(1)

A mean-value theorem:

If $f_1(x,y)$ and $f_2(x,y)$ are continuous in a neighbourhood of the point (a,b), and if the absolute values of h and k are sufficiently small, then there exist numbers θ_1 and θ_2, each between 0 and 1, such that

$$f(a+h,b+k)-f(a,b)=h\,f_1(a+\theta_1 h,b+k)+k\,f_2(a,b+\theta_2 k).$$

<u>Theorem.</u> If f_1 and f_2 are continuous in a neighbourhood of the point (a,b), then f is differentiable at (a,b).

<u>Proof.</u> Using mean-value theorem and

$$\left|\frac{h}{\sqrt{h^2+k^2}}\right|\le 1,\quad \left|\frac{k}{\sqrt{h^2+k^2}}\right|\le 1,$$ we estimate

$$\left|\frac{f(a+h,b+k)-f(a,b)-h\,f_1(a,b)-k\,f_2(a,b)}{\sqrt{h^2+k^2}}\right|$$

$$=\left|\frac{h}{\sqrt{h^2+k^2}}\left(f_1(a+\theta_1 h,b+k)-f_1(a,b)\right)+\right.$$
$$\left.+\frac{k}{\sqrt{h^2+k^2}}\left(f_2(a,b+\theta_2 k)-f_2(a,b)\right)\right|$$

$$\leq |f_1(a+\theta_1 h, b+k) - f_1(a,b)| + |f_2(a, b+\theta_2 k) - f_2(a,b)|$$

Since f_1 and f_2 are continuous at (a,b), each of these terms approaches 0 as $h \to 0$, $k \to 0$.

<u>Example</u> Calculate

$$f(x+h, y+k) - f(x,y) - f_1(x,y)h - f_2(x,y)k$$

if $f(x,y) = x^3 + xy^2$.

Solution. $f_1(x,y) = 3x^2 + y^2$, $f_2(x,y) = 2xy$,

So

$$f(x+h, y+k) - f(x,y) - f_1(x,y)h - f_2(x,y)k$$

$$= (x+h)^3 + (x+h)(y+k)^2 - x^3 - xy^2 - (3x^2+y^2)h - 2xyk$$

$$= 3xh^2 + h^3 + 2yhk + hk^2 + xk^2.$$

$(h,k) \to (0,0) \Longrightarrow \sqrt{h^2+k^2} \to 0.$

So,

$$\lim_{(h,k)\to(0,0)} \frac{3xh^2 + h^3 + 2yhk + hk^2 + xk^2}{\sqrt{h^2+k^2}} = 0.$$

f is difble.

Theorem. Let $z = f(x,y)$, where $x = u(s,t)$, $y = v(s,t)$. Suppose that

i) $u(a,b) = p$, $v(a,b) = q$

ii) the first partial derivatives of u and v exist ~~and~~ at the point (a,b) and

iii) f is dif-ble at (p,q).

Then, $z = w(s,t) = f(u(s,t), v(s,t))$ has first partial derivatives w.r.t s and t at (a,b) and

$$w_1(a,b) = f_1(p,q) \cdot u_1(a,b) + f_2(p,q) \cdot v_1(a,b)$$

$$w_2(a,b) = f_1(p,q) \cdot u_2(a,b) + f_2(p,q) \cdot v_2(a,b).$$

That is,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

Differentials

Definition. If the first partial derivatives of a function $z = f(x_1, \dots, x_n)$ exist at a point, we ~~say~~ construct a differential dz or df of the function at that point in a manner similar to that used for functions of one variable

$$dz = df = \frac{\partial z}{\partial x_1} dx_1 + \frac{\partial z}{\partial x_2} dx_2 + \dots + \frac{\partial z}{\partial x_n} dx_n$$

$$= f_1(x_1, \dots, x_n)\, dx_1 + \dots + f_n(x_1, \dots, x_n)\, dx_n.$$

Exercises

1) Find approximate value for the following functions at the points indicated

a) $f(x,y) = x^2y^3$ at $(3.1, 0.9)$

b) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ at $(3.01, 2.99)$

c) $f(x,y) = x e^{y+x^2}$ at $(2.05, -3.92)$

2) Calculate $\frac{dz}{dt}$ given that $z = txy^2$, $x = t + \ln(y+t^2)$, $y = e^t$.

3) Calculate $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ if

$z = \tan^{-1}\left(\frac{u}{v}\right)$, $u = 2x+y$, $v = 3x-y$.

4) Assume that $f(x,y)$ has continuous first partial derivatives. Find

a) $\frac{\partial}{\partial x} f(y^2, x^2)$, b) $\frac{\partial}{\partial y} f(y f(x,t), f(y,t))$

5) If $z = f(x,y)$, where $x = 2s+3t$, $y = 3s-2t$, find

$$\frac{\partial^2 z}{\partial s \partial t}$$

6) If $x = t\sin s$, $y = t\cos s$, find

$$\frac{\partial^2}{\partial s \partial t} f(x,y)$$

7) Find $\frac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2)$ in terms of partial derivatives of the function f.

8) Find all the second partial derivatives of the given functions

a) $z = x^2(1+y^2)$

b) $z = \sqrt{x^2+y^2}$

c) $z = xe^y - ye^x$

9) Find all the first partial derivatives of the given functions at the points

a) $f(x,y) = \ln(1+e^{xy})$, $P=(2,0)$

b) $f(x,y,z) = x^{(y\ln z)}$, $P=(e,2,e)$

10) If $z = \frac{x+y}{x-y}$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

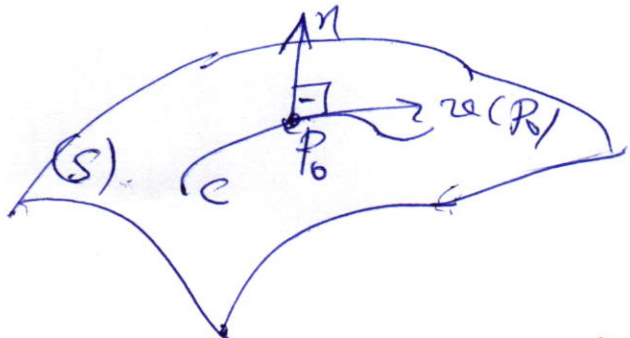
11) If $z = f(x^2+y^2)$ where f is any dif.ble function of one variable, ~~show~~ evaluate $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = ?$

12) Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

The Tangent Plane to a surface

Let S be the graph of the equation
$F(x,y,z)=0$.

Let $P_0=(x_0,y_0,z_0)$ be any fixed point of S and let the curve C with parametric equation
$$x=x(t),\ y=y(t),\ z=z(t),\ \alpha<t<\beta$$
be any curve on S going through P_0.



Assume that C is smooth which means that the functions $x(t), y(t)$ and $z(t)$ are continuously differentiable on (α,β) and satisfy the condition
$$(x'(t))^2+(y'(t))^2+(z'(t))^2\neq 0.$$

Let t_0 be the value of the parameter t corresponding to the point P_0 so that $P_0=(x(t_0),y(t_0),z(t_0))=(x_0,y_0,z_0)$ and let $F(x,y,z)$ be differentiable at P_0 with partial derivatives not all zero

Since C lies on S, we have

$$F(x(t), y(t), z(t)) = 0$$

Using chain rule at t_0,

$$F_x(x_0, y_0, z_0)\, x'(t_0) + F_y(x_0, y_0, z_0)\, y'(t_0) + F_z(x_0, y_0, z_0)\, z'(t_0) = 0$$

This means that dot product of the vectors

$$x'(t_0)\, i + y'(t_0)\, j + z'(t_0)\, k \quad \text{and}$$

$$n = F_x(x_0, y_0, z_0)\, i + F_y(x_0, y_0, z_0)\, j + F_z(x_0, y_0, z_0)\, k$$

is zero. they are perpendicular

1) Tangent Plane equation:

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

2) The normal line to a surface

$$\frac{x - x_0}{F_x(P_0)} = \frac{y - y_0}{F_y(P_0)} = \frac{z - z_0}{F_z(P_0)}.$$

Example. Find the tangent plane and normal line to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{at the point}$$

$$P_0 = (x_0, y_0, z_0)$$

Solution

$$F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$

$$F_x=\frac{2x}{a^2},\quad F_y=\frac{2y}{b^2},\quad F_z=\frac{2z}{c^2}.$$

$$F_x(P_0)=\frac{2x_0}{a^2},\quad F_y(P_0)=\frac{2y_0}{b^2},\quad F_z(P_0)=\frac{2z_0}{c^2}$$

The tangent plane is

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)+\frac{2z_0}{c^2}(z-z_0)=0$$

$$\Rightarrow\quad \frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=\underbrace{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}}_{1}$$

$$\Rightarrow\quad \frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1$$

The normal line is

$$\frac{x-x_0}{x_0/a^2}=\frac{y-y_0}{y_0/b^2}=\frac{z-z_0}{z_0/c^2}.$$