The Chain Rule:

Let f be differentiable at x, and let g be differentiable at $f(x)$. Then the composite function $g \circ f$ is differentiable at x, and its derivative is given by

$$\frac{d}{dx}(g \circ f)(x) = g'(f(x)) \cdot f'(x).$$

or Let $y = g(f(x))$, $u = f(x)$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Examples

① $y = \sin x \rightarrow y' = \cos x$

② $y = \cos x \rightarrow y' = -\sin x$

③ $y = \tan x = \frac{\sin x}{\cos x} \rightarrow y' = \sec^2 x = 1 + \tan^2 x$

④ $y = \cot x = \frac{\cos x}{\sin x} \rightarrow y' = -\csc^2 x$

⑤ $y = \sec x = \frac{1}{\cos x} \rightarrow y' = \tan x . \sec x$

⑥ $y = \csc x = \frac{1}{\sin x} \rightarrow y' = -\cot x . \csc x$

(7) $y = \frac{1}{(x^2+2)^3} \longrightarrow y' = ?$

Let $z = x^2+2$, $y = z^{-3} = \frac{1}{z^3}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{-3}{z^4} \cdot 2x = \frac{-6x}{(x^2+2)^4}$$

(8) $y = (1+5x)^{10} \longrightarrow y' = ?$

Let $u = 1+5x$.

$y = u^{10}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 10 \cdot u^9 \cdot 5$$

$$= 50 \cdot (1+5x)^9.$$

(9) $y = \sin(\sin x). \longrightarrow y' = ?$

Let $u = \sin x$, $y = \sin u$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{dy}{dx} = \cos u \cdot \cos x$$

$$= \cos(\sin x) \cdot \cos x$$

(10) $y = \cos(2x^3-1) \longrightarrow y' = ?$

Let $u = 2x^3-1$, $y = \cos u$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{dy}{dx} = -\sin u \cdot 6x^2$$

$$= -\sin(2x^3-1) \cdot 6x^2$$

Example. Let $y = x^r$, $r \in \mathbb{Q}$.
$y' = ?$

Let $r = \frac{m}{n}$, $m, n \in \mathbb{Z}$.

$$y = x^r = x^{m/n} \rightarrow y^n = x^m.$$

Differentrate,

$$n y^{n-1} \cdot y' = m x^{m-1}.$$

$$\rightarrow y' = \frac{m}{n} \cdot \frac{x^{m-1}}{y^{n-1}} = \frac{m}{n} \cdot \frac{x^{m-1}}{(x^{m/n})^{n-1}}$$

$$\rightarrow y' = \frac{m}{n} \cdot x^{\frac{m}{n} - 1}.$$

$$\boxed{y' = r \cdot x^{r-1}.}$$

Ex. $y = x^{2/3} + x^{3/4} \rightarrow y' = ?$

$$y' = \frac{2}{3} x^{2/3 - 1} + \frac{3}{4} x^{3/4 - 1}.$$

$$= \frac{2}{3} x^{-1/3} + \frac{3}{4} x^{-1/4}.$$

(4)

The tangent line to a curve

Let $y=f(x)$, $P=(a,f(a))$.

Tangent line equation:

$$y-f(a)=f'(a)(x-a).$$

If $f'(a)=0$, then $y=f(a)$.

The normal line to a curve.

Let $y=f(x)$, $P=(a,f(a))$.

$$y-f(a)=-\frac{1}{f'(a)}(x-a),\quad f'(a)\neq 0.$$