Normed Spaces

Definition. Let X be a vector space. A function $\|\cdot\|: X \to \mathbb{R}$ is called a norm if the following conditions are satisfied:

i) $\|x\| \geq 0$ for all $x \in X$.

ii) $\|x\| = 0 \iff x = 0$.

iii) $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and scalar λ.

iv) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. And X is called a normed space and denote it by $(X, \|\cdot\|)$.

Examples.

1. Let $X = \mathbb{R}$.

Define $\|x\| = |x|$ for all $x \in X$.

Let's show $\|\cdot\|$ a norm.

i) $\|x\| = |x| \geq 0$ for all $x \in X$. By the definition of ~~a~~ absolute value.

ii) $\|x\| = |x| = 0 \iff x = 0$

iii) $\|\lambda x\| = |\lambda x| = |\lambda| |x| = |\lambda| \|x\|$ for all $x \in X$, scalar λ.

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(iv) $\|x+y\| = |x+y|$

$\leq |x| + |y| = \|x\| + \|y\|$ for all $x, y \in X$.

Therefore, $\|\cdot\|$ is a norm on X.

2) Let $S \neq \emptyset$ be a set and $B(S) = \{ f \mid f : S \to \mathbb{R}$ is a bounded function$\}$.

Define $\|f\| = \sup_{x \in S} |f(x)|$ for every $f \in B(S)$.

$B(S)$ is a normed space.

i) $\|f\| = \sup_{x \in S} |f(x)| \geq 0$ ✓

ii) $0 \leq |f(x)| \leq \sup_{x \in S} |f(x)| = 0$

$\Rightarrow |f(x)| = 0 \Rightarrow f(x) = 0 \ \forall x \in S$

$\Rightarrow f = 0$.

If $f = 0$, then $f(x) = 0$, and $|f(x)| = 0 \Rightarrow \sup_{x \in S} |f(x)| = 0$

$\Rightarrow \|f\| = 0$.

(3)

iii) $\|\lambda f\| = \sup_{x\in S} |(\lambda f)(x)| = \sup_{x\in S} |\lambda f(x)|$

$= \sup_{x\in S} |\lambda|\,|f(x)| = |\lambda| \sup_{x\in S} |f(x)|$

$= |\lambda|\, \|f\|$ $\forall f \in B(S)$ and scalar λ.

(iv) Take $f, g \in B(S)$.

$\|f+g\| = \sup_{x\in S} |(f+g)(x)| = \sup_{x\in S} |f(x)+g(x)|$

$|f(x)+g(x)| \leq |f(x)| + |g(x)|$

$\leq \sup_{x\in S} |f(x)| + \sup_{x\in S} |g(x)|$

$\Rightarrow \sup_{x\in S} |f(x)+g(x)| \leq \sup_{x\in S} |f(x)| + \sup_{x\in S} |g(x)|$

$\Rightarrow \|f+g\| \leq \|f\| + \|g\|$.

Therefore, $\|\cdot\|$ is a norm on $B(S)$.

3) $c_0 = \{x = (x_n) \mid \lim_{n \to \infty} x_n = 0\}$.

Define $\|x\| = \sup_n |x_n|$ for every $x = (x_n) \in c_0$.

c_0 is a normed space.

4) $c = \{x = (x_n) \mid (x_n) \text{ converges}\}$.

Define $\|x\| = \sup_n |x_n|$ for every $x = (x_n) \in c$.

c is a normed space.

5) $l_\infty = \{x = (x_n) \mid (x_n) \text{ is a bounded sequence}\}$

Define $\|x\| = \sup_n |x_n|$ for every $x = (x_n) \in l_\infty$.

l_∞ is a normed space.

6) $C(K) = \{f \mid f : K \to \mathbb{R}$ is a continuous function on a compact Hausdorff space $K\}$

Define $\|f\| = \sup_{x \in K} |f(x)|$ for all $f \in C(K)$.

Lemma Let $0 \le a$, $0 \le b$.
If $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$, then

$$a^{1/p} \, b^{1/q} \le \frac{a}{p} + \frac{b}{q}.$$

<u>Proof:</u> If $a = 0$ or $b = 0$, then inequality holds.

So, let $0 < a$, $0 < b$.

Define $f(x) = \alpha(x-1) - (x^{a} - 1)$
for $\alpha \in [0,1]$.

Consider $f(x) \geq 0$ for all $x \in [1, \infty)$.

Hence, $x^{\alpha} - 1 \le \alpha(x-1)$.

Suppose $a \le b$. Then choose
$x = \frac{b}{a}$, $\alpha = \frac{1}{q}$.

If ~~we choose~~ $b < a$, then we choose $x = \frac{a}{b}$, $\alpha = \frac{1}{p}$.

Then, we see that inequality holds.

Let $1 \le p < \infty$.

$$l_p = \left\{ x = (x_n) : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

Holder's Inequality.

Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

If $x = (x_n) \in l_p$ and $y = (y_n) \in l_q$,

then

$$\sum_{n=1}^{\infty} |x_n y_n| \le \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p} \left[\sum_{n=1}^{\infty} |y_n|^q\right]^{1/q}.$$

holds.

Proof. Suppose $\sum_{n=1}^{\infty} |x_n|^p \ne 0$ and $\sum_{n=1}^{\infty} |y_n|^q \ne 0$. For $n \in \mathbb{N}$, by the previous lemma,

$$\frac{|x_n|\,|y_n|}{\left[\sum_{n=1}^{\infty} |x_m|^p\right]^{1/p} \left[\sum_{n=1}^{\infty} |y_m|^q\right]^{1/q}} \le \frac{1}{p} \frac{|x_n|^p}{\sum_{m=1}^{\infty} |x_m|^p} + \frac{1}{q} \frac{|y_n|^q}{\sum_{m=1}^{\infty} |y_m|^q}$$

Here, $a = \dfrac{|x_n|^p}{\sum_{m=1}^{\infty} |x_m|^p}$, $b = \dfrac{|y_n|^q}{\sum_{m=1}^{\infty} |y_m|^q}$.

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In inequality, sum $\sum_{n=1}^{\infty} \Rightarrow$

$$\frac{\sum_{n=1}^{\infty} |x_n y_n|}{\left[\sum_{m=1}^{\infty} |x_m|^p\right]^{1/p} \left[\sum_{m=1}^{\infty} |y_m|^q\right]^{1/q}} \leq \frac{1}{p} + \frac{1}{q} = 1$$

This implies

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left[\sum_{m=1}^{\infty} |x_m|^p\right]^{1/p} \left[\sum_{m=1}^{\infty} |y_m|^q\right]^{1/q}$$

Write $m=n$.

Therefore,

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=1}^{\infty} |y_n|^q\right)^{1/q}$$

Minkowski's Inequality:

Let $1 \leq p < \infty$ and $x = (x_n) \in l_p$ $y = (y_n) \in l_p$. Then

$$\left[\sum_{n=1}^{\infty} |x_n + y_n|^p\right]^{1/p} \leq \left[\sum_{n=1}^{\infty} |x_n|^p\right]^{1/p} + \left[\sum_{n=1}^{\infty} |y_n|^p\right]^{1/p}$$

holds.