.$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x = 2.$$

$\lim_{x \to 2} f(x)$ does not exist.

Example. Evaluate $\lim_{x \to 0} \frac{\sin|x|}{x}$

Solution

$$\lim_{\substack{x \to 0^+ \\ x > 0}} \frac{\sin x}{x} = 1.$$

$$\lim_{\substack{x \to 0^- \\ x < 0}} \frac{\sin(-x)}{x} = -\lim_{x \to 0^-} \frac{\sin x}{x} = -1$$

Since $1 \neq -1$, $\lim_{x \to 0} \frac{\sin|x|}{x}$ does not exist.