0 Review of Algebra

0.1 Sets of Real Numbers

0.2 Some Properties of Real Numbers

0.3 Exponents and Radicals

0.4 Operations with Algebraic Expressions

0.5 Factoring

0.6 Fractions

0.7 Equations, in Particular Linear Equations

0.8 Quadratic Equations

Lesley Griffith works for a yacht supply company in Antibes, France. Often, she needs to examine receipts in which only the total paid is reported and then determine the amount of the total which was French value-added tax, known as TVA for "Taxe à la Value Ajouté". The French TVA rate is 19.6%. A lot of Lesley's business comes from Italian suppliers and purchasers, so she also must deal with the similar problem of receipts containing Italian sales tax at 18%.

A problem of this kind demands a formula, but many people are able to work through a particular instance of the problem, using specified numbers, without knowing the formula. Thus if Lesley has a 200-Euro French receipt, she might reason as follows: If the item cost 100 Euros before tax, then the receipt total would be for 119.6 Euros with tax of 19.6—and then with a leap of faith—*tax in a receipt total of 200 is to 200 as 19.6 is to 119.6*. Stated mathematically,

$$\frac{\text{tax in } 200}{200} = \frac{19.6}{119.6} \approx 0.164 = 16.4\%$$

At this point it is fairly clear that the amount of TVA in a 200-Euro receipt is about 16.4% of 200 Euros, which is 32.8 Euros. In fact, many people will now guess that

$$\text{tax in } R = R\left(\frac{p}{100+p}\right)$$

gives the tax in a receipt R, when the tax rate is $p\%$. Thus, if Lesley feels comfortable with her deduction, she can multiply her Italian receipts by $\frac{18}{118}$ to determine the tax they contain.

Of course, most people do not remember formulas for very long and are uncomfortable basing a monetary calculation on a leap of faith. The purpose of this chapter is to review the algebra necessary for you to construct your own formulas, *with confidence,* as needed. In particular, we will derive Lesley's formula, without a mysterious invocation of proportion, from principles with which everybody is familiar. This usage of algebra will appear throughout the book, in the course of making *general calculations with variable quantities*.

In this chapter we will review real numbers and algebraic expressions and the basic operations on them. The chapter is designed to provide a brief review of some terms and methods of symbolic calculation. No doubt you have been exposed to much of this material before. However, because these topics are important in handling the mathematics that comes later, an immediate second exposure to them may be beneficial. Devote whatever time is necessary to the sections in which you need review.

Objective

To become familiar with sets, the classification of real numbers, and the real-number line.

0.1 Sets of Real Numbers

A **set** is a collection of objects. For example, we can speak of the set of even numbers between 5 and 11, namely, 6, 8, and 10. An object in a set is called an **element** of that set. If this sounds a little circular, do not worry. The words *set* and *element* are like *line* and *point* in plane geometry. We cannot hope to define them in more primitive terms. It is only with practice in using them that we come to understand their meaning. The situation is also rather like the way in which a child learns a first language. Without knowing any words, a child infers the meaning of a few very simple words and ultimately uses these to build a working vocabulary. None of us needs to understand the mechanics of this process in order to learn how to speak. In the same way, it is possible to learn practical mathematics without becoming embroiled in the issue of undefined primitive terms.

One way to specify a set is by listing its elements, in any order, inside braces. For example, the previous set is $\{6, 8, 10\}$, which we could denote by a letter such as A, allowing us to write $A = \{6, 8, 10\}$. Note that $\{8, 10, 6\}$ also denotes the same set, as does $\{6, 8, 10, 10\}$. *A set is determined by its elements* and neither rearrangements nor repetitions in a listing affect the set. A set A is said to be a subset of a set B if and only if every element of A is also an element of B. For example, if $A = \{6, 8, 10\}$ and $B = \{6, 8, 10, 12\}$, then A is a subset of B.

Certain sets of numbers have special names. The numbers 1, 2, 3, and so on form the set of **positive integers:**

$$\textit{set of positive integers} = \{1, 2, 3, \ldots\}$$

The three dots mean that the listing of elements is unending, although we do know what the elements are.

The positive integers, together with 0 and the **negative integers** $-1, -2, -3, \ldots$, form the set of **integers:**

$$\textit{set of integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

CAUTION!

The reason for $q \neq 0$ is that we cannot divide by zero.

The set of **rational numbers** consists of numbers, such as $\frac{1}{2}$ and $\frac{5}{3}$, that can be written as a quotient of two integers. That is, a rational number is a number that can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. (The symbol "$\neq$" is read "is not equal to.") For example, the numbers $\frac{19}{20}$, $\frac{-2}{7}$, and $\frac{-6}{-2}$ are rational. We remark that $\frac{2}{4}$, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{-4}{-8}$, 0.5, and 50% all represent the same rational number. The integer 2 is rational, since $2 = \frac{2}{1}$. In fact, every integer is rational.

Every integer is a rational number.

All rational numbers can be represented by decimal numbers that *terminate,* such as $\frac{3}{4} = 0.75$ and $\frac{3}{2} = 1.5$, or by *nonterminating repeating decimal numbers* (composed of a group of digits that repeats without end), such as $\frac{2}{3} = 0.666\ldots$, $\frac{-4}{11} = -0.3636\ldots$, and $\frac{2}{15} = 0.1333\ldots$. Numbers represented by *nonterminating nonrepeating* decimals are called **irrational numbers.** An irrational number cannot be written as an integer divided by an integer. The numbers π (pi) and $\sqrt{2}$ are examples of irrational numbers. Together, the rational numbers and the irrational numbers form the set of **real numbers.**

Every rational number is a real number.

The real numbers consist of all decimal numbers.

Real numbers can be represented by points on a line. First we choose a point on the line to represent zero. This point is called the *origin.* (See Figure 0.1.) Then a standard measure of distance, called a *unit distance,* is chosen and is successively marked off both to the right and to the left of the origin. With each point on the line we associate a directed distance, which depends on the position of the point with respect to the origin. Positions to the right of the origin are considered positive (+) and positions to the left are negative (−). For example, with the point $\frac{1}{2}$ unit to the right of the origin there

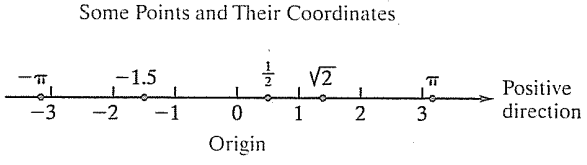


FIGURE 0.1 The real-number line.

corresponds the number $\frac{1}{2}$, which is called the **coordinate** of that point. Similarly, the coordinate of the point 1.5 units to the left of the origin is -1.5. In Figure 0.1, the coordinates of some points are marked. The arrowhead indicates that the direction to the right along the line is considered the positive direction.

To each point on the line there corresponds a unique real number, and to each real number there corresponds a unique point on the line. There is a *one-to-one correspondence* between points on the line and real numbers. We call such a line with coordinates marked off a **real-number line.** We feel free to treat real numbers as points on a real-number line and vice versa.

PROBLEMS 0.1

In Problems 1–12, determine the truth of each statement. If the statement is false, give a reason why that is so.

1. -13 is an integer.
2. $\frac{-2}{7}$ is rational.
3. -3 is a positive integer.
4. 0 is not rational.
5. $\sqrt{3}$ is rational.
6. $\frac{7}{0}$ is a rational number.
7. $\sqrt{25}$ is not a positive integer.
8. $\sqrt{2}$ is a real number.
9. $\frac{0}{0}$ is rational.
10. π is a positive integer.
11. -3 is to the right of -4 on the real-number line.
12. Every integer is positive or negative.

0.2 Some Properties of Real Numbers

Objective

To name, illustrate, and relate properties of the real numbers in terms of their operations.

We now state a few important properties of the real numbers. Let a, b, and c be real numbers.

1. The Transitive Property of Equality

If $a = b$ and $b = c$, then $a = c$.

Thus, two numbers that are both equal to a third number are equal to each other. For example, if $x = y$ and $y = 7$, then $x = 7$.

2. The Closure Properties of Addition and Multiplication

For all real numbers a and b, there are unique real numbers $a + b$ and ab.

This means that any two numbers can be added and multiplied, and the result in each case is a real number.

3. The Commutative Properties of Addition and Multiplication

$$a + b = b + a \quad \text{and} \quad ab = ba$$

This means that two numbers can be added or multiplied in any order. For example, $3 + 4 = 4 + 3$ and $(7)(-4) = (-4)(7)$.

4. The Associative Properties of Addition and Multiplication

$$a + (b + c) = (a + b) + c \quad \text{and} \quad a(bc) = (ab)c$$

This means that in addition or multiplication, numbers can be grouped in any order. For example, $2 + (3 + 4) = (2 + 3) + 4$; in both cases, the sum is 9. Similarly, $2x + (x + y) = (2x + x) + y$ and $6(\frac{1}{3} \cdot 5) = (6 \cdot \frac{1}{3}) \cdot 5$.

5. The Identity Properties

There are unique real numbers denoted 0 and 1 such that, for each real number a,

$$0 + a = a \quad \text{and} \quad 1a = a$$

6. The Inverse Properties

For each real number a, there is a unique real number denoted $-a$ such that

$$a + (-a) = 0$$

The number $-a$ is called the **negative** of a.

For example, since $6 + (-6) = 0$, the negative of 6 is -6. The negative of a number is not necessarily a negative number. For example, the negative of -6 is 6, since $(-6) + (6) = 0$. That is, the negative of -6 is 6, so we can write $-(-6) = 6$.

For each real number a, *except 0*, there is a unique real number denoted a^{-1} such that

$$a \cdot a^{-1} = 1$$

The number a^{-1} is called the **reciprocal** of a.

CAUTION!

Zero does not have a reciprocal because there is no number that when multiplied by 0 gives 1.

Thus, all numbers *except 0* have a reciprocal. You may recall that a^{-1} can be written $\frac{1}{a}$. For example, the reciprocal of 3 is $\frac{1}{3}$, since $3(\frac{1}{3}) = 1$. Hence, $\frac{1}{3}$ is the reciprocal of 3. The reciprocal of $\frac{1}{3}$ is 3, since $(\frac{1}{3})(3) = 1$. *The reciprocal of 0 is not defined.*

7. The Distributive Properties

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca$$

For example, although $2(3 + 4) = 2(7) = 14$, we can also write

$$2(3 + 4) = 2(3) + 2(4) = 6 + 8 = 14$$

Similarly,

$$(2 + 3)(4) = 2(4) + 3(4) = 8 + 12 = 20$$

and

$$x(z + 4) = x(z) + x(4) = xz + 4x$$

The distributive property can be extended to the form

$$a(b + c + d) = ab + ac + ad$$

In fact, it can be extended to sums involving any number of terms.

Subtraction is defined in terms of addition:

$$a - b \quad \text{means} \quad a + (-b)$$

where $-b$ is the negative of b. Thus, $6 - 8$ means $6 + (-8)$.

In a similar way, we define **division** in terms of multiplication. If $b \neq 0$, then

$$a \div b \quad \text{means} \quad a(b^{-1})$$

Usually we write either $\frac{a}{b}$ or a/b for $a \div b$. Since $b^{-1} = \frac{1}{b}$,

$$\frac{a}{b} = a(b^{-1}) = a\left(\frac{1}{b}\right)$$

$\frac{a}{b}$ means a times the reciprocal of b.

Thus, $\frac{3}{5}$ means 3 times $\frac{1}{5}$, where $\frac{1}{5}$ is the reciprocal of 5. Sometimes we refer to $\frac{a}{b}$ as the *ratio* of a to b. We remark that since 0 does not have a reciprocal, **division by 0 is not defined.**

The following examples show some manipulations involving the preceding properties.

EXAMPLE 1 Applying Properties of Real Numbers

a. $x(y-3z+2w)=(y-3z+2w)x$, by the commutative property of multiplication.

b. By the associative property of multiplication, $3(4\cdot 5)=(3\cdot 4)5$. Thus, the result of multiplying 3 by the product of 4 and 5 is the same as the result of multiplying the product of 3 and 4 by 5. In either case, the result is 60.

c. *Show that* $a(b\cdot c)\neq(ab)\cdot(ac)$

Solution: To show that the negation of a general statement is false, it suffices to provide a *counterexample*. Here, taking $a=2$ and $b=1=c$, we see that that $a(b\cdot c)=2$ while $(ab)\cdot(ac)=4$.

Now Work Problem 9 ◁

EXAMPLE 2 Applying Properties of Real Numbers

a. Show that $2-\sqrt{2}=-\sqrt{2}+2$.

Solution: By the definition of subtraction, $2-\sqrt{2}=2+(-\sqrt{2})$. However, by the commutative property of addition, $2+(-\sqrt{2})=-\sqrt{2}+2$. Hence, by the transitive property of equality, $2-\sqrt{2}=-\sqrt{2}+2$. Similarly, it is clear that, for any a and b, we have

$$a-b=-b+a$$

b. Show that $(8+x)-y=8+(x-y)$.

Solution: Beginning with the left side, we have

$$\begin{aligned}(8+x)-y&=(8+x)+(-y) && \text{definition of subtraction}\\ &=8+[x+(-y)] && \text{associative property}\\ &=8+(x-y) && \text{definition of subtraction}\end{aligned}$$

Hence, by the transitive property of equality,

$$(8+x)-y=8+(x-y)$$

Similarly, for all a, b, and c, we have

$$(a+b)-c=a+(b-c)$$

c. Show that $3(4x+2y+8)=12x+6y+24$.

Solution: By the distributive property,

$$3(4x+2y+8)=3(4x)+3(2y)+3(8)$$

But by the associative property of multiplication,

$$3(4x)=(3\cdot 4)x=12x \quad \text{and similarly} \quad 3(2y)=6y$$

Thus, $3(4x+2y+8)=12x+6y+24$

Now Work Problem 21 ◁

EXAMPLE 3 Applying Properties of Real Numbers

a. Show that $\dfrac{ab}{c}=a\left(\dfrac{b}{c}\right)$ for $c\neq 0$.

Solution: By the definition of division,

$$\frac{ab}{c}=(ab)\cdot\frac{1}{c} \text{ for } c\neq 0$$

But by the associative property,

$$(ab)\cdot\frac{1}{c}=a\left(b\cdot\frac{1}{c}\right)$$

However, by the definition of division, $b\cdot\frac{1}{c}=\frac{b}{c}$. Thus,

$$\frac{ab}{c}=a\left(\frac{b}{c}\right)$$

We can also show that $\frac{ab}{c}=\left(\frac{a}{c}\right)b$.

b. Show that $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$ *for* $c\neq 0$.

Solution: By the definition of division and the distributive property,

$$\frac{a+b}{c}=(a+b)\frac{1}{c}=a\cdot\frac{1}{c}+b\cdot\frac{1}{c}$$

However,

$$a\cdot\frac{1}{c}+b\cdot\frac{1}{c}=\frac{a}{c}+\frac{b}{c}$$

Hence,

$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$$

Now Work Problem 27 ◁

Finding the product of several numbers can be done by considering products of numbers taken just two at a time. For example, to find the product of x, y, and z, we could first multiply x by y and then multiply that product by z; that is, we find $(xy)z$. Alternatively, we could multiply x by the product of y and z; that is, we find $x(yz)$. The associative property of multiplication guarantees that both results are identical, regardless of how the numbers are grouped. Thus, it is not ambiguous to write xyz. This concept can be extended to more than three numbers and applies equally well to addition.

Not only should you be able to manipulate real numbers, you should also be aware of, and familiar with, the terminology involved.

The following list states important properties of real numbers that you should study thoroughly. Being able to manipulate real numbers is essential to your success in mathematics. A numerical example follows each property. **All denominators are assumed to be different from zero** (but for emphasis we have been explicit about these restrictions).

Property	*Example(s)*
1. $a-b=a+(-b)$	$2-7=2+(-7)=-5$
2. $a-(-b)=a+b$	$2-(-7)=2+7=9$
3. $-a=(-1)(a)$	$-7=(-1)(7)$
4. $a(b+c)=ab+ac$	$6(7+2)=6\cdot 7+6\cdot 2=54$
5. $a(b-c)=ab-ac$	$6(7-2)=6\cdot 7-6\cdot 2=30$
6. $-(a+b)=-a-b$	$-(7+2)=-7-2=-9$
7. $-(a-b)=-a+b$	$-(2-7)=-2+7=5$
8. $-(-a)=a$	$-(-2)=2$
9. $a(0)=0$	$2(0)=0$
10. $(-a)(b)=-(ab)=a(-b)$	$(-2)(7)=-(2\cdot 7)=2(-7)=-14$
11. $(-a)(-b)=ab$	$(-2)(-7)=2\cdot 7=14$
12. $\frac{a}{1}=a$	$\frac{7}{1}=7,\frac{-2}{1}=-2$
13. $\frac{a}{b}=a\left(\frac{1}{b}\right)$ for $b\neq 0$	$\frac{2}{7}=2\left(\frac{1}{7}\right)$

Property	*Example(s)*
14. $\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}$ for $b \neq 0$	$\frac{2}{-7} = -\frac{2}{7} = \frac{-2}{7}$
15. $\frac{-a}{-b} = \frac{a}{b}$ for $b \neq 0$	$\frac{-2}{-7} = \frac{2}{7}$
16. $\frac{0}{a} = 0$ for $a \neq 0$	$\frac{0}{7} = 0$
17. $\frac{a}{a} = 1$ for $a \neq 0$	$\frac{2}{2} = 1, \frac{-5}{-5} = 1$
18. $a\left(\frac{b}{a}\right) = b$ for $a \neq 0$	$2\left(\frac{7}{2}\right) = 7$
19. $a \cdot \frac{1}{a} = 1$ for $a \neq 0$	$2 \cdot \frac{1}{2} = 1$
20. $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $b, d \neq 0$	$\frac{2}{3} \cdot \frac{4}{5} = \frac{2 \cdot 4}{3 \cdot 5} = \frac{8}{15}$
21. $\frac{ab}{c} = \left(\frac{a}{c}\right)b = a\left(\frac{b}{c}\right)$ for $c \neq 0$	$\frac{2 \cdot 7}{3} = \frac{2}{3} \cdot 7 = 2 \cdot \frac{7}{3}$
22. $\frac{a}{bc} = \frac{a}{b} \cdot \frac{1}{c} = \frac{1}{b} \cdot \frac{a}{c}$ for $b, c \neq 0$	$\frac{2}{3 \cdot 7} = \frac{2}{3} \cdot \frac{1}{7} = \frac{1}{3} \cdot \frac{2}{7}$
23. $\frac{a}{b} = \frac{a}{b} \cdot \frac{c}{c} = \frac{ac}{bc}$ for $b, c \neq 0$	$\frac{2}{7} = \left(\frac{2}{7}\right)\left(\frac{5}{5}\right) = \frac{2 \cdot 5}{7 \cdot 5}$
24. $\frac{a}{b(-c)} = \frac{a}{(-b)(c)} = \frac{-a}{bc} =$ $\frac{-a}{(-b)(-c)} = -\frac{a}{bc}$ for $b, c \neq 0$	$\frac{2}{3(-5)} = \frac{2}{(-3)(5)} = \frac{-2}{3(5)} =$ $\frac{-2}{(-3)(-5)} = -\frac{2}{3(5)} = -\frac{2}{15}$
25. $\frac{a(-b)}{c} = \frac{(-a)b}{c} = \frac{ab}{-c} = 0$ $\frac{(-a)(-b)}{-c} = -\frac{ab}{c}$ for $c \neq 0$	$\frac{2(-3)}{5} = \frac{(-2)(3)}{5} = \frac{2(3)}{-5} =$ $\frac{(-2)(-3)}{-5} = -\frac{2(3)}{5} = -\frac{6}{5}$
26. $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ for $c \neq 0$	$\frac{2}{9} + \frac{3}{9} = \frac{2+3}{9} = \frac{5}{9}$
27. $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$ for $c \neq 0$	$\frac{2}{9} - \frac{3}{9} = \frac{2-3}{9} = \frac{-1}{9}$
28. $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ for $b, d \neq 0$	$\frac{4}{5} + \frac{2}{3} = \frac{4 \cdot 3 + 5 \cdot 2}{5 \cdot 3} = \frac{22}{15}$
29. $\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$ for $b, d \neq 0$	$\frac{4}{5} - \frac{2}{3} = \frac{4 \cdot 3 - 5 \cdot 2}{5 \cdot 3} = \frac{2}{15}$
30. $\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}$ for $b, c, d \neq 0$	$\frac{\frac{2}{3}}{\frac{7}{5}} = \frac{2}{3} \div \frac{7}{5} = \frac{2}{3} \cdot \frac{5}{7} = \frac{2 \cdot 5}{3 \cdot 7} = \frac{10}{21}$
31. $\frac{\frac{a}{b}}{c} = a \div \frac{b}{c} = a \cdot \frac{c}{b} = \frac{ac}{b}$ for $b, c \neq 0$	$\frac{\frac{2}{3}}{5} = 2 \div \frac{3}{5} = 2 \cdot \frac{5}{3} = \frac{2 \cdot 5}{3} = \frac{10}{3}$
32. $\frac{\frac{a}{b}}{c} = \frac{a}{b} \div c = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc}$ for $b, c \neq 0$	$\frac{\frac{2}{3}}{5} = \frac{2}{3} \div 5 = \frac{2}{3} \cdot \frac{1}{5} = \frac{2}{3 \cdot 5} = \frac{2}{15}$

Property 23 could be called the **fundamental principle of fractions,** which states that *multiplying or dividing both the numerator and denominator of a fraction by the same nonzero number results in a fraction that is equal to the original fraction.* Thus,

$$\frac{7}{\frac{1}{8}} = \frac{7 \cdot 8}{\frac{1}{8} \cdot 8} = \frac{56}{1} = 56$$

By Properties 28 and 23, we have

$$\frac{2}{5} + \frac{4}{15} = \frac{2 \cdot 15 + 5 \cdot 4}{5 \cdot 15} = \frac{50}{75} = \frac{2 \cdot 25}{3 \cdot 25} = \frac{2}{3}$$

We can also do this problem by converting $\frac{2}{5}$ and $\frac{4}{15}$ into fractions that have the same denominators and then using Property 26. The fractions $\frac{2}{5}$ and $\frac{4}{15}$ can be written with a common denominator of $5 \cdot 15$:

$$\frac{2}{5} = \frac{2 \cdot 15}{5 \cdot 15} \quad \text{and} \quad \frac{4}{15} = \frac{4 \cdot 5}{15 \cdot 5}$$

However, 15 is the *least* such common denominator and is called the *least common denominator* (LCD) of $\frac{2}{5}$ and $\frac{4}{15}$. Thus,

$$\frac{2}{5} + \frac{4}{15} = \frac{2 \cdot 3}{5 \cdot 3} + \frac{4}{15} = \frac{6}{15} + \frac{4}{15} = \frac{6+4}{15} = \frac{10}{15} = \frac{2}{3}$$

Similarly,

$$\begin{aligned} \frac{3}{8} - \frac{5}{12} &= \frac{3 \cdot 3}{8 \cdot 3} - \frac{5 \cdot 2}{12 \cdot 2} \qquad \text{LCD} = 24 \\ &= \frac{9}{24} - \frac{10}{24} = \frac{9-10}{24} \\ &= -\frac{1}{24} \end{aligned}$$

PROBLEMS 0.2

In Problems 1–10, determine the truth of each statement.

1. Every real number has a reciprocal.
2. The reciprocal of $\frac{7}{3}$ is $\frac{3}{7}$.
3. The negative of 7 is $\frac{-1}{7}$.
4. $1(x \cdot y) = (1 \cdot x)(1 \cdot y)$
5. $-x + y = -y + x$
6. $(x+2)(4) = 4x + 8$
7. $\frac{x+2}{2} = \frac{x}{2} + 1$
8. $3\left(\frac{x}{4}\right) = \frac{3x}{4}$
9. $2(x \cdot y) = (2x) \cdot (2y)$
10. $x(4y) = 4xy$

In Problems 11–20, state which properties of the real numbers are being used.

11. $2(x+y) = 2x + 2y$
12. $(x+5) + y = y + (x+5)$
13. $2(3y) = (2 \cdot 3)y$
14. $\frac{a}{b} = \frac{1}{b} \cdot a$
15. $5(b-a) = (a-b)(-5)$
16. $y + (x+y) = (y+x) + y$
17. $8 - y = 8 + (-y)$
18. $5(4+7) = 5(7+4)$
19. $(2+a)b = 2b + ba$
20. $(-1)(-3+4) = (-1)(-3) + (-1)(4)$

In Problems 20–26, show that the statements are true by using properties of the real numbers.

21. $2x(y-7) = 2xy - 14x$
22. $(a-b) + c = a + (c-b)$
23. $(x+y)(2) = 2x + 2y$
24. $a[b + (c+d)] = a[(d+b) + c]$
25. $x[(2y+1)+3] = 2xy + 4x$
26. $(1+a)(b+c) = b + c + ab + ac$
27. Show that $x(y - z + w) = xy - xz + xw$.
[*Hint*: $b + c + d = (b + c) + d$.]

Simplify each of the following, if possible.

28. $-2 + (-4)$
29. $-a + b$
30. $6 + (-4)$
31. $7 - 2$
32. $7 - (-4)$
33. $-5 - (-13)$
34. $-(-a) + (-b)$
35. $(-2)(9)$
36. $7(-9)$
37. $(-2)(-12)$
38. $19(-1)$
39. $\dfrac{\frac{-1}{-1}}{a}$
40. $-(-6+x)$
41. $-7(x)$
42. $-12(x-y)$

43. $-[-6+(-y)]$ **44.** $-3 \div 3a$ **45.** $-9 \div (-27)$

46. $(-a) \div (-b)$ **47.** $2(-6+2)$ **48.** $3[-2(3)+6(2)]$

49. $(-a)(-b)(-1)$ **50.** $(-12)(-12)$ **51.** $X(1)$

52. $3(x-4)$ **53.** $4(5+x)$ **54.** $-(x-y)$

55. $0(-x)$ **56.** $8\left(\frac{1}{11}\right)$ **57.** $\frac{5}{1}$

58. $\frac{14x}{21y}$ **59.** $\frac{2x}{-2}$ **60.** $\frac{2}{3} \cdot \frac{1}{x}$

61. $\frac{a}{c}(3b)$ **62.** $(5a)\left(\frac{7}{5a}\right)$ **63.** $\frac{-aby}{-ax}$

64. $\frac{a}{b} \cdot \frac{1}{c}$ **65.** $\frac{2}{x} \cdot \frac{5}{y}$ **66.** $\frac{1}{2}+\frac{1}{3}$

67. $\frac{5}{12}+\frac{3}{4}$ **68.** $\frac{3}{10}-\frac{7}{15}$ **69.** $\frac{a}{b}+\frac{c}{b}$

70. $\frac{X}{\sqrt{5}}-\frac{Y}{\sqrt{5}}$ **71.** $\frac{3}{2}-\frac{1}{4}+\frac{1}{6}$ **72.** $\frac{2}{5}-\frac{3}{8}$

73. $\frac{\frac{6}{x}}{y}$ **74.** $\frac{\frac{l}{w}}{m}$ **75.** $\frac{\frac{-x}{y^2}}{\frac{z}{xy}}$

76. $\frac{7}{0}$ **77.** $\frac{0}{7}$ **78.** $\frac{0}{0}$

0.3 Exponents and Radicals

Objective

To review positive integral exponents, the zero exponent, negative integral exponents, rational exponents, principal roots, radicals, and the procedure of rationalizing the denominator.

The product $x \cdot x \cdot x$ of 3 x's is abbreviated x^3. In general, for n a positive integer, x^n is the abbreviation for the product of n x's. The letter n in x^n is called the *exponent*, and x is called the *base*. More specifically, if n is a positive integer, we have

1. $x^n = \underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}}$
2. $x^{-n} = \frac{1}{x^n} = \frac{1}{\underbrace{x \cdot x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}}}$ for $x \neq 0$
3. $\frac{1}{x^{-n}} = x^n$ for $x \neq 0$
4. $x^0 = 1$

CAUTION!

Some authors say that 0^0 is not defined. However, $0^0 = 1$ is a consistent and often useful definition.

EXAMPLE 1 Exponents

a. $\left(\frac{1}{2}\right)^4 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{16}$

b. $3^{-5} = \frac{1}{3^5} = \frac{1}{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} = \frac{1}{243}$

c. $\frac{1}{3^{-5}} = 3^5 = 243$

d. $2^0 = 1, \pi^0 = 1, (-5)^0 = 1$

e. $x^1 = x$

Now Work Problem 5 ◁

If $r^n = x$, where n is a positive integer, then r is an nth *root* of x. Second roots, the case $n = 2$, are called *square roots*; and third roots, the case $n = 3$, are called *cube roots*. For example, $3^2 = 9$, so 3 is a square root of 9. Since $(-3)^2 = 9$, -3 is also a square root of 9. Similarly, -2 is a cube root of -8, since $(-2)^3 = -8$, while 5 is a fourth root of 625 since $5^4 = 625$.

Some numbers do not have an nth root that is a real number. For example, since the square of any real number is nonnegative: there is no real number that is a square root of -4.

The **principal nth root**[1] of x is the nth root of x that is positive if x is positive and is negative if x is negative and n is odd. We denote the principal nth root of x by $\sqrt[n]{x}$. Thus,

$$\sqrt[n]{x} \text{ is } \begin{cases} \text{positive if } x \text{ is positive} \\ \text{negative if } x \text{ is negative and } n \text{ is odd} \end{cases}$$

[1] Our use of "principal nth root" does not agree with that in advanced books.

For example, $\sqrt[2]{9} = 3$, $\sqrt[3]{-8} = -2$, and $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$. We define $\sqrt[n]{0} = 0$.

The symbol $\sqrt[n]{x}$ is called a **radical.** Here n is the *index,* x is the *radicand,* and $\sqrt{\ }$ is the *radical sign.* With principal square roots, we usually omit the index and write $\sqrt{x}$ instead of $\sqrt[2]{x}$. Thus, $\sqrt{9} = 3$.

CAUTION!

Although both 2 and -2 are square roots of 4, the **principal** square root of 4 is 2, not -2. Hence, $\sqrt{4} = 2$.

If x is positive, the expression $x^{p/q}$, where p and q are integers, with no common factors, and q is positive, is defined to be $\sqrt[q]{x^p}$. Hence,

$$x^{3/4} = \sqrt[4]{x^3};\ 8^{2/3} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$$

$$4^{-1/2} = \sqrt[2]{4^{-1}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

Here are the basic laws of exponents and radicals:[2]

Law	*Example(s)*
1. $x^m \cdot x^n = x^{m+n}$	$2^3 \cdot 2^5 = 2^8 = 256;\ x^2 \cdot x^3 = x^5$
2. $x^0 = 1$	$2^0 = 1$
3. $x^{-n} = \frac{1}{x^n}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
4. $\frac{1}{x^{-n}} = x^n$	$\frac{1}{2^{-3}} = 2^3 = 8;\ \frac{1}{x^{-5}} = x^5$
5. $\frac{x^m}{x^n} = x^{m-n} = \frac{1}{x^{n-m}}$	$\frac{2^{12}}{2^8} = 2^4 = 16;\ \frac{x^8}{x^{12}} = \frac{1}{x^4}$
6. $\frac{x^m}{x^m} = 1$	$\frac{2^4}{2^4} = 1$
7. $(x^m)^n = x^{mn}$	$(2^3)^5 = 2^{15};\ (x^2)^3 = x^6$
8. $(xy)^n = x^n y^n$	$(2 \cdot 4)^3 = 2^3 \cdot 4^3 = 8 \cdot 64 = 512$
9. $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$
10. $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$
11. $x^{1/n} = \sqrt[n]{x}$	$3^{1/5} = \sqrt[5]{3}$
12. $x^{-1/n} = \frac{1}{x^{1/n}} = \frac{1}{\sqrt[n]{x}}$	$4^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$
13. $\sqrt[n]{x}\sqrt[n]{y} = \sqrt[n]{xy}$	$\sqrt[3]{9}\sqrt[3]{2} = \sqrt[3]{18}$
14. $\frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}}$	$\frac{\sqrt[3]{90}}{\sqrt[3]{10}} = \sqrt[3]{\frac{90}{10}} = \sqrt[3]{9}$
15. $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$	$\sqrt[3]{\sqrt[4]{2}} = \sqrt[12]{2}$
16. $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$	$8^{2/3} = \sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 2^2 = 4$
17. $(\sqrt[m]{x})^m = x$	$(\sqrt[8]{7})^8 = 7$

EXAMPLE 2 **Exponents and Radicals**

CAUTION!

When computing $x^{m/n}$, it is often easier to first find $\sqrt[n]{x}$ and then raise the result to the mth power. Thus, $(-27)^{4/3} = (\sqrt[3]{-27})^4 = (-3)^4 = 81$.

a. By Law 1,

$$x^6x^8 = x^{6+8} = x^{14}$$

$$a^3b^2a^5b = a^3a^5b^2b^1 = a^8b^3$$

$$x^{11}x^{-5} = x^{11-5} = x^6$$

$$z^{2/5}z^{3/5} = z^1 = z$$

$$xx^{1/2} = x^1x^{1/2} = x^{3/2}$$

[2]Although some laws involve restrictions, they are not vital to our discussion.

b. By Law 16,

$$\left(\frac{1}{4}\right)^{3/2} = \left(\sqrt{\frac{1}{4}}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

c. $\left(-\frac{8}{27}\right)^{4/3} = \left(\sqrt[3]{\frac{-8}{27}}\right)^4 = \left(\frac{\sqrt[3]{-8}}{\sqrt[3]{27}}\right)^4$ Laws 16 and 14

$$= \left(\frac{-2}{3}\right)^4$$

$$= \frac{(-2)^4}{3^4} = \frac{16}{81}$$ Law 9

d. $(64a^3)^{2/3} = 64^{2/3}(a^3)^{2/3}$ Law 8

$= (\sqrt[3]{64})^2 a^2$ Laws 16 and 7

$= (4)^2 a^2 = 16a^2$

Now Work Problem 39 ◁

Rationalizing the denominator of a fraction is a procedure in which a fraction having a radical in its denominator is expressed as an equal fraction without a radical in its denominator. We use the fundamental principle of fractions, as Example 3 shows.

EXAMPLE 3 Rationalizing Denominators

a. $\frac{2}{\sqrt{5}} = \frac{2}{5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^{1/2} \cdot 5^{1/2}} = \frac{2 \cdot 5^{1/2}}{5^1} = \frac{2\sqrt{5}}{5}$

b. $\frac{2}{\sqrt[6]{3x^5}} = \frac{2}{\sqrt[6]{3} \cdot \sqrt[6]{x^5}} = \frac{2}{3^{1/6}x^{5/6}} = \frac{2 \cdot 3^{5/6}x^{1/6}}{3^{1/6}x^{5/6} \cdot 3^{5/6}x^{1/6}}$ for $x \neq 0$

$$= \frac{2(3^5 x)^{1/6}}{3x} = \frac{2\sqrt[6]{3^5 x}}{3x}$$

Now Work Problem 63 ◁

The following examples illustrate various applications of the laws of exponents and radicals. All denominators are understood to be nonzero.

EXAMPLE 4 Exponents

a. Eliminate negative exponents in $\frac{x^{-2}y^3}{z^{-2}}$ for $x \neq 0, z \neq 0$.

Solution: $\frac{x^{-2}y^3}{z^{-2}} = x^{-2} \cdot y^3 \cdot \frac{1}{z^{-2}} = \frac{1}{x^2} \cdot y^3 \cdot z^2 = \frac{y^3 z^2}{x^2}$

By comparing our answer with the original expression, we conclude that we can bring a factor of the numerator down to the denominator, and vice versa, by changing the sign of the exponent.

b. Simplify $\frac{x^2y^7}{x^3y^5}$ for $x \neq 0, y \neq 0$.

Solution: $\frac{x^2y^7}{x^3y^5} = \frac{y^{7-5}}{x^{3-2}} = \frac{y^2}{x}$

c. Simplify $(x^5y^8)^5$.

Solution: $(x^5y^8)^5 = (x^5)^5(y^8)^5 = x^{25}y^{40}$

d. Simplify $(x^{5/9}y^{4/3})^{18}$.

Solution: $(x^{5/9}y^{4/3})^{18} = (x^{5/9})^{18}(y^{4/3})^{18} = x^{10}y^{24}$

e. Simplify $\left(\dfrac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5$ for $z \neq 0$.

Solution: $\left(\dfrac{x^{1/5}y^{6/5}}{z^{2/5}}\right)^5 = \dfrac{(x^{1/5}y^{6/5})^5}{(z^{2/5})^5} = \dfrac{xy^6}{z^2}$

f. Simplify $\dfrac{x^3}{y^2} \div \dfrac{x^6}{y^5}$ for $x \neq 0, y \neq 0$.

Solution: $\dfrac{x^3}{y^2} \div \dfrac{x^6}{y^5} = \dfrac{x^3}{y^2} \cdot \dfrac{y^5}{x^6} = \dfrac{y^3}{x^3}$

Now Work Problem 51 ◁

EXAMPLE 5 Exponents

a. For $x \neq 0$ and $y \neq 0$, eliminate negative exponents in $x^{-1} + y^{-1}$ and simplify.

Solution: $x^{-1} + y^{-1} = \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{y+x}{xy}$

b. Simplify $x^{3/2} - x^{1/2}$ by using the distributive law.

Solution: $x^{3/2} - x^{1/2} = x^{1/2}(x-1)$

c. For $x \neq 0$, eliminate negative exponents in $7x^{-2} + (7x)^{-2}$.

Solution: $7x^{-2} + (7x)^{-2} = \dfrac{7}{x^2} + \dfrac{1}{(7x)^2} = \dfrac{7}{x^2} + \dfrac{1}{49x^2}$

d. For $x \neq 0$ and $y \neq 0$, eliminate negative exponents in $(x^{-1} - y^{-1})^{-2}$.

Solution: $(x^{-1} - y^{-1})^{-2} = \left(\dfrac{1}{x} - \dfrac{1}{y}\right)^{-2} = \left(\dfrac{y-x}{xy}\right)^{-2}$

$$= \left(\frac{xy}{y-x}\right)^2 = \frac{x^2y^2}{(y-x)^2}$$

e. Apply the distributive law to $x^{2/5}(y^{1/2} + 2x^{6/5})$.

Solution: $x^{2/5}(y^{1/2} + 2x^{6/5}) = x^{2/5}y^{1/2} + 2x^{8/5}$

Now Work Problem 41 ◁

EXAMPLE 6 Radicals

a. Simplify $\sqrt[4]{48}$.

Solution: $\sqrt[4]{48} = \sqrt[4]{16 \cdot 3} = \sqrt[4]{16}\,\sqrt[4]{3} = 2\sqrt[4]{3}$

b. Rewrite $\sqrt{2+5x}$ without using a radical sign.

Solution: $\sqrt{2+5x} = (2+5x)^{1/2}$

c. Rationalize the denominator of $\dfrac{\sqrt[5]{2}}{\sqrt[3]{6}}$ and simplify.

Solution: $\dfrac{\sqrt[5]{2}}{\sqrt[3]{6}} = \dfrac{2^{1/5} \cdot 6^{2/3}}{6^{1/3} \cdot 6^{2/3}} = \dfrac{2^{3/15}6^{10/15}}{6} = \dfrac{(2^3 6^{10})^{1/15}}{6} = \dfrac{\sqrt[15]{2^3 6^{10}}}{6}$

d. Simplify $\dfrac{\sqrt{20}}{\sqrt{5}}$.

Solution:
$$\frac{\sqrt{20}}{\sqrt{5}} = \sqrt{\frac{20}{5}} = \sqrt{4} = 2$$

Now Work Problem 71 ◁

EXAMPLE 7 Radicals

a. Simplify $\sqrt[3]{x^6y^4}$.

Solution:
$$\sqrt[3]{x^6y^4} = \sqrt[3]{(x^2)^3y^3y} = \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{y^3} \cdot \sqrt[3]{y}$$
$$= x^2y\sqrt[3]{y}$$

b. Simplify $\sqrt{\dfrac{2}{7}}$.

Solution:
$$\sqrt{\frac{2}{7}} = \sqrt{\frac{2 \cdot 7}{7 \cdot 7}} = \sqrt{\frac{14}{7^2}} = \frac{\sqrt{14}}{\sqrt{7^2}} = \frac{\sqrt{14}}{7}$$

c. Simplify $\sqrt{250} - \sqrt{50} + 15\sqrt{2}$.

Solution:
$$\sqrt{250} - \sqrt{50} + 15\sqrt{2} = \sqrt{25 \cdot 10} - \sqrt{25 \cdot 2} + 15\sqrt{2}$$
$$= 5\sqrt{10} - 5\sqrt{2} + 15\sqrt{2}$$
$$= 5\sqrt{10} + 10\sqrt{2}$$

d. If x is any real number, simplify $\sqrt{x^2}$.

Solution:
$$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, $\sqrt{2^2} = 2$ and $\sqrt{(-3)^2} = -(-3) = 3$.

Now Work Problem 75 ◁

PROBLEMS 0.3

In Problems 1–14, simplify and express all answers in terms of positive exponents.

1. $(2^3)(2^2)$ **2.** x^6x^9 **3.** a^5a^2

4. z^3zz^2 **5.** $\dfrac{x^3x^5}{y^9y^5}$ **6.** $(x^{12})^4$

7. $\dfrac{(a^3)^7}{(b^4)^5}$ **8.** $\left(\dfrac{w}{w^3}\right)^7$ **9.** $(2x^2y^3)^3$

10. $\left(\dfrac{w^2s^3}{y^2}\right)^2$ **11.** $\dfrac{x^9}{x^5}$ **12.** $\left(\dfrac{2a^4}{7b^5}\right)^6$

13. $\dfrac{(x^2)^5}{(x^3)^2x^4}$ **14.** $\dfrac{(x^2)^3(x^3)^2}{(x^3)^4}$

In Problems 15–28, evaluate the expressions.

15. $\sqrt{25}$ **16.** $\sqrt[4]{81}$ **17.** $\sqrt[7]{-128}$

18. $\sqrt[3]{0.027}$ **19.** $\sqrt[4]{\dfrac{1}{16}}$ **20.** $\sqrt[3]{-\dfrac{8}{27}}$

21. $(49)^{1/2}$ **22.** $(64)^{1/3}$ **23.** $27^{2/3}$

24. $(9)^{-5/2}$ **25.** $(32)^{-2/5}$ **26.** $(0.09)^{-1/2}$

27. $\left(\dfrac{1}{32}\right)^{4/5}$ **28.** $\left(-\dfrac{27}{64}\right)^{4/3}$

In Problems 29–40, simplify the expressions.

29. $\sqrt{50}$ **30.** $\sqrt[3]{54}$ **31.** $\sqrt[3]{2x^3}$

32. $\sqrt{4x}$ **33.** $\sqrt{25y^6}$ **34.** $\sqrt[4]{\dfrac{x}{16}}$

35. $2\sqrt{8} - 5\sqrt{27} + \sqrt[3]{128}$ **36.** $\sqrt{\dfrac{3}{13}}$

37. $(9z^4)^{1/2}$ **38.** $(27x^6)^{4/3}$

39. $\left(\dfrac{27t^3}{8}\right)^{2/3}$ **40.** $\left(\dfrac{256}{x^{12}}\right)^{-3/4}$

In Problems 41–52, write the expressions in terms of positive exponents only. Avoid all radicals in the final form. For example,

$$y^{-1}\sqrt{x} = \frac{x^{1/2}}{y}$$

41. $\dfrac{a^5b^{-3}}{c^2}$ **42.** $\sqrt[5]{x^2y^3z^{-10}}$ **43.** $2a^{-1}b^{-3}$

44. $x + y^{-1}$ **45.** $(3t)^{-2}$ **46.** $(3 - z)^{-4}$

47. $\sqrt[5]{5x^2}$ **48.** $(X^2Y^{-2})^{-2}$ **49.** $\sqrt{x} - \sqrt{y}$

50. $\dfrac{u^{-2}v^{-6}w^3}{vw^{-5}}$ **51.** $x^2\sqrt[4]{xy^{-2}z^3}$ **52.** $\sqrt[4]{a^{-3}b^{-2}a^5b^{-4}}$

In Problems 53–58, rewrite the exponential forms using radicals.

53. $(a+b-c)^{2/3}$ **54.** $(ab^2c^3)^{3/4}$

55. $x^{-4/5}$ **56.** $2x^{1/2}-(2y)^{1/2}$

57. $3w^{-3/5}-(3w)^{-3/5}$ **58.** $((x^{-5})^{1/3})^{1/4}$

In Problems 59–68, rationalize the denominators.

59. $\dfrac{6}{\sqrt{5}}$ **60.** $\dfrac{3}{\sqrt[4]{8}}$ **61.** $\dfrac{4}{\sqrt{2x}}$

62. $\dfrac{y}{\sqrt{2y}}$ **63.** $\dfrac{1}{\sqrt[3]{2a}}$ **64.** $\dfrac{2}{3\sqrt[3]{y^2}}$

65. $\dfrac{\sqrt{12}}{\sqrt{3}}$ **66.** $\dfrac{\sqrt{18}}{\sqrt{2}}$ **67.** $\dfrac{\sqrt[5]{2}}{\sqrt[4]{a^2b}}$

68. $\dfrac{\sqrt{3}}{\sqrt[3]{2}}$

In Problems 69–90, simplify. Express all answers in terms of positive exponents. Rationalize the denominator where necessary to avoid fractional exponents in the denominator.

69. $2x^2y^{-3}x^4$ **70.** $\dfrac{3}{u^{5/2}v^{1/2}}$

71. $\dfrac{\sqrt{243}}{\sqrt{3}}$ **72.** $\{[(3a^3)^2]^{-5}\}^{-2}$

73. $\dfrac{3^0}{(3^{-3}x^{1/3}y^{-3})^2}$ **74.** $\dfrac{\sqrt{s^5}}{\sqrt[3]{s^2}}$

75. $\sqrt[3]{x^2yz^3}\sqrt[3]{xy^2}$ **76.** $(\sqrt[4]{3})^8$

77. $3^2(32)^{-2/5}$ **78.** $(\sqrt[5]{a^2b})^{3/5}$

79. $(2x^{-1}y^2)^2$ **80.** $\dfrac{3}{\sqrt[3]{y}\sqrt[4]{x}}$

81. $\sqrt{x}\sqrt{x^2y^3}\sqrt{xy^2}$ **82.** $\sqrt{75k^4}$

83. $\dfrac{(a^2b^{-3}c^4)^5}{(a^{-1}c^{-2})^{-3}}$ **84.** $\sqrt[3]{7(49)}$

85. $\dfrac{(x^2)^3}{x^4}\div\left[\dfrac{x^3}{(x^3)^2}\right]^2$ **86.** $\sqrt{(-6)(-6)}$

87. $-\dfrac{8s^{-2}}{2s^3}$ **88.** $(x^3y^{-4}\sqrt{z})^5$

89. $(3x^3y^2\div 2y^2z^{-3})^4$ **90.** $\dfrac{1}{\left(\dfrac{\sqrt{2}x^{-2}}{\sqrt{16}x^3}\right)^2}$

Objective

To add, subtract, multiply, and divide algebraic expressions. To define a polynomial, to use special products, and to use long division to divide polynomials.

0.4 Operations with Algebraic Expressions

If numbers, represented by symbols, are combined by any or all of the operations of addition, subtraction, multiplication, division, exponentiation, and extraction of roots, then the resulting expression is called an *algebraic expression.*

EXAMPLE 1 **Algebraic Expressions**

a. $\sqrt[3]{\dfrac{3x^3-5x-2}{10-x}}$ is an algebraic expression in the variable x.

b. $10-3\sqrt{y}+\dfrac{5}{7+y^2}$ is an algebraic expression in the variable y.

c. $\dfrac{(x+y)^3-xy}{y}+2$ is an algebraic expression in the variables x and y.

◁

The algebraic expression $5ax^3-2bx+3$ consists of three *terms:* $+5ax^3$, $-2bx$, and $+3$. Some of the *factors* of the first term, $5ax^3$, are $5, a, x, x^2, x^3, 5ax$, and ax^2. Also, $5a$ is the coefficient of x^3, and 5 is the *numerical coefficient* of ax^3. If a and b represent fixed numbers throughout a discussion, then a and b are called *constants.*

Algebraic expressions with exactly one term are called *monomials.* Those having exactly two terms are *binomials,* and those with exactly three terms are *trinomials.* Algebraic expressions with more than one term are called *multinomials.* Thus, the multinomial $2x-5$ is a binomial; the multinomial $3\sqrt{y}+2y-4y^2$ is a trinomial.

A *polynomial in* x is an algebraic expression of the form[3]

$$c_nx^n+c_{n-1}x^{n-1}+\cdots+c_1x+c_0$$

where n is a nonnegative integer and the coefficients $c_0, c_1, \ldots, c_n$ are constants with $c_n\neq 0$. We call n the *degree* of the polynomial. Hence, $4x^3-5x^2+x-2$ is a polynomial

CAUTION!

The words *polynomial* and *multinomial* should not be used interchangeably. A polynomial is a special kind of multinomial. For example, $\sqrt{x}+2$ is a multinomial, but not a polynomial. On the other hand, $x+2$ is a polynomial and hence a multinomial.

[3]The three dots indicate all other terms that are understood to be included in the sum.

in x of degree 3, and $y^5 - 2$ is a polynomial in y of degree 5. A nonzero constant is a polynomial of degree zero; thus, 5 is a polynomial of degree zero. The constant 0 is considered to be a polynomial; however, no degree is assigned to it.

In the following examples, we illustrate operations with algebraic expressions.

EXAMPLE 2 Adding Algebraic Expressions

Simplify $(3x^2y - 2x + 1) + (4x^2y + 6x - 3)$.

Solution: We first remove the parentheses. Next, using the commutative property of addition, we gather all similar terms together. *Similar terms* are terms that differ only by their numerical coefficients. In this example, $3x^2y$ and $4x^2y$ are similar, as are the pairs $-2x$ and $6x$, and 1 and -3. Thus,

$$\begin{aligned}(3x^2y - 2x + 1) + (4x^2y + 6x - 3) &= 3x^2y - 2x + 1 + 4x^2y + 6x - 3 \\ &= 3x^2y + 4x^2y - 2x + 6x + 1 - 3\end{aligned}$$

By the distributive property,

$$3x^2y + 4x^2y = (3 + 4)x^2y = 7x^2y$$

and

$$-2x + 6x = (-2 + 6)x = 4x$$

Hence, $(3x^2y - 2x + 1) + (4x^2y + 6x - 3) = 7x^2y + 4x - 2$

Now Work Problem 3 ◁

EXAMPLE 3 Subtracting Algebraic Expressions

Simplify $(3x^2y - 2x + 1) - (4x^2y + 6x - 3)$.

Solution: Here we apply the definition of subtraction and the distributive property:

$$\begin{aligned}&(3x^2y - 2x + 1) - (4x^2y + 6x - 3) \\ &= (3x^2y - 2x + 1) + (-1)(4x^2y + 6x - 3) \\ &= (3x^2y - 2x + 1) + (-4x^2y - 6x + 3) \\ &= 3x^2y - 2x + 1 - 4x^2y - 6x + 3 \\ &= 3x^2y - 4x^2y - 2x - 6x + 1 + 3 \\ &= (3 - 4)x^2y + (-2 - 6)x + 1 + 3 \\ &= -x^2y - 8x + 4\end{aligned}$$

Now Work Problem 13 ◁

EXAMPLE 4 Removing Grouping Symbols

Simplify $3\{2x[2x + 3] + 5[4x^2 - (3 - 4x)]\}$.

Solution: We first eliminate the innermost grouping symbols (the parentheses). Then we repeat the process until all grouping symbols are removed—combining similar terms whenever possible. We have

$$\begin{aligned}3\{2x[2x + 3] + 5[4x^2 - (3 - 4x)]\} &= 3\{2x[2x + 3] + 5[4x^2 - 3 + 4x]\} \\ &= 3\{4x^2 + 6x + 20x^2 - 15 + 20x\} \\ &= 3\{24x^2 + 26x - 15\} \\ &= 72x^2 + 78x - 45\end{aligned}$$

Observe that properly paired parentheses are the only grouping symbols needed.

$$3\{2x[2x+3]+5[4x^2-(3-4x)]\} = 3(2x(2x+3)+5(4x^2-(3-4x)))$$

The optional use of brackets and braces sometimes adds clarity.

Now Work Problem 15 ◁

The distributive property is the key tool in multiplying expressions. For example, to multiply $ax + c$ by $bx + d$ we can consider $ax + c$ to be a single number and then use the distributive property:

$$(ax+c)(bx+d) = (ax+c)bx + (ax+c)d$$

Using the distributive property again, we have

$$\begin{aligned}(ax+c)bx + (ax+c)d &= abx^2 + cbx + adx + cd \\ &= abx^2 + (ad+cb)x + cd\end{aligned}$$

Thus, $(ax + c)(bx + d) = abx^2 + (ad + cb)x + cd$. In particular, if $a = 2, b = 1$, $c = 3$, and $d = -2$, then

$$\begin{aligned}(2x+3)(x-2) &= 2(1)x^2 + [2(-2)+3(1)]x + 3(-2) \\ &= 2x^2 - x - 6\end{aligned}$$

We now give a list of special products that may be obtained from the distributive property and are useful in multiplying algebraic expressions.

Special Products

1. $x(y+z) = xy + xz$	distributive property
2. $(x+a)(x+b) = x^2 + (a+b)x + ab$	
3. $(ax+c)(bx+d) = abx^2 + (ad+cb)x + cd$	
4. $(x+a)^2 = x^2 + 2ax + a^2$	square of a sum
5. $(x-a)^2 = x^2 - 2ax + a^2$	square of a difference
6. $(x+a)(x-a) = x^2 - a^2$	product of sum and difference
7. $(x+a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$	cube of a sum
8. $(x-a)^3 = x^3 - 3ax^2 + 3a^2x - a^3$	cube of a difference

EXAMPLE 5 **Special Products**

a. By Rule 2,

$$\begin{aligned}(x+2)(x-5) &= [x+2][x+(-5)] \\ &= x^2 + (2-5)x + 2(-5) \\ &= x^2 - 3x - 10\end{aligned}$$

b. By Rule 3,

$$\begin{aligned}(3z+5)(7z+4) &= 3\cdot 7z^2 + (3\cdot 4 + 5\cdot 7)z + 5\cdot 4 \\ &= 21z^2 + 47z + 20\end{aligned}$$

c. By Rule 5,

$$\begin{aligned}(x-4)^2 &= x^2 - 2(4)x + 4^2 \\ &= x^2 - 8x + 16\end{aligned}$$

d. By Rule 6,

$$\begin{aligned}(\sqrt{y^2+1}+3)(\sqrt{y^2+1}-3) &= (\sqrt{y^2+1})^2 - 3^2 \\ &= (y^2+1) - 9 \\ &= y^2 - 8\end{aligned}$$

e. By Rule 7,

$$\begin{aligned}(3x+2)^3 &= (3x)^3 + 3(2)(3x)^2 + 3(2)^2(3x) + (2)^3 \\ &= 27x^3 + 54x^2 + 36x + 8\end{aligned}$$

Now Work Problem 19 ◁

EXAMPLE 6 **Multiplying Multinomials**

Find the product $(2t-3)(5t^2+3t-1)$.

Solution: We treat $2t-3$ as a single number and apply the distributive property twice:

$$\begin{aligned}(2t-3)(5t^2+3t-1) &= (2t-3)5t^2 + (2t-3)3t - (2t-3)1 \\ &= 10t^3 - 15t^2 + 6t^2 - 9t - 2t + 3 \\ &= 10t^3 - 9t^2 - 11t + 3\end{aligned}$$

Now Work Problem 35 ◁

In Example 3(b) of Section 0.2, we showed that $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$. Similarly, $\dfrac{a-b}{c} = \dfrac{a}{c} - \dfrac{b}{c}$. Using these results, we can divide a multinomial by a monomial by dividing each term in the multinomial by the monomial.

EXAMPLE 7 **Dividing a Multinomial by a Monomial**

a. $\dfrac{x^3+3x}{x} = \dfrac{x^3}{x} + \dfrac{3x}{x} = x^2 + 3$

b. $$\begin{aligned}\frac{4z^3-8z^2+3z-6}{2z} &= \frac{4z^3}{2z} - \frac{8z^2}{2z} + \frac{3z}{2z} - \frac{6}{2z} \\ &= 2z^2 - 4z + \frac{3}{2} - \frac{3}{z}\end{aligned}$$

Now Work Problem 47 ◁

Long Division

To divide a polynomial by a polynomial, we use so-called long division when the degree of the divisor is less than or equal to the degree of the dividend, as the next example shows.

EXAMPLE 8 **Long Division**

Divide $2x^3 - 14x - 5$ by $x - 3$.

Solution: Here $2x^3 - 14x - 5$ is the *dividend* and $x - 3$ is the *divisor*. To avoid errors, it is best to write the dividend as $2x^3 + 0x^2 - 14x - 5$. Note that the powers of x are in

decreasing order. We have

$$\begin{array}{r} 2x^2+6x+4 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x-3\overline{)2x^3+0x^2-14x-5} \leftarrow \text{dividend} \\ \underline{2x^3-6x^2} \\ 6x^2-14x \\ \underline{6x^2-18x} \\ 4x-5 \\ \underline{4x-12} \\ 7 \leftarrow \text{remainder} \end{array}$$

Note that we divided x (the first term of the divisor) into $2x^3$ and got $2x^2$. Then we multiplied $2x^2$ by $x-3$, getting $2x^3-6x^2$. After subtracting $2x^3-6x^2$ from $2x^3+0x^2$, we obtained $6x^2$ and then "brought down" the term $-14x$. This process is continued until we arrive at 7, the *remainder*. We always stop when the remainder is 0 or is a polynomial whose degree is less than the degree of the divisor. Our answer can be written as

$$2x^2+6x+4+\frac{7}{x-3}$$

That is, the answer to the question

$$\frac{\text{dividend}}{\text{divisor}}=?$$

has the form

$$\text{quotient}+\frac{\text{remainder}}{\text{divisor}}$$

A way of checking a division is to verify that

$$(\text{quotient})(\text{divisor})+\text{remainder}=\text{dividend}$$

By using this equation, you should be able to verify the result of the example.

Now Work Problem 51 ◁

PROBLEMS 0.4

Perform the indicated operations and simplify.

1. $(8x-4y+2)+(3x+2y-5)$
2. $(4a^2-2ab+3)+(5c-3ab+7)$
3. $(8t^2-6s^2)+(4s^2-2t^2+6)$
4. $(\sqrt{x}+2\sqrt{x})+(\sqrt{x}+3\sqrt{x})$
5. $(\sqrt{a}+2\sqrt{3b})-(\sqrt{c}-3\sqrt{3b})$
6. $(3a+7b-9)-(5a+9b+21)$
7. $(7x^2+5xy+\sqrt{2})-(2z-2xy+\sqrt{2})$
8. $(\sqrt{x}+2\sqrt{x})-(\sqrt{x}+3\sqrt{x})$
9. $(\sqrt{x}+\sqrt{2y})-(\sqrt{x}+\sqrt{3z})$
10. $4(2z-w)-3(w-2z)$
11. $3(3x+3y-7)-3(8x-2y+2)$
12. $(4s-5t)+(-2s-5t)+(s+9)$
13. $5(x^2-y^2)+x(y-3x)-4y(2x+7y)$
14. $2-[3+4(s-3)]$
15. $2\{3[3(x^2+2)-2(x^2-5)]\}$
16. $4\{3(t+5)-t[1-(t+1)]\}$
17. $-2(3u^2(2u+2)-2(u^2-(5-2u)))$
18. $-\{-3[2a+2b-2]+5[2a+3b]-a[2(b+5)]\}$
19. $(x+4)(x+5)$
20. $(u+2)(u+5)$
21. $(w+2)(w-5)$
22. $(x-4)(x+7)$
23. $(2x+3)(5x+2)$
24. $(t-5)(2t+7)$
25. $(X+2Y)^2$
26. $(2x-1)^2$
27. $(7-X)^2$
28. $(\sqrt{x}-1)(2\sqrt{x}+5)$
29. $(\sqrt{3x}+5)^2$
30. $(\sqrt{y}-3)(\sqrt{y}+3)$
31. $(2s-1)(2s+1)$
32. $(a^2+2b)(a^2-2b)$
33. $(x^2-3)(x+4)$
34. $(x+1)(x^2+x+3)$
35. $(x^2-4)(3x^2+2x-1)$
36. $(3y-2)(4y^3+2y^2-3y)$
37. $t\{3(t+2)(t-4)+5[3t(t-7)]\}$
38. $[(2z+1)(2z-1)](4z^2+1)$
39. $(x+y+2)(3x+2y-4)$
40. $(x^2+x+1)^2$
41. $(2a+3)^3$
42. $(2a-3)^3$
43. $(2x-3)^3$
44. $(x+2y)^3$
45. $\dfrac{z^2-18z}{z}$
46. $\dfrac{2x^3-7x+4}{x}$
47. $\dfrac{6u^5+9u^3-1}{3u^2}$
48. $\dfrac{(3y-4)-(9y+5)}{3y}$

49. $(x^2 + 5x - 3) \div (x + 5)$

50. $(x^2 - 5x + 4) \div (x - 4)$

51. $(3x^3 - 2x^2 + x - 3) \div (x + 2)$

52. $(x^4 + 3x^2 + 2) \div (x + 1)$

53. $x^3 \div (x + 2)$

54. $(6x^2 + 8x + 1) \div (2x + 3)$

55. $(3x^2 - 4x + 3) \div (3x + 2)$

56. $(z^3 + z^2 + z) \div (z^2 - z + 1)$

Objective

To state the basic rules for factoring and apply them to factor expressions.

0.5 Factoring

If two or more expressions are multiplied together, the expressions are called *factors* of the product. Thus, if $c = ab$, then a and b are both factors of the product c. The process by which an expression is written as a product of its factors is called *factoring*.

Listed next are rules for factoring expressions, most of which arise from the special products discussed in Section 0.4. The right side of each identity is the factored form of the left side.

Rules for Factoring

1. $xy + xz = x(y + z)$	common factor
2. $x^2 + (a + b)x + ab = (x + a)(x + b)$	
3. $abx^2 + (ad + cb)x + cd = (ax + c)(bx + d)$	
4. $x^2 + 2ax + a^2 = (x + a)^2$	perfect-square trinomial
5. $x^2 - 2ax + a^2 = (x - a)^2$	perfect-square trinomial
6. $x^2 - a^2 = (x + a)(x - a)$	difference of two squares
7. $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$	sum of two cubes
8. $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$	difference of two cubes

When factoring a polynomial, we usually choose factors that themselves are polynomials. For example, $x^2 - 4 = (x+2)(x-2)$. We will not write $x-4$ as $(\sqrt{x}+2)(\sqrt{x}-2)$ unless it allows us to simplify other calculations.

Always factor as completely as you can. For example,

$$2x^2 - 8 = 2(x^2 - 4) = 2(x + 2)(x - 2)$$

EXAMPLE 1 **Common Factors**

a. Factor $3k^2x^2 + 9k^3x$ completely.

Solution: Since $3k^2x^2 = (3k^2x)(x)$ and $9k^3x = (3k^2x)(3k)$, each term of the original expression contains the common factor $3k^2x$. Thus, by Rule 1,

$$3k^2x^2 + 9k^3x = 3k^2x(x + 3k)$$

Note that although $3k^2x^2 + 9k^3x = 3(k^2x^2 + 3k^3x)$, we do not say that the expression is completely factored, since $k^2x^2 + 3k^3x$ can still be factored.

b. Factor $8a^5x^2y^3 - 6a^2b^3yz - 2a^4b^4xy^2z^2$ completely.

Solution: $8a^5x^2y^3 - 6a^2b^3yz - 2a^4b^4xy^2z^2 = 2a^2y(4a^3x^2y^2 - 3b^3z - a^2b^4xyz^2)$

Now Work Problem 5 ◁

EXAMPLE 2 **Factoring Trinomials**

a. Factor $3x^2 + 6x + 3$ completely.

Solution: First we remove a common factor. Then we factor the resulting expression completely. Thus, we have

$$\begin{aligned} 3x^2 + 6x + 3 &= 3(x^2 + 2x + 1) \\ &= 3(x + 1)^2 \end{aligned}$$ Rule 4

b. Factor $x^2 - x - 6$ completely.

Solution: If this trinomial factors into the form $(x + a)(x + b)$, which is a product of two binomials, then we must determine the values of a and b. Since $(x + a)(x + b) = x^2 + (a + b)x + ab$, it follows that

$$x^2 + (-1)x + (-6) = x^2 + (a + b)x + ab$$

By equating corresponding coefficients, we want

$$a + b = -1 \quad \text{and} \quad ab = -6$$

If $a = -3$ and $b = 2$, then both conditions are met and hence

$$x^2 - x - 6 = (x - 3)(x + 2)$$

As a check, it is wise to multiply the right side to see if it agrees with the left side.

c. Factor $x^2 - 7x + 12$ completely.

Solution: $x^2 - 7x + 12 = (x - 3)(x - 4)$

Now Work Problem 9 ◁

EXAMPLE 3 Factoring

The following is an assortment of expressions that are completely factored. The numbers in parentheses refer to the rules used.

a. $x^2 + 8x + 16 = (x + 4)^2$ (4)

b. $9x^2 + 9x + 2 = (3x + 1)(3x + 2)$ (3)

c. $6y^3 + 3y^2 - 18y = 3y(2y^2 + y - 6)$ (1)
$= 3y(2y - 3)(y + 2)$ (3)

d. $x^2 - 6x + 9 = (x - 3)^2$ (5)

e. $z^{1/4} + z^{5/4} = z^{1/4}(1 + z)$ (1)

f. $x^4 - 1 = (x^2 + 1)(x^2 - 1)$ (6)
$= (x^2 + 1)(x + 1)(x - 1)$ (6)

g. $x^{2/3} - 5x^{1/3} + 4 = (x^{1/3} - 1)(x^{1/3} - 4)$ (2)

h. $ax^2 - ay^2 + bx^2 - by^2 = a(x^2 - y^2) + b(x^2 - y^2)$ (1), (1)
$= (x^2 - y^2)(a + b)$ (1)
$= (x + y)(x - y)(a + b)$ (6)

i. $8 - x^3 = (2)^3 - (x)^3 = (2 - x)(4 + 2x + x^2)$ (8)

j. $x^6 - y^6 = (x^3)^2 - (y^3)^2 = (x^3 + y^3)(x^3 - y^3)$ (6)
$= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)$ (7), (8)

◁

Note in Example 3(f) that $x^2 - 1$ is factorable, but $x^2 + 1$ is not. In Example 3(h), note that the common factor of $x^2 - y^2$ was not immediately evident.

PROBLEMS 0.5

Factor the following expressions completely.

1. $5bx + 5b$

2. $6y^2 - 4y$

3. $10xy + 5xz$

4. $3x^2y - 9x^3y^3$

5. $8a^3bc - 12ab^3cd + 4b^4c^2d^2$

6. $5r^2st^2 + 10r^3s^2t^3 - 15r^2t^2$

7. $z^2 - 49$

8. $x^2 - x - 6$

9. $p^2 + 4p + 3$

10. $s^2 - 6s + 8$

11. $25y^2 - 4$

12. $x^2 + 2x - 24$

13. $a^2 + 12a + 35$

14. $4t^2 - 9s^2$

15. $x^2 + 6x + 9$

16. $t^2 - 18t + 72$

17. $5x^2 + 25x + 30$

18. $3t^2 + 12t - 15$

19. $3x^2 - 3$

20. $9y^2 - 18y + 8$

21. $5x^2 + 16x + 3$
22. $4x^2 - x - 3$
23. $12s^3 + 10s^2 - 8s$
24. $9z^2 + 30z + 25$
25. $u^{13/5}v - 4u^{3/5}v^3$
26. $4x^{6/5} - 1$
27. $2x^3 + 2x^2 - 12x$
28. $x^2y^2 - 4xy + 4$
29. $(4x + 2)^2$
30. $2x^2(2x - 4x^2)^2$
31. $x^3y^2 - 16x^2y + 64x$
32. $(5x^2 + 2x) + (10x + 4)$
33. $(x^3 - 4x) + (8 - 2x^2)$
34. $(x^2 - 1) + (x^2 - x - 2)$
35. $(y^4 + 8y^3 + 16y^2) - (y^2 + 8y + 16)$
36. $t^3u - 3tu + t^2w^2 - 3w^2$
37. $b^3 + 64$
38. $x^3 - 1$
39. $x^6 - 1$
40. $27 + 8x^3$
41. $(x + 4)^3(x - 2) + (x + 4)^2(x - 2)^2$
42. $(a + 5)^3(a + 1)^2 + (a + 5)^2(a + 1)^3$
43. $P(1 + r) + P(1 + r)r$
44. $(X - 3I)(3X + 5I) - (3X + 5I)(X + 2I)$
45. $x^4 - 16$
46. $256y^4 - z^4$
47. $y^8 - 1$
48. $t^4 - 4$
49. $X^4 + 4X^2 - 5$
50. $x^4 - 10x^2 + 9$
51. $a^4b - 8a^2b + 16b$
52. $4x^3 - 6x^2 - 4x$

Objective

To simplify, add, subtract, multiply, and divide algebraic fractions. To rationalize the denominator of a fraction.

0.6 Fractions

Students should take particular care in studying *fractions*. In everyday life, numerical fractions often disappear from view with the help of calculators. However, understanding how to manipulate fractions of algebraic expressions is an essential prerequisite for calculus. Most calculators are of no help!

Simplifying Fractions

By using the fundamental principle of fractions (Section 0.2), we may be able to simplify algebraic expressions that are fractions. That principle allows us to multiply or divide both the numerator and the denominator of a fraction by the same nonzero quantity. The resulting fraction will be equal to the original one. The fractions that we consider are assumed to have nonzero denominators. Thus, all the factors of the denominators in our examples are assumed to be nonzero. This will often mean that certain values are excluded for the variables that occur in the denominators.

EXAMPLE 1 **Simplifying Fractions**

a. Simplify $\dfrac{x^2 - x - 6}{x^2 - 7x + 12}$.

Solution: First, we completely factor both the numerator and the denominator:

$$\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)}$$

Dividing both numerator and denominator by the common factor $x - 3$, we have

$$\frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} = \frac{1(x + 2)}{1(x - 4)} = \frac{x + 2}{x - 4} \quad \text{for } x \neq 3$$

Usually, we just write

$$\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{\overset{1}{\cancel{(x - 3)}}(x + 2)}{\underset{1}{\cancel{(x - 3)}}(x - 4)} = \frac{x + 2}{x - 4}$$

or

$$\frac{x^2 - x - 6}{x^2 - 7x + 12} = \frac{(x - 3)(x + 2)}{(x - 3)(x - 4)} = \frac{x + 2}{x - 4} \quad \text{for } x \neq 3$$

The process of eliminating the common factor $x - 3$ is commonly referred to as "cancellation." We issued a blanket statement before this example that all fractions are assumed to have nonzero denominators and that this requires excluding certain values for the variables. Observe that, nevertheless, we explicitly wrote "for $x \neq 3$".

This is because the expression to the right of the equal sign, $\frac{x+2}{x-4}$, *is defined for* $x = 3$. Its value is -5 but we want to make it quite clear that the expression to the left of the equal sign *is not defined for* $x = 3$.

b. Simplify $\frac{2x^2+6x-8}{8-4x-4x^2}$.

Solution:
$$\frac{2x^2+6x-8}{8-4x-4x^2} = \frac{2(x^2+3x-4)}{4(2-x-x^2)} = \frac{2(x-1)(x+4)}{4(1-x)(2+x)}$$
$$= \frac{2(x-1)(x+4)}{2(2)[(-1)(x-1)](2+x)}$$
$$= \frac{x+4}{-2(2+x)} \quad \text{for } x \neq 1$$

The simplified expression is defined for $x = 1$ but since the original expression is not defined for $x = 1$, we explicitly exclude this value.

Now Work Problem 3 ◁

Multiplication and Division of Fractions

The rule for multiplying $\frac{a}{b}$ by $\frac{c}{d}$ is

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

EXAMPLE 2 **Multiplying Fractions**

a. $\frac{x}{x+2} \cdot \frac{x+3}{x-5} = \frac{x(x+3)}{(x+2)(x-5)}$

b. $\frac{x^2-4x+4}{x^2+2x-3} \cdot \frac{6x^2-6}{x^2+2x-8} = \frac{[(x-2)^2][6(x+1)(x-1)]}{[(x+3)(x-1)][(x+4)(x-2)]}$
$$= \frac{6(x-2)(x+1)}{(x+3)(x+4)} \quad \text{for } x \neq 1, 2$$

Note that we explicitly excluded the values which make the "cancelled factors" 0. While the final expression is defined for these values, the original expression is not.

Now Work Problem 9 ◁

To divide $\frac{a}{b}$ by $\frac{c}{d}$, where $b \neq 0$, $d \neq 0$, and $c \neq 0$, we have

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

In short, to divide by a fraction we invert the divisor and multiply.

EXAMPLE 3 **Dividing Fractions**

a. $\frac{x}{x+2} \div \frac{x+3}{x-5} = \frac{x}{x+2} \cdot \frac{x-5}{x+3} = \frac{x(x-5)}{(x+2)(x+3)}$

b. $\frac{\frac{x-5}{x-3}}{2x} = \frac{\frac{x-5}{x-3}}{\frac{2x}{1}} = \frac{x-5}{x-3} \cdot \frac{1}{2x} = \frac{x-5}{2x(x-3)}$

c. $\frac{\frac{4x}{x^2-1}}{\frac{2x^2+8x}{x-1}} = \frac{4x}{x^2-1} \cdot \frac{x-1}{2x^2+8x} = \frac{4x(x-1)}{[(x+1)(x-1)][2x(x+4)]}$
$$= \frac{2}{(x+1)(x+4)} \quad \text{for } x \neq 1$$

Why did we write "for $x \neq 1$"?

Now Work Problem 11 ◁

Rationalizing the Denominator

Sometimes the denominator of a fraction has two terms and involves square roots, such as $2-\sqrt{3}$ or $\sqrt{5}+\sqrt{2}$. The denominator may then be rationalized by multiplying by an expression that makes the denominator a difference of two squares. For example,

$$\frac{4}{\sqrt{5}+\sqrt{2}} = \frac{4}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$
$$= \frac{4(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{4(\sqrt{5}-\sqrt{2})}{5-2}$$
$$= \frac{4(\sqrt{5}-\sqrt{2})}{3}$$

Rationalizing the *numerator* is a similar procedure.

EXAMPLE 4 **Rationalizing Denominators**

a. $$\frac{x}{\sqrt{2}-6} = \frac{x}{\sqrt{2}-6} \cdot \frac{\sqrt{2}+6}{\sqrt{2}+6} = \frac{x(\sqrt{2}+6)}{(\sqrt{2})^2-6^2}$$
$$= \frac{x(\sqrt{2}+6)}{2-36} = -\frac{x(\sqrt{2}+6)}{34}$$

b. $$\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} \cdot \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$
$$= \frac{(\sqrt{5}-\sqrt{2})^2}{5-2} = \frac{5-2\sqrt{5}\sqrt{2}+2}{3} = \frac{7-2\sqrt{10}}{3}$$

Now Work Problem 53 ◁

Addition and Subtraction of Fractions

In Example 3(b) of Section 0.2, it was shown that $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$. That is, if we add two fractions having a common denominator, then the result is a fraction whose denominator is the common denominator. The numerator is the sum of the numerators of the original fractions. Similarly, $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.

EXAMPLE 5 **Adding and Subtracting Fractions**

a. $$\frac{p^2-5}{p-2}+\frac{3p+2}{p-2} = \frac{(p^2-5)+(3p+2)}{p-2}$$
$$= \frac{p^2+3p-3}{p-2}$$

b. $$\frac{x^2-5x+4}{x^2+2x-3} - \frac{x^2+2x}{x^2+5x+6} = \frac{(x-1)(x-4)}{(x-1)(x+3)} - \frac{x(x+2)}{(x+2)(x+3)}$$
$$= \frac{x-4}{x+3} - \frac{x}{x+3} = \frac{(x-4)-x}{x+3} = -\frac{4}{x+3} \quad \text{for } x \neq -2, 1$$

c. $$\frac{x^2+x-5}{x-7} - \frac{x^2-2}{x-7} + \frac{-4x+8}{x^2-9x+14} = \frac{x^2+x-5}{x-7} - \frac{x^2-2}{x-7} + \frac{-4}{x-7}$$
$$= \frac{(x^2+x-5)-(x^2-2)+(-4)}{x-7}$$
$$= \frac{x-7}{x-7} = 1 \quad \text{for } x \neq 2$$

Why did we write "for $x \neq 2$"?

Now Work Problem 29 ◁

To add (or subtract) two fractions with *different* denominators, use the fundamental principle of fractions to rewrite the fractions as fractions that have the same denominator. Then proceed with the addition (or subtraction) by the method just described.

For example, to find

$$\frac{2}{x^3(x-3)} + \frac{3}{x(x-3)^2}$$

we can convert the first fraction to an equal fraction by multiplying the numerator and denominator by $x - 3$:

$$\frac{2(x-3)}{x^3(x-3)^2}$$

and we can convert the second fraction by multiplying the numerator and denominator by x^2:

$$\frac{3x^2}{x^3(x-3)^2}$$

These fractions have the same denominator. Hence,

$$\frac{2}{x^3(x-3)} + \frac{3}{x(x-3)^2} = \frac{2(x-3)}{x^3(x-3)^2} + \frac{3x^2}{x^3(x-3)^2}$$

$$= \frac{3x^2 + 2x - 6}{x^3(x-3)^2}$$

We could have converted the original fractions into equal fractions with *any* common denominator. However, we chose to convert them into fractions with the denominator $x^3(x-3)^2$. This is the **least common denominator (LCD)** of the fractions $2/[x^3(x-3)]$ and $3/[x(x-3)^2]$.

In general, to find the LCD of two or more fractions, first factor each denominator completely. *The LCD is the product of each of the distinct factors appearing in the denominators, each raised to the highest power to which it occurs in any single denominator.*

EXAMPLE 6 **Adding and Subtracting Fractions**

a. Subtract: $\dfrac{t}{3t+2} - \dfrac{4}{t-1}$.

Solution: The LCD is $(3t + 2)(t - 1)$. Thus, we have

$$\frac{t}{(3t+2)} - \frac{4}{t-1} = \frac{t(t-1)}{(3t+2)(t-1)} - \frac{4(3t+2)}{(3t+2)(t-1)}$$

$$= \frac{t(t-1) - 4(3t+2)}{(3t+2)(t-1)}$$

$$= \frac{t^2 - t - 12t - 8}{(3t+2)(t-1)} = \frac{t^2 - 13t - 8}{(3t+2)(t-1)}$$

b. Add: $\dfrac{4}{q-1} + 3$.

Solution: The LCD is $q - 1$.

$$\frac{4}{q-1} + 3 = \frac{4}{q-1} + \frac{3(q-1)}{q-1}$$

$$= \frac{4 + 3(q-1)}{q-1} = \frac{3q+1}{q-1}$$

Now Work Problem 33 ◁

EXAMPLE 7 **Subtracting Fractions**

$$\frac{x-2}{x^2+6x+9}-\frac{x+2}{2(x^2-9)}$$

$$=\frac{x-2}{(x+3)^2}-\frac{x+2}{2(x+3)(x-3)} \qquad [\text{LCD} = 2(x+3)^2(x-3)]$$

$$=\frac{(x-2)(2)(x-3)}{(x+3)^2(2)(x-3)}-\frac{(x+2)(x+3)}{2(x+3)(x-3)(x+3)}$$

$$=\frac{(x-2)(2)(x-3)-(x+2)(x+3)}{2(x+3)^2(x-3)}$$

$$=\frac{2(x^2-5x+6)-(x^2+5x+6)}{2(x+3)^2(x-3)}$$

$$=\frac{2x^2-10x+12-x^2-5x-6}{2(x+3)^2(x-3)}$$

$$=\frac{x^2-15x+6}{2(x+3)^2(x-3)}$$

Now Work Problem 39 ◁

Example 8 is important for later work. Note that we explicitly assume $h \neq 0$.

EXAMPLE 8 **Combined Operations with Fractions**

Simplify $\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}$, where $h \neq 0$.

Solution: First we combine the fractions in the numerator and obtain

$$\frac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h}=\frac{\dfrac{x}{x(x+h)}-\dfrac{x+h}{x(x+h)}}{h}=\frac{\dfrac{x-(x+h)}{x(x+h)}}{h}$$

$$=\frac{\dfrac{-h}{x(x+h)}}{\dfrac{h}{1}}=\frac{-h}{x(x+h)h}=-\frac{1}{x(x+h)}$$

Now Work Problem 47 ◁

PROBLEMS 0.6

In Problems 1–6, simplify.

1. $\dfrac{a^2-9}{a^2-3a}$
2. $\dfrac{x^2-3x-10}{x^2-4}$
3. $\dfrac{x^2-9x+20}{x^2+x-20}$
4. $\dfrac{3x^2-27x+24}{2x^3-16x^2+14x}$
5. $\dfrac{15x^2+x-2}{3x^2+20x-7}$
6. $\dfrac{12x^2-19x+4}{6x^2-17x+12}$

In Problems 7–48, perform the operations and simplify as much as possible.

7. $\dfrac{y^2}{y-3}\cdot\dfrac{-1}{y+2}$
8. $\dfrac{t^2-9}{t^2+3t}\cdot\dfrac{t^2}{t^2-6t+9}$
9. $\dfrac{ax-b}{x-c}\cdot\dfrac{c-x}{ax+b}$
10. $\dfrac{a^2-b^2}{a-b}\cdot\dfrac{a^2-2ab+b^2}{2a+2b}$
11. $\dfrac{2x-2}{x^2-2x-8}\div\dfrac{x^2-1}{x^2+5x+4}$
12. $\dfrac{x^2+2x}{3x^2-18x+24}\div\dfrac{x^2-x-6}{x^2-4x+4}$
13. $\dfrac{\dfrac{X^2}{8}}{\dfrac{X}{4}}$
14. $\dfrac{\dfrac{3x^2}{7x}}{\dfrac{x}{14}}$
15. $\dfrac{\dfrac{15u}{v^3}}{\dfrac{3u}{v^4}}$
16. $\dfrac{\dfrac{c+d}{c}}{\dfrac{c-d}{2c}}$
17. $\dfrac{\dfrac{4x}{3}}{2x}$
18. $\dfrac{4x}{\dfrac{3}{2x}}$
19. $\dfrac{\dfrac{-9x^3}{x}}{3}$
20. $\dfrac{\dfrac{21t^5}{t^2}}{-7}$
21. $\dfrac{x-3}{\dfrac{x^2-7x+12}{x-4}}$

22. $\dfrac{\dfrac{x^2+6x+9}{x}}{x+3}$

23. $\dfrac{\dfrac{10x^3}{x^2-1}}{\dfrac{5x}{x+1}}$

24. $\dfrac{\dfrac{x^2-x-6}{x^2-9}}{\dfrac{x^2-4}{x^2+2x-3}}$

25. $\dfrac{\dfrac{x^2+8x+12}{x^2+9x+18}}{\dfrac{x^2-3x-10}{x^2-2x-15}}$

26. $\dfrac{\dfrac{(x+3)^2}{4x-3}}{\dfrac{7x+21}{9-16x^2}}$

27. $\dfrac{\dfrac{4x^2-9}{x^2+3x-4}}{\dfrac{2x-3}{1-x^2}}$

28. $\dfrac{\dfrac{6x^2y+7xy-3y}{xy-x+5y-5}}{\dfrac{x^3y+4x^2y}{xy-x+4y-4}}$

29. $\dfrac{x^2}{x+3}+\dfrac{5x+6}{x+3}$

30. $\dfrac{-1}{x-1}+\dfrac{x}{x-1}$

31. $\dfrac{2}{t}+\dfrac{1}{3t}$

32. $\dfrac{9}{X^3}-\dfrac{1}{X^2}$

33. $1-\dfrac{x^3}{x^3-1}$

34. $\dfrac{4}{s+4}+s$

35. $\dfrac{1}{3x-1}+\dfrac{x}{x+1}$

36. $\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}$

37. $\dfrac{1}{x^2-2x-3}+\dfrac{1}{x^2-9}$

38. $\dfrac{4}{2x^2-7x-4}-\dfrac{x}{2x^2-9x+4}$

39. $\dfrac{4}{x-1}-3+\dfrac{-3x^2}{5-4x-x^2}$

40. $\dfrac{x+1}{2x^2+3x-2}-\dfrac{x-1}{3x^2+5x-2}+\dfrac{1}{3x-1}$

41. $(1+x^{-1})^2$

42. $(x^{-1}+y^{-1})^2$

43. $(x^{-1}-y)^{-1}$

44. $(a+b^{-1})^2$

45. $\dfrac{5+\dfrac{2}{x}}{3}$

46. $\dfrac{\dfrac{x+3}{x}}{x-\dfrac{9}{x}}$

47. $\dfrac{3-\dfrac{1}{2x}}{x+\dfrac{x}{x+2}}$

48. $\dfrac{\dfrac{x-1}{x^2+5x+6}-\dfrac{1}{x+2}}{3+\dfrac{x-7}{3}}$

In Problems 49 and 50, perform the indicated operations, but do not rationalize the denominators.

49. $\dfrac{3}{\sqrt[3]{x+h}}-\dfrac{3}{\sqrt[3]{x}}$

50. $\dfrac{x\sqrt{x}}{\sqrt{3+x}}+\dfrac{2}{\sqrt{x}}$

In Problems 51–60, simplify, and express your answer in a form that is free of radicals in the denominator.

51. $\dfrac{1}{2+\sqrt{3}}$

52. $\dfrac{1}{1-\sqrt{2}}$

53. $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{6}}$

54. $\dfrac{5}{\sqrt{6}+\sqrt{7}}$

55. $\dfrac{2\sqrt{3}}{\sqrt{3}+\sqrt{5}}$

56. $\dfrac{2\sqrt{5}}{\sqrt{3}-\sqrt{7}}$

57. $\dfrac{3}{t+\sqrt{7}}$

58. $\dfrac{x-3}{\sqrt{x}-1}+\dfrac{4}{\sqrt{x}-1}$

59. $\dfrac{5}{2+\sqrt{3}}-\dfrac{4}{1-\sqrt{2}}$

60. $\dfrac{5}{\sqrt{x}+3}\cdot\dfrac{x^2}{4}$

0.7 Equations, in Particular Linear Equations

Objective

To discuss equivalent equations and to develop techniques for solving linear equations, including literal equations as well as fractional and radical equations that lead to linear equations.

Equations

An **equation** is a statement that two expressions are equal. The two expressions that make up an equation are called its **sides**. They are separated by the **equality sign**, $=$.

EXAMPLE 1 **Examples of Equations**

a. $x+2=3$

b. $x^2+3x+2=0$

c. $\dfrac{y}{y-4}=6$

d. $w=7-z$

◁

In Example 1, each equation contains at least one variable. A **variable** is a symbol that can be replaced by any one of a set of different numbers. The most popular symbols for variables are letters from the latter part of the alphabet, such as x, y, z, w, and t. Hence, equations (a) and (c) are said to be in the variables x and y, respectively. Equation (d) is in the variables w and z. In the equation $x+2=3$, the numbers 2 and 3 are called *constants*. They are fixed numbers.

Here we discuss restrictions on variables.

We *never* allow a variable in an equation to have a value for which any expression in that equation is undefined. For example, in

$$\frac{y}{y-4} = 6$$

y cannot be 4, because this would make the denominator zero; while in

$$\sqrt{x-3} = 9$$

we cannot have $x-3$ negative because we cannot take square roots of negative numbers. In some equations, the allowable values of a variable are restricted for physical reasons. For example, if the variable q represents quantity sold, negative values of q do not make sense.

To *solve* an equation means to find all values of its variables for which the equation is true. These values are called *solutions* of the equation and are said to *satisfy* the equation. When only one variable is involved, a solution is also called a *root*. The set of all solutions is called the *solution set* of the equation. Sometimes a letter representing an unknown quantity in an equation is simply called an *unknown*. Let us illustrate these terms.

EXAMPLE 2 **Terminology for Equations**

a. In the equation $x + 2 = 3$, the variable x is the unknown. The only value of x that satisfies the equation is obviously 1. Hence, 1 is a root and the solution set is $\{1\}$.

b. -2 is a root of $x^2 + 3x + 2 = 0$ because substituting -2 for x makes the equation true: $(-2)^2 + 3(-2) + 2 = 0$. Hence -2 is an element of the solution set but in this case it is not the only one. There is one more. Can you find it?

c. $w = 7 - z$ is an equation in two unknowns. One solution is the pair of values $w = 4$ and $z = 3$. However, there are infinitely many solutions. Can you think of another?

Now Work Problem 1 ◁

Equivalent Equations

Two equations are said to be *equivalent* if they have exactly the same solutions, which means, precisely, that the solution set of one is equal to the solution set of the other. Solving an equation may involve performing operations on it. We prefer that any such operation result in an equivalent equation. Here are three operations that guarantee equivalence:

1. Adding (subtracting) the same polynomial to (from) both sides of an equation, where the polynomial is in the same variable as that occurring in the equation.

For example, if $-5x = 5 - 6x$, then adding $6x$ to both sides gives the equivalent equation $-5x + 6x = 5 - 6x + 6x$, which in turn is equivalent to $x = 5$.

CAUTION!

Equivalence is not guaranteed if both sides are multiplied or divided by an expression involving a variable.

2. Multiplying (dividing) both sides of an equation by the same nonzero constant.

For example, if $10x = 5$, then dividing both sides by 10 gives the equivalent equation $\frac{10x}{10} = \frac{5}{10}$, equivalently, $x = \frac{1}{2}$.

3. Replacing either side of an equation by an equal expression.

For example, if the equation is $x(x + 2) = 3$, then replacing the left side by the equal expression $x^2 + 2x$ gives the equivalent equation $x^2 + 2x = 3$.

We repeat: Applying Operations 1–3 guarantees that the resulting equation is equivalent to the given one. However, sometimes in solving an equation we have to apply

Here we discuss restrictions on variables.

We *never* allow a variable in an equation to have a value for which any expression in that equation is undefined. For example, in

$$\frac{y}{y-4} = 6$$

y cannot be 4, because this would make the denominator zero; while in

$$\sqrt{x-3} = 9$$

we cannot have $x-3$ negative because we cannot take square roots of negative numbers. In some equations, the allowable values of a variable are restricted for physical reasons. For example, if the variable q represents quantity sold, negative values of q do not make sense.

To *solve* an equation means to find all values of its variables for which the equation is true. These values are called *solutions* of the equation and are said to *satisfy* the equation. When only one variable is involved, a solution is also called a *root*. The set of all solutions is called the *solution set* of the equation. Sometimes a letter representing an unknown quantity in an equation is simply called an *unknown*. Let us illustrate these terms.

EXAMPLE 2 **Terminology for Equations**

a. In the equation $x + 2 = 3$, the variable x is the unknown. The only value of x that satisfies the equation is obviously 1. Hence, 1 is a root and the solution set is $\{1\}$.

b. -2 is a root of $x^2 + 3x + 2 = 0$ because substituting -2 for x makes the equation true: $(-2)^2 + 3(-2) + 2 = 0$. Hence -2 is an element of the solution set but in this case it is not the only one. There is one more. Can you find it?

c. $w = 7 - z$ is an equation in two unknowns. One solution is the pair of values $w = 4$ and $z = 3$. However, there are infinitely many solutions. Can you think of another?

Now Work Problem 1 ◁

Equivalent Equations

Two equations are said to be *equivalent* if they have exactly the same solutions, which means, precisely, that the solution set of one is equal to the solution set of the other. Solving an equation may involve performing operations on it. We prefer that any such operation result in an equivalent equation. Here are three operations that guarantee equivalence:

1. Adding (subtracting) the same polynomial to (from) both sides of an equation, where the polynomial is in the same variable as that occurring in the equation.

For example, if $-5x = 5 - 6x$, then adding $6x$ to both sides gives the equivalent equation $-5x + 6x = 5 - 6x + 6x$, which in turn is equivalent to $x = 5$.

CAUTION!

Equivalence is not guaranteed if both sides are multiplied or divided by an expression involving a variable.

2. Multiplying (dividing) both sides of an equation by the same nonzero constant.

For example, if $10x = 5$, then dividing both sides by 10 gives the equivalent equation $\frac{10x}{10} = \frac{5}{10}$, equivalently, $x = \frac{1}{2}$.

3. Replacing either side of an equation by an equal expression.

For example, if the equation is $x(x + 2) = 3$, then replacing the left side by the equal expression $x^2 + 2x$ gives the equivalent equation $x^2 + 2x = 3$.

We repeat: Applying Operations 1–3 guarantees that the resulting equation is equivalent to the given one. However, sometimes in solving an equation we have to apply

operations other than 1–3. These operations may *not* necessarily result in equivalent equations. They include the following:

Operations That May Not Produce Equivalent Equations

4. Multiplying both sides of an equation by an expression involving the variable.
5. Dividing both sides of an equation by an expression involving the variable.
6. Raising both sides of an equation to equal powers.

Operation 6 includes taking roots of both sides.

Let us illustrate the last three operations. For example, by inspection, the only root of $x - 1 = 0$ is 1. Multiplying each side by x (Operation 4) gives $x^2 - x = 0$, which is satisfied if x is 0 or 1. (Check this by substitution.) But 0 *does not* satisfy the *original* equation. Thus, the equations are not equivalent.

Continuing, you can check that the equation $(x - 4)(x - 3) = 0$ is satisfied when x is 4 or when x is 3. Dividing both sides by $x - 4$ (Operation 5) gives $x - 3 = 0$, whose only root is 3. Again, we do not have equivalence, since in this case a root has been "lost." Note that when x is 4, division by $x - 4$ implies division by 0, an invalid operation.

Finally, squaring each side of the equation $x = 2$ (Operation 6) gives $x^2 = 4$, which is true if $x = 2$ or if $x = -2$. But -2 is not a root of the given equation.

From our discussion, it is clear that when Operations 4–6 are performed, we must be careful about drawing conclusions concerning the roots of a given equation. Operations 4 and 6 *can* produce an equation with more roots. Thus, you should check whether or not each "solution" obtained by these operations satisfies the *original* equation. Operation 5 *can* produce an equation with fewer roots. In this case, any "lost" root may never be determined. Thus, avoid Operation 5 whenever possible.

In summary, an equation may be thought of as a set of restrictions on any variable in the equation. Operations 4–6 may increase or decrease the number of restrictions, giving solutions different from those of the original equation. However, Operations 1–3 never affect the restrictions.

Linear Equations

The principles presented so far will now be demonstrated in the solution of a *linear equation.*

Definition

A ***linear equation*** in the variable x is an equation that is equivalent to one that can be written in the form

$$ax + b = 0 \tag{1}$$

where a and b are constants and $a \neq 0$.

A linear equation is also called a first-degree equation or an equation of degree one, since the highest power of the variable that occurs in Equation (1) is the first.

To solve a linear equation, we perform operations on it until we have an equivalent equation whose solutions are obvious. This means an equation in which the variable is isolated on one side, as the following examples show.

EXAMPLE 3 Solving a Linear Equation

Solve $5x - 6 = 3x$.

Solution: We begin by getting the terms involving x on one side and the constant on the other. Then we solve for x by the appropriate mathematical operation. We have

$$5x - 6 = 3x$$

$$
\begin{aligned}
5x - 6 + (-3x) &= 3x + (-3x) && \text{adding } -3x \text{ to both sides}\\
2x - 6 &= 0 && \text{simplifying, that is, Operation 3}\\
2x - 6 + 6 &= 0 + 6 && \text{adding 6 to both sides}\\
2x &= 6 && \text{simplifying}\\
\frac{2x}{2} &= \frac{6}{2} && \text{dividing both sides by 2}\\
x &= 3
\end{aligned}
$$

Clearly, 3 is the only root of the last equation. Since each equation is equivalent to the one before it, we conclude that 3 must be the only root of $5x - 6 = 3x$. That is, the solution set is $\{3\}$. We can describe the first step in the solution as moving a term from one side of an equation to the other while changing its sign; this is commonly called *transposing*. Note that since the original equation can be put in the form $2x + (-6) = 0$, it is a linear equation.

Now Work Problem 23 ◁

EXAMPLE 4 **Solving a Linear Equation**

Solve $2(p + 4) = 7p + 2$.

Solution: First, we remove parentheses. Then we collect similar terms and solve. We have

$$
\begin{aligned}
2(p + 4) &= 7p + 2\\
2p + 8 &= 7p + 2 && \text{distributive property}\\
2p &= 7p - 6 && \text{subtracting 8 from both sides}\\
-5p &= -6 && \text{subtracting } 7p \text{ from both sides}\\
p &= \frac{-6}{-5} && \text{dividing both sides by } -5\\
p &= \frac{6}{5}
\end{aligned}
$$

Now Work Problem 27 ◁

EXAMPLE 5 **Solving a Linear Equation**

Solve $\dfrac{7x + 3}{2} - \dfrac{9x - 8}{4} = 6$.

Solution: We first clear the equation of fractions by multiplying *both* sides by the LCD, which is 4. Then we use various algebraic operations to obtain a solution. Thus,

$$
\begin{aligned}
4\left(\frac{7x + 3}{2} - \frac{9x - 8}{4}\right) &= 4(6)\\
4 \cdot \frac{7x + 3}{2} - 4 \cdot \frac{9x - 8}{4} &= 24 && \text{distributive property}\\
2(7x + 3) - (9x - 8) &= 24 && \text{simplifying}\\
14x + 6 - 9x + 8 &= 24 && \text{distributive property}\\
5x + 14 &= 24 && \text{simplifying}\\
5x &= 10 && \text{subtracting 14 from both sides}\\
x &= 2 && \text{dividing both sides by 5}
\end{aligned}
$$

Now Work Problem 31 ◁

CAUTION! The distributive property requires that *both* terms within the parentheses be multiplied by 4.

Every linear equation has exactly one root. The root of $ax + b = 0$ is $x = -\dfrac{b}{a}$.

Each equation in Examples 3–5 has one and only one root. This is true of every linear equation in one variable.

Literal Equations

Equations in which some of the constants are not specified, but are represented by letters, such as a, b, c, or d, are called **literal equations,** and the letters are called **literal constants**. For example, in the literal equation $x + a = 4b$, we can consider a and b to be literal constants. Formulas, such as $I = Prt$, that express a relationship between certain quantities may be regarded as literal equations. If we want to express a particular letter in a formula in terms of the others, this letter is considered the unknown.

EXAMPLE 6 Solving Literal Equations

a. The equation $I = Prt$ is the formula for the simple interest I on a principal of P dollars at the annual interest rate of r for a period of t years. Express r in terms of I, P, and t.

Solution: Here we consider r to be the unknown. To isolate r, we divide both sides by Pt. We have

$$I = Prt$$
$$\frac{I}{Pt} = \frac{Prt}{Pt}$$
$$\frac{I}{Pt} = r \text{ so } r = \frac{I}{Pt}$$

When we divided both sides by Pt, we assumed that $Pt \neq 0$, since we cannot divide by 0. Similar assumptions will be made in solving other literal equations.

b. The equation $S = P + Prt$ is the formula for the value S of an investment of a principal of P dollars at a simple annual interest rate of r for a period of t years. Solve for P.

Solution:

$$S = P + Prt$$
$$S = P(1 + rt) \qquad \text{factoring}$$
$$\frac{S}{1 + rt} = P \qquad \text{dividing both sides by } 1 + rt$$

Now Work Problem 87 ◁

EXAMPLE 7 Solving a Literal Equation

Solve $(a + c)x + x^2 = (x + a)^2$ for x.

Solution: We first simplify the equation and then get all terms involving x on one side:

$$(a + c)x + x^2 = (x + a)^2$$
$$ax + cx + x^2 = x^2 + 2ax + a^2$$
$$ax + cx = 2ax + a^2$$
$$cx - ax = a^2$$
$$x(c - a) = a^2$$
$$x = \frac{a^2}{c - a}$$

Now Work Problem 89 ◁

EXAMPLE 8 Solving the "Tax in a Receipt" Problem

We will recall Lesley Griffith's problem from the opening paragraphs of this chapter. We will generalize the problem so as to illustrate further the use of literal equations.

Lesley has a receipt for an amount R. She knows that the percentage sales tax rate is p. She wants to know the amount that was paid in sales tax. Certainly,

$$\text{price} + \text{tax} = \text{receipt} \tag{2}$$

Writing P for the price (which she does not yet know), the tax is $(p/100)P$ so that she has

$$P + \frac{p}{100}P = R$$
$$P\left(1 + \frac{p}{100}\right) = R$$
$$P\left(\frac{100+p}{100}\right) = R$$
$$P = \frac{100R}{100+p}$$

It follows that the tax paid is

$$R - P = R - \frac{100R}{100+p} = R\left(1 - \frac{100}{100+p}\right) = R\left(\frac{p}{100+p}\right)$$

where you should check the manipulations with fractions, supplying more details if necessary. Recall that the French tax rate is 19.6% and the Italian tax rate is 18%. We conclude that Lesley has only to multiply a French receipt by $\frac{19.6}{119.6} \approx 0.16388$ to determine the tax it contains, while for an Italian receipt she should multiply the amount by $\frac{18}{118}$. It should be noted that working from the simple conceptual equation (2) we have been able to avoid the *assumptions* about proportionality that we made at the beginning of this chapter.

Now Work Problem 107 ◁

Fractional Equations

A **fractional equation** is an equation in which an unknown is in a denominator. We illustrate that solving such a nonlinear equation may lead to a linear equation.

EXAMPLE 9 Solving a Fractional Equation

Solve $\dfrac{5}{x-4} = \dfrac{6}{x-3}$.

Solution:

Strategy We first write the equation in a form that is free of fractions. Then we use standard algebraic techniques to solve the resulting equation.

Multiplying both sides by the LCD, $(x-4)(x-3)$, we have

$$(x-4)(x-3)\left(\frac{5}{x-4}\right) = (x-4)(x-3)\left(\frac{6}{x-3}\right)$$
$$5(x-3) = 6(x-4) \qquad \text{linear equation}$$
$$5x - 15 = 6x - 24$$
$$9 = x$$

In the first step, we multiplied each side by an expression involving the *variable* x. As we mentioned in this section, this means that we are not guaranteed that the last equation is equivalent to the *original* equation. Thus, we must check whether or not 9 satisfies the *original* equation. Since

$$\frac{5}{9-4} = \frac{5}{5} = 1 \quad \text{and} \quad \frac{6}{9-3} = \frac{6}{6} = 1$$

we see that 9 indeed satisfies the original equation.

Now Work Problem 55 ◁

An alternative solution that avoids multiplying both sides by the LCD is as follows:

$$\frac{5}{x-4} - \frac{6}{x-3} = 0$$

Assuming that x is neither 3 nor 4 and combining fractions gives

$$\frac{9-x}{(x-4)(x-3)} = 0$$

A fraction can be 0 only when its numerator is 0 and its denominator is not. Hence, $x = 9$.

Some equations that are not linear do not have any solutions. In that case, we say that the solution set is the **empty set**, which we denote by $\emptyset$. Example 10 will illustrate.

EXAMPLE 10 Solving Fractional Equations

a. Solve $\dfrac{3x+4}{x+2} - \dfrac{3x-5}{x-4} = \dfrac{12}{x^2-2x-8}$.

Solution: Observing the denominators and noting that

$$x^2 - 2x - 8 = (x+2)(x-4)$$

we conclude that the LCD is $(x+2)(x-4)$. Multiplying both sides by the LCD, we have

$$(x+2)(x-4)\left(\frac{3x+4}{x+2} - \frac{3x-5}{x-4}\right) = (x+2)(x-4)\cdot\frac{12}{(x+2)(x-4)}$$

$$(x-4)(3x+4) - (x+2)(3x-5) = 12$$

$$3x^2 - 8x - 16 - (3x^2 + x - 10) = 12$$

$$3x^2 - 8x - 16 - 3x^2 - x + 10 = 12$$

$$-9x - 6 = 12$$

$$-9x = 18$$

$$x = -2 \tag{3}$$

However, the *original* equation is not defined for $x = -2$ (we cannot divide by zero), so there are no roots. Thus, the solution set is $\emptyset$. Although -2 is a solution of Equation (3), it is not a solution of the *original* equation.

b. Solve $\dfrac{4}{x-5} = 0$.

Solution: The only way a fraction can equal zero is for the numerator to equal zero (and the denominator to not equal zero). Since the numerator, 4, is never 0, the solution set is $\emptyset$.

Now Work Problem 49 ◁

EXAMPLE 11 Literal Equation

If $s = \dfrac{u}{au+v}$, express u in terms of the remaining letters; that is, solve for u.

Solution:

Strategy Since the unknown, u, occurs in the denominator, we first clear fractions and then solve for u.

$$s = \frac{u}{au+v}$$

$$s(au+v) = u \qquad \text{multiplying both sides by } au+v$$

$$sau + sv = u$$

$$sau - u = -sv$$

$$u(sa-1) = -sv$$

$$u = \frac{-sv}{sa-1} = \frac{sv}{1-sa}$$

Now Work Problem 91 ◁

Radical Equations

A **radical equation** is one in which an unknown occurs in a radicand. The next two examples illustrate the techniques employed to solve such equations.

EXAMPLE 12 Solving a Radical Equation

Solve $\sqrt{x^2+33}-x=3$.

Solution: To solve this radical equation, we raise both sides to the same power to eliminate the radical. This operation does *not* guarantee equivalence, so we must check any resulting "solutions." We begin by isolating the radical on one side. Then we square both sides and solve using standard techniques. Thus,

$$\begin{aligned}\sqrt{x^2+33}&=x+3\\ x^2+33&=(x+3)^2 \qquad \text{squaring both sides}\\ x^2+33&=x^2+6x+9\\ 24&=6x\\ 4&=x\end{aligned}$$

You should show by substitution that 4 is indeed a root.

Now Work Problem 79 ◁

With some radical equations, you may have to raise both sides to the same power more than once, as Example 13 shows.

EXAMPLE 13 Solving a Radical Equation

Solve $\sqrt{y-3}-\sqrt{y}=-3$.

The reason we want one radical on each side is to eliminate squaring a binomial with two different radicals.

Solution: When an equation has two terms involving radicals, first write the equation so that one radical is on each side, if possible. Then square and solve. We have

$$\begin{aligned}\sqrt{y-3}&=\sqrt{y}-3\\ y-3&=y-6\sqrt{y}+9 \qquad \text{squaring both sides}\\ 6\sqrt{y}&=12\\ \sqrt{y}&=2\\ y&=4 \qquad \text{squaring both sides}\end{aligned}$$

Substituting 4 into the left side of the *original* equation gives $\sqrt{1}-\sqrt{4}$, which is -1. Since this does not equal the right side, -3, there is no solution. That is, the solution set is $\emptyset$.

Now Work Problem 77 ◁

PROBLEMS 0.7

In Problems 1–6, determine by substitution which of the given numbers, if any, satisfy the given equation.

1. $9x-x^2=0;\ 1,\ 0$
2. $12-7x=-x^2;\ 4, 3$
3. $z+3(z-4)=5;\ \frac{17}{4},\ 4$
4. $x^2+x-6=0;\ 2, 3$
5. $x(6+x)-2(x+1)-5x=4;\ -2,\ 0$
6. $x(x+1)^2(x+2)=0;\ 0, -1,\ 2$

In Problems 7–16, determine what operations were applied to the first equation to obtain the second. State whether or not the operations guarantee that the equations are equivalent. Do not solve the equations.

7. $x-5=4x+10;\ x=4x+15$
8. $8x-4=16;\ x-\frac{1}{2}=2$
9. $x=5;\ x^4=625$

10. $2x^2+4=5x-7; x^2+2=\frac{5}{2}x-\frac{7}{2}$

11. $x^2-2x=0; x-2=0$

12. $\frac{2}{x-2}+x=x^2; 2+x(x-2)=x^2(x-2)$

13. $\frac{x^2-1}{x-1}=3; x^2-1=3(x-1)$

14. $(x+2)(x+1)=(x+3)(x+1); x+2=x+3$

15. $\frac{2x(3x+1)}{2x-3}=2x(x+4); 3x+1=(x+4)(2x-3)$

16. $2x^2-9=x; x^2-\frac{1}{2}x=\frac{9}{2}$

In Problems 17–80, solve the equations.

17. $4x=10$

18. $0.2x=7$

19. $7y^2=0$

20. $2x-4x=-5$

21. $-8x=12-20$

22. $4-7x=3$

23. $5x-3=9$

24. $\sqrt{3}x+2=11$

25. $7x+7=2(x+1)$

26. $4s+3s-1=41$

27. $5(p-7)-2(3p-4)=3p$

28. $t=2-2[2t-3(1-t)]$

29. $\frac{x}{7}=3x+5$

30. $\frac{5y}{7}-\frac{6}{7}=2-4y$

31. $7+\frac{4x}{9}=\frac{x}{2}$

32. $\frac{x}{3}-4=\frac{x}{5}$

33. $r=\frac{4}{3}r-5$

34. $\frac{2x}{11}+\frac{11x}{2}=4$

35. $3x+\frac{x}{5}-5=\frac{1}{5}+5x$

36. $y-\frac{y}{2}+\frac{y}{3}-\frac{y}{4}=\frac{y}{5}$

37. $\frac{2y-3}{4}=\frac{6y+7}{3}$

38. $\frac{t}{4}+\frac{5}{3}t=\frac{7}{2}(t-1)$

39. $t+\frac{t}{3}-\frac{t}{4}+\frac{t}{36}=10$

40. $\frac{7+2(x+1)}{3}=\frac{6x}{5}$

41. $\frac{x+2}{3}-\frac{2-x}{6}=x-2$

42. $\frac{x}{5}+\frac{2(x-4)}{10}=7$

43. $\frac{9}{5}(3-x)=\frac{3}{4}(x-3)$

44. $\frac{2x-7}{3}+\frac{8x-9}{14}=\frac{3x-5}{21}$

45. $\frac{4}{3}(5x-2)=7[x-(5x-2)]$

46. $(2x-5)^2+(3x-3)^2=13x^2-5x+7$

47. $\frac{5}{x}=25$

48. $\frac{4}{x-1}=2$

49. $\frac{5}{x+3}=0$

50. $\frac{3x-5}{x-3}=0$

51. $\frac{3}{5-2x}=\frac{7}{2}$

52. $\frac{x+3}{x}=\frac{2}{5}$

53. $\frac{q}{5q-4}=\frac{1}{3}$

54. $\frac{5q}{3-q}=2$

55. $\frac{1}{p-1}=\frac{2}{p-2}$

56. $\frac{2x-3}{4x-5}=6$

57. $\frac{1}{x}+\frac{1}{7}=\frac{3}{7}$

58. $\frac{2}{x-1}=\frac{3}{x-2}$

59. $\frac{2t+1}{2t+3}=\frac{3t-1}{3t+4}$

60. $\frac{x+2}{x-1}+\frac{x+1}{3-x}=0$

61. $\frac{y-6}{y}-\frac{6}{y}=\frac{y+6}{y-6}$

62. $\frac{y-2}{y+2}=\frac{y-2}{y+3}$

63. $\frac{-5}{2x-3}=\frac{7}{3-2x}+\frac{11}{3x+5}$

64. $\frac{1}{x+1}+\frac{2}{x-3}=\frac{-6}{3-2x}$

65. $\frac{9}{x-3}=\frac{3x}{x-3}$

66. $\frac{x}{x+3}-\frac{x}{x-3}=\frac{3x-4}{x^2-9}$

67. $\sqrt{x+5}=4$

68. $\sqrt{z-2}=3$

69. $\sqrt{2x+3}-4=0$

70. $4-\sqrt{3x+1}=0$

71. $\sqrt{\frac{x}{2}+1}=\frac{2}{3}$

72. $(x+6)^{1/2}=7$

73. $\sqrt{4x-6}=\sqrt{x}$

74. $\sqrt{x+1}=\sqrt{2x-3}$

75. $(x-5)^{3/4}=27$

76. $\sqrt{y^2-9}=9-y$

77. $\sqrt{y}+\sqrt{y+2}=3$

78. $\sqrt{x}-\sqrt{x+1}=1$

79. $\sqrt{a^2+2a}=2+a$

80. $\sqrt{\frac{1}{w}}-\sqrt{\frac{2}{5w-2}}=0$

In Problems 81–92, express the indicated symbol in terms of the remaining symbols.

81. $I=Prt;\ r$

82. $P\left(1+\frac{p}{100}\right)-R=0;\ P$

83. $p=8q-1;\ q$

84. $p=10-2q;\ q$

85. $S=P(1+rt);\ r$

86. $r=\frac{2mI}{B(n+1)};I$

87. $A=\frac{R[1-(1+i)^{-n}]}{i};\ R$

88. $S=\frac{R[(1+i)^n-1]}{i};\ R$

89. $S=P(1+r)^n;\ r$

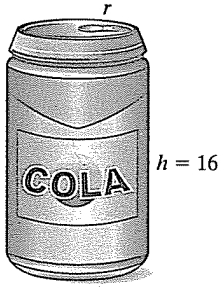
90. $\frac{x-a}{b-x}=\frac{x-b}{a-x};\ x$

91. $r=\frac{2mI}{B(n+1)};\ n$

92. $\frac{1}{p}+\frac{1}{q}=\frac{1}{f};\ q$

93. Geometry Use the formula $P=2l+2w$ to find the length l of a rectangle whose perimeter P is 660 m and whose width w is 160 m.

94. Geometry Use the formula $V=\pi r^2h$ to find the radius r of a cola can whose volume V is 355 ml and whose height h is 16 cm.



95. Sales Tax A salesperson needs to calculate the cost of an item with a sales tax of 8.25%. Write an equation that represents the total cost c of an item costing x dollars.

96. Revenue A day care center's total monthly revenue from the care of x toddlers is given by $r = 450x$, and its total monthly costs are given by $c = 380x + 3500$. How many toddlers need to be enrolled each month to break even? In other words, when will revenue equal costs?

97. Straight-Line Depreciation If you purchase an item for business use, in preparing your income tax you may be able to spread out its expense over the life of the item. This is called *depreciation*. One method of depreciation is *straight-line depreciation*, in which the annual depreciation is computed by dividing the cost of the item, less its estimated salvage value, by its useful life. Suppose the cost is C dollars, the useful life is N years, and there is no salvage value. Then the value V (in dollars) of the item at the end of n years is given by

$$V = C\left(1 - \frac{n}{N}\right)$$

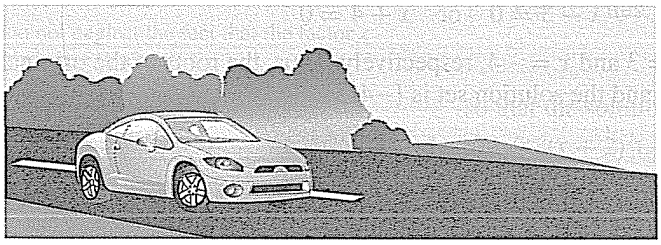
If new office furniture is purchased for \$3200, has a useful life of 8 years, and has no salvage value, after how many years will it have a value of \$2000?

98. Radar Beam When radar is used on a highway to determine the speed of a car, a radar beam is sent out and reflected from the moving car. The difference F (in cycles per second) in frequency between the original and reflected beams is given by

$$F = \frac{vf}{334.8}$$

where v is the speed of the car in miles per hour and f is the frequency of the original beam (in megacycles per second).

Suppose you are driving along a highway with a speed limit of 65 mi/h. A police officer aims a radar beam with a frequency of 2500 megacycles per second at your car, and the officer observes the difference in frequency to be 495 cycles per second. Can the officer claim that you were speeding?



99. Savings Bronwyn wants to buy a house, so she has decided to save one quarter of her salary. Bronwyn earns \$47.00 per hour and receives an extra \$28.00 a week because she declined company benefits. She wants to save at least \$550.00 each week. How many hours must she work each week to achieve her goal?

100. Predator–Prey Relation To study a predator–prey relationship, an experiment[4] was conducted in which a blindfolded subject, the "predator," stood in front of a 3-ft-square table on which uniform sandpaper discs, the "prey," were placed. For 1 minute the "predator" searched for the discs by tapping with a finger. Whenever a disc was found, it was removed and searching resumed. The experiment was repeated for various disc densities (number of discs per 9 ft^2). It was estimated that if y is the number of discs picked up in 1 minute when x discs are on the table, then

$$y = a(1 - by)x$$

where a and b are constants. Solve this equation for y.

101. Prey Density In a certain area, the number y of moth larvae consumed by a single predatory beetle over a given period of time is given by

$$y = \frac{1.4x}{1 + 0.09x}$$

where x is the *prey density* (the number of larvae per unit of area). What prey density would allow a beetle to survive if it needs to consume 10 larvae over the given period?

102. Store Hours Suppose the ratio of the number of hours a video store is open to the number of daily customers is constant. When the store is open 8 hours, the number of customers is 92 less than the maximum number of customers. When the store is open 10 hours, the number of customers is 46 less than the maximum number of customers. Write an equation describing this situation, and find the maximum number of daily customers.

103. Travel Time The time it takes a boat to travel a given distance upstream (against the current) can be calculated by dividing the distance by the difference of the speed of the boat and the speed of the current. Write an equation that calculates the time t it takes a boat moving at a speed r against a current c to travel a distance d. Solve your equation for c.

104. Wireless Tower A wireless tower is 100 meters tall. An engineer determines electronically that the distance from the top of the tower to a nearby house is 2 meters greater than the horizontal distance from the base of the tower to the house. Determine the distance from the base of the tower to the house.

105. Automobile Skidding Police have used the formula $s = \sqrt{30fd}$ to estimate the speed s (in miles per hour) of a car if it skidded d feet when stopping. The literal number f is the coefficient of friction, determined by the kind of road (such as concrete, asphalt, gravel, or tar) and whether the road is wet or dry. Some values of f are given in Table 0.1. At 45 mi/h, about how many feet will a car skid on a dry concrete road? Give your answer to the nearest foot.

Table 0.1

	Concrete	Tar
Wet	0.4	0.5
Dry	0.8	1.0

106. Interest Earned Allison Bennett discovers that she has \$1257 in an off-shore account that she has not used for a year. The interest rate was 7.3% compounded annually. How much interest did she earn from that account over the last year?

[4] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

107. Tax in a Receipt In 2006 Nova Scotia consumers paid HST, *harmonized sales tax*, of 15%. Tom Wood traveled from Alberta, which has only federal GST, *goods and services tax*, of 7% to Nova Scotia for a chemistry conference. When he later submitted his expense claims in Alberta, the comptroller was puzzled to find that her usual multiplier of $\frac{7}{107}$ to determine tax in a receipt did not produce correct results. What percentage of Tom's Nova Scotia receipts were HST?

0.8 Quadratic Equations

Objective

To solve quadratic equations by factoring or by using the quadratic formula.

To learn how to solve certain classical problems, we turn to methods of solving *quadratic equations*.

Definition

A ***quadratic equation*** in the variable x is an equation that can be written in the form

$$ax^2 + bx + c = 0 \tag{1}$$

where a, b, and c are constants and $a \neq 0$.

A quadratic equation is also called a *second-degree equation* or an *equation of degree two,* since the highest power of the variable that occurs is the second. Whereas a linear equation has only one root, a quadratic equation may have two different roots.

Solution by Factoring

A useful method of solving quadratic equations is based on factoring, as the following example shows.

EXAMPLE 1 Solving a Quadratic Equation by Factoring

a. Solve $x^2 + x - 12 = 0$.

Solution: The left side factors easily:

$$(x - 3)(x + 4) = 0$$

Think of this as two quantities, $x - 3$ and $x + 4$, whose product is zero. **Whenever the product of two or more quantities is *zero*, at least one of the quantities *must* be zero.** This means that either

$$x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$

Solving these gives $x = 3$ and $x = -4$, respectively. Thus, the roots of the original equation are 3 and -4, and the solution set is $\{-4, 3\}$.

b. Solve $6w^2 = 5w$.

Solution: We write the equation as

$$6w^2 - 5w = 0$$

so that one side is 0. Factoring gives

$$w(6w - 5) = 0$$

Setting each factor equal to 0, we have

$$w = 0 \quad \text{or} \quad 6w - 5 = 0$$

$$w = 0 \quad \text{or} \quad 6w = 5$$

Thus, the roots are $w = 0$ and $w = \frac{5}{6}$. Note that if we had divided both sides of $6w^2 = 5w$ by w and obtained $6w = 5$, our only solution would be $w = \frac{5}{6}$. That is, we would lose the root $w = 0$. This is in line with our discussion of Operation 5 in Section 0.7 and sheds light on the problem with Operation 5. One way of approaching the possibilities for a variable quantity, w is to observe that *either* $w \neq 0$ *or* $w = 0$. In the first case we are free to divide by w. *In this case*, the original

CAUTION!

We do not divide both sides by w (a variable) since equivalence is not guaranteed and we may "lose" a root.

equation is *equivalent* to $6w = 5$, whose only solution is $w = \frac{5}{6}$. Now turning to *the other case*, $w = 0$, we are obliged to examine whether it is also a solution of the original equation—and in *this* problem it is.

Now Work Problem 3 ◁

EXAMPLE 2 **Solving a Quadratic Equation by Factoring**

CAUTION!

You should approach a problem like this with caution. If the product of two quantities is equal to -2, it is not true that at least one of the quantities must be -2. Why?

Solve $(3x - 4)(x + 1) = -2$.

Solution: We first multiply the factors on the left side:

$$3x^2 - x - 4 = -2$$

Rewriting this equation so that 0 appears on one side, we have

$$3x^2 - x - 2 = 0$$

$$(3x + 2)(x - 1) = 0$$

$$x = -\frac{2}{3},\ 1$$

Now Work Problem 7 ◁

Some equations that are not quadratic may be solved by factoring, as Example 3 shows.

EXAMPLE 3 **Solving a Higher-Degree Equation by Factoring**

a. Solve $4x - 4x^3 = 0$.

Solution: This is called a *third-degree equation*. We proceed to solve it as follows:

$$4x - 4x^3 = 0$$

$$4x(1 - x^2) = 0 \qquad \text{factoring}$$

$$4x(1 - x)(1 + x) = 0 \qquad \text{factoring}$$

CAUTION!

Do not neglect the fact that the factor x gives rise to a root.

Setting each factor equal to 0 gives $4 = 0$ (impossible), $x = 0, 1 - x = 0$, or $1 + x = 0$. Thus,

$$x = 0 \text{ or } x = 1 \text{ or } x = -1$$

so that the solution set is $\{-1, 0, 1\}$.

b. Solve $x(x + 2)^2(x + 5) + x(x + 2)^3 = 0$.

Solution: Factoring $x(x + 2)^2$ from both terms on the left side, we have

$$x(x + 2)^2[(x + 5) + (x + 2)] = 0$$

$$x(x + 2)^2(2x + 7) = 0$$

Hence, $x = 0, x + 2 = 0$, or $2x + 7 = 0$, from which it follows that the solution set is $\{-\frac{7}{2}, -2, 0\}$.

Now Work Problem 23 ◁

EXAMPLE 4 **A Fractional Equation Leading to a Quadratic Equation**

Solve

$$\frac{y + 1}{y + 3} + \frac{y + 5}{y - 2} = \frac{7(2y + 1)}{y^2 + y - 6} \tag{2}$$

Solution: Multiplying both sides by the LCD, $(y+3)(y-2)$, we get

$$(y-2)(y+1)+(y+3)(y+5)=7(2y+1) \tag{3}$$

Since Equation (2) was multiplied by an expression involving the variable y, remember (from Section 0.7) that Equation (3) is not necessarily equivalent to Equation (2). After simplifying Equation (3), we have

$$2y^2-7y+6=0 \qquad \text{quadratic equation}$$
$$(2y-3)(y-2)=0 \qquad \text{factoring}$$

We have shown that *if* y satisfies the original equation *then* $y=\frac{3}{2}$ or $y=2$. Thus, $\frac{3}{2}$ and 2 are the only *possible* roots of the given equation. But 2 cannot be a root of Equation (2), since substitution leads to a denominator of 0. However, you should check that $\frac{3}{2}$ does indeed satisfy the *original* equation. Hence, its only root is $\frac{3}{2}$.

Now Work Problem 63 ◁

EXAMPLE 5 **Solution by Factoring**

CAUTION! Do not hastily conclude that the solution of $x^2=3$ consists of $x=\sqrt{3}$ only.

Solve $x^2=3$.

Solution:

$$x^2=3$$
$$x^2-3=0$$

Factoring, we obtain

$$(x-\sqrt{3})(x+\sqrt{3})=0$$

Thus $x-\sqrt{3}=0$ or $x+\sqrt{3}=0$, so $x=\pm\sqrt{3}$.

Now Work Problem 9 ◁

A more general form of the equation $x^2=3$ is $u^2=k$. In the same manner as the preceding, we can show that

$$\text{If } u^2=k \quad \text{then} \quad u=\pm\sqrt{k}. \tag{4}$$

Quadratic Formula

Solving quadratic equations by factoring can be quite difficult, as is evident by trying that method on $0.7x^2-\sqrt{2}x-8\sqrt{5}=0$. However, there is a formula called the *quadratic formula* that gives the roots of any quadratic equation.

Quadratic Formula

The roots of the quadratic equation $ax^2+bx+c=0$, where a, b, and c are constants and $a\neq 0$, are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Actually, the quadratic formula is not hard to derive if one first writes the quadratic equation in the form

$$x^2+\frac{b}{a}x+\frac{c}{a}=0$$

and then as

$$\left(x+\frac{b}{2a}\right)^2-K^2=0$$

for a number K, as yet to be determined. This leads to

$$\left(x+\frac{b}{2a}-K\right)\left(x+\frac{b}{2a}+K\right)=0$$

which in turn leads to $x=-\frac{b}{2a}+K$ or $x=-\frac{b}{2a}-K$ by the methods already under consideration. It is not hard to see what K must be, but it requires further thought to understand how one could discover the value of K without knowing the answer in advance.

EXAMPLE 6 **A Quadratic Equation with Two Real Roots**

Solve $4x^2-17x+15=0$ by the quadratic formula.

Solution: Here $a=4, b=-17$, and $c=15$. Thus,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-(-17)\pm\sqrt{(-17)^2-4(4)(15)}}{2(4)}$$

$$=\frac{17\pm\sqrt{49}}{8}=\frac{17\pm7}{8}$$

The roots are $\frac{17+7}{8}=\frac{24}{8}=3$ and $\frac{17-7}{8}=\frac{10}{8}=\frac{5}{4}$.

Now Work Problem 31 ◁

EXAMPLE 7 **A Quadratic Equation with One Real Root**

Solve $2+6\sqrt{2}y+9y^2=0$ by the quadratic formula.

Solution: Look at the arrangement of the terms. Here $a=9, b=6\sqrt{2}$, and $c=2$. Hence,

$$y=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-6\sqrt{2}\pm\sqrt{0}}{2(9)}$$

Thus,

$$y=\frac{-6\sqrt{2}+0}{18}=-\frac{\sqrt{2}}{3} \quad \text{or} \quad y=\frac{-6\sqrt{2}-0}{18}=-\frac{\sqrt{2}}{3}$$

Therefore, the only root is $-\frac{\sqrt{2}}{3}$.

Now Work Problem 33 ◁

EXAMPLE 8 **A Quadratic Equation with No Real Solution**

Solve $z^2+z+1=0$ by the quadratic formula.

Solution: Here $a=1, b=1$, and $c=1$. The roots are

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{-3}}{2}$$

Now, $\sqrt{-3}$ denotes a number whose square is -3. However, no such real number exists, since the square of any real number is nonnegative. Thus, the equation has no real roots.[5]

Now Work Problem 37 ◁

---

5 $\frac{-1\pm\sqrt{-3}}{2}$ can be expressed as $\frac{-1\pm i\sqrt{3}}{2}$, where $i=\sqrt{-1}$ is called the *imaginary unit*. We emphasize that $i=\sqrt{-1}$ is not a real number. Complex numbers are numbers of the form $a+ib$, with a and b real, but they are not discussed in this book.

This describes the nature of the roots of a quadratic equation.

From Examples 6–8, you can see that a quadratic equation has either two different real roots, exactly one real root, or no real roots, depending on whether $b^2 - 4ac$ is positive, zero, or negative, respectively.

Quadratic-Form Equation

Sometimes an equation that is not quadratic can be transformed into a quadratic equation by an appropriate substitution. In this case, the given equation is said to have **quadratic form**. The next example will illustrate.

EXAMPLE 9 **Solving a Quadratic-Form Equation**

Solve $\frac{1}{x^6} + \frac{9}{x^3} + 8 = 0$.

Solution: This equation can be written as

$$\left(\frac{1}{x^3}\right)^2 + 9\left(\frac{1}{x^3}\right) + 8 = 0$$

so it is quadratic in $1/x^3$ and hence has quadratic form. Substituting the variable w for $1/x^3$ gives a quadratic equation in the variable w, which we can then solve:

$$w^2 + 9w + 8 = 0$$

$$(w + 8)(w + 1) = 0$$

$$w = -8 \quad \text{or} \quad w = -1$$

CAUTION!

Do not assume that -8 and -1 are solutions of the *original* equation.

Returning to the variable x, we have

$$\frac{1}{x^3} = -8 \quad \text{or} \quad \frac{1}{x^3} = -1$$

Thus,

$$x^3 = -\frac{1}{8} \text{ or } x^3 = -1$$

from which it follows that

$$x = -\frac{1}{2} \text{ or } x = -1$$

Checking, we find that these values of x satisfy the original equation.

Now Work Problem 49 ◁

PROBLEMS 0.8

In Problems 1–30, solve by factoring.

1. $x^2 - 4x + 4 = 0$
2. $t^2 + 3t + 2 = 0$
3. $t^2 - 6t + 8 = 0$
4. $x^2 + 3x - 10 = 0$
5. $x^2 - 2x - 3 = 0$
6. $x^2 - 16 = 0$
7. $u^2 - 13u = -36$
8. $2z^2 + 8z + 8 = 0$
9. $x^2 - 4 = 0$
10. $3u^2 - 6u = 0$
11. $t^2 - 5t = 0$
12. $x^2 + 9x = -14$
13. $9x^2 + 4 = -12x$
14. $2z^2 + 9z = 5$
15. $v(3v - 5) = -2$
16. $2 + x - 6x^2 = 0$
17. $-x^2 + 3x + 10 = 0$
18. $\frac{1}{5}u^2 = \frac{2}{5}u$
19. $2p^2 = 3p$
20. $-r^2 - r + 12 = 0$
21. $x(x + 4)(x - 1) = 0$
22. $(w - 3)^2(w + 1)^2 = 0$
23. $s^3 - 16s = 0$
24. $x^3 - 4x^2 - 5x = 0$
25. $6x^3 + 5x^2 - 4x = 0$
26. $(x + 1)^2 - 5x + 1 = 0$
27. $(x - 3)(x^2 - 4) = 0$
28. $7(z^2 - 3z - 4)(z + 7) = 0$
29. $p(p - 3)^2 - 4(p - 3)^3 = 0$
30. $x^4 - 3x^2 + 2 = 0$

In Problems 31–44, find all real roots by using the quadratic formula.

31. $x^2 + 2x - 24 = 0$
32. $x^2 - 2x - 15 = 0$
33. $16x^2 - 40x + 25 = 0$
34. $q^2 - 5q = 0$
35. $p^2 - 2p - 7 = 0$
36. $2 - 2x + x^2 = 0$
37. $4 - 2n + n^2 = 0$
38. $u^2 - u = 1$
39. $4x^2 + 5x - 2 = 0$
40. $w^2 - 2w + 1 = 0$

1 Applications and More Algebra

1.1 Applications of Equations

1.2 Linear Inequalities

1.3 Applications of Inequalities

1.4 Absolute Value

1.5 Summation Notation

1.6 Sequences

Chapter 1 Review

EXPLORE & EXTEND
Variable-Quality Recording

In this chapter, we will apply equations to various practical situations. We will then do the same with inequalities, which are statements that one quantity is less than ($<$), greater than ($>$), less than or equal to ($\leq$), or greater than or equal to ($\geq$) some other quantity.

One application of inequalities is the regulation of sporting equipment. Dozens of baseballs are used in a typical major league game and it would be unrealistic to expect that every ball weigh exactly $5\frac{1}{8}$ ounces. But it is reasonable to require that each one weigh no less than 5 ounces and no more than $5\frac{1}{4}$ ounces, which is how 1.09 of the Official Rules of Major League Baseball reads. (See `http://mlb.mlb.com/` and look up "official rules".) Note that *no less than* is a synonym for *greater than or equal to* while *no more than* is a synonym for *less than or equal to*. In translating English statements into mathematics, we advocate avoiding the negative wordings as a first step. In any event, we have

$$\text{ball weight} \geq 5 \text{ ounces} \quad \text{and} \quad \text{ball weight} \leq 5\frac{1}{4} \text{ ounces}$$

which can be combined to give

$$5 \text{ ounces} \leq \text{ball weight} \leq 5\frac{1}{4} \text{ ounces}$$

which reads nicely as saying that the ball must weigh between 5 and $5\frac{1}{4}$ ounces (where *between* here includes the extreme values).

Another inequality applies to the sailboats used in the America's Cup race. The America's Cup Class (ACC) for yachts was defined until 30 January 2009 by

$$\frac{L + 1.25\sqrt{S} - 9.8\sqrt[3]{DSP}}{0.686} \leq 24.000 \text{ m}$$

The "$\leq$" signifies that the expression on the left must come out as less than or equal to the 24 m on the right. The L, S, and DSP were themselves specified by complicated formulas, but roughly, L stood for length, S for sail area, and DSP for displacement (the hull volume below the waterline).

The ACC formula gave yacht designers some latitude. Suppose a yacht had $L = 20.2$ m, $S = 282$ m^2, and $DSP = 16.4$ m^3. Since the formula is an inequality, the designer could reduce the sail area while leaving the length and displacement unchanged. Typically, however, values of L, S, and DSP were used that made the expression on the left as close to 24 m as possible.

In addition to applications of linear equations and inequalities, this chapter will review the concept of absolute value and introduce sequences and summation notation.

Objective

To model situations described by linear or quadratic equations.

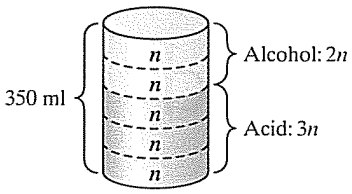


FIGURE 1.1 Chemical solution (Example 1).

CAUTION!

Note that the solution to an equation is not necessarily the solution to the problem posed.

1.1 Applications of Equations

In most cases, to solve practical problems you must translate the relationships stated in the problems into mathematical symbols. This is called *modeling*. The following examples illustrate basic techniques and concepts. Examine each of them carefully before going to the problems.

EXAMPLE 1 Mixture

A chemist must prepare 350 ml of a chemical solution made up of two parts alcohol and three parts acid. How much of each should be used?

Solution: Let n be the number of milliliters in each part. Figure 1.1 shows the situation. From the diagram, we have

$$2n + 3n = 350$$
$$5n = 350$$
$$n = \frac{350}{5} = 70$$

But $n = 70$ is *not* the answer to the original problem. Each *part* has 70 ml. The amount of alcohol is $2n = 2(70) = 140$, and the amount of acid is $3n = 3(70) = 210$. Thus, the chemist should use 140 ml of alcohol and 210 ml of acid. This example shows how helpful a diagram can be in setting up a word problem.

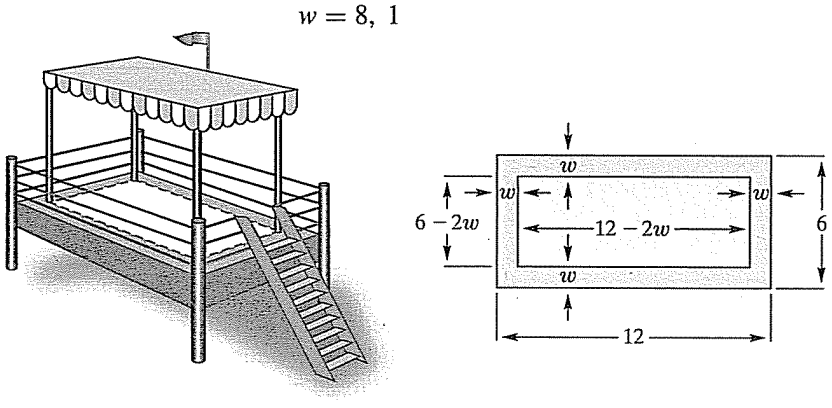
Now Work Problem 5 ◁

EXAMPLE 2 Observation Deck

A rectangular observation deck overlooking a scenic valley is to be built. [See Figure 1.2(a).] The deck is to have dimensions 6 m by 12 m. A rectangular shelter with area 40 m^2 is to be centered over the deck. The uncovered part of the deck is to be a walkway of uniform width. How wide should this walkway be?

Solution: A diagram of the deck is shown in Figure 1.2(b). Let w be the width (in meters) of the walkway. Then the part of the deck under the shelter has dimensions $12 - 2w$ by $6 - 2w$. Since its area must be 40 m^2, where area = (length)(width), we have

$$(12 - 2w)(6 - 2w) = 40$$
$$72 - 36w + 4w^2 = 40 \qquad \text{multiplying}$$
$$4w^2 - 36w + 32 = 0$$
$$w^2 - 9w + 8 = 0 \qquad \text{dividing both sides by 4}$$
$$(w - 8)(w - 1) = 0$$
$$w = 8,\ 1$$



(a) (b)

FIGURE 1.2 Deck walkway (Example 2).

Although 8 is a solution to the equation, it is *not* a solution to our problem, because one of the dimensions of the deck itself is only 6 m. Thus, the only possible solution is that the walkway be 1 m wide.

Now Work Problem 7 ◁

The key words introduced here are *fixed cost, variable cost, total cost, total revenue*, and *profit*. This is the time to gain familiarity with these terms because they recur throughout the book.

In the next example, we refer to some business terms relative to a manufacturing firm. **Fixed cost** is the sum of all costs that are independent of the level of production, such as rent, insurance, and so on. This cost must be paid whether or not output is produced. **Variable cost** is the sum of all costs that are dependent on the level of output, such as labor and material. **Total cost** is the sum of variable cost and fixed cost:

$$\text{total cost} = \text{variable cost} + \text{fixed cost}$$

Total revenue is the money that the manufacturer receives for selling the output:

$$\text{total revenue} = (\text{price per unit})\,(\text{number of units sold})$$

Profit is total revenue minus total cost:

$$\text{profit} = \text{total revenue} - \text{total cost}$$

EXAMPLE 3 Profit

The Anderson Company produces a product for which the variable cost per unit is \$6 and fixed cost is \$80,000. Each unit has a selling price of \$10. Determine the number of units that must be sold for the company to earn a profit of \$60,000.

Solution: Let q be the number of units that must be sold. (In many business problems, q represents quantity.) Then the variable cost (in dollars) is $6q$. The *total* cost for the business is therefore $6q + 80{,}000$. The total revenue from the sale of q units is $10q$. Since

$$\text{profit} = \text{total revenue} - \text{total cost}$$

our model for this problem is

$$60{,}000 = 10q - (6q + 80{,}000)$$

Solving gives

$$60{,}000 = 10q - 6q - 80{,}000$$
$$4q = 140{,}000$$
$$q = 35{,}000$$

Thus, 35,000 units must be sold to earn a profit of \$60,000.

Now Work Problem 9 ◁

EXAMPLE 4 Pricing

Sportcraft manufactures denim clothing and is planning to sell its new line of jeans to retail outlets. The cost to the retailer will be \$60 per pair of jeans. As a convenience to the retailer, Sportcraft will attach a price tag to each pair. What amount should be marked on the price tag so that the retailer can reduce this price by 20% during a sale and still make a profit of 15% on the cost?

Solution: Here we use the fact that

Note that price = cost + profit.

$$\text{selling price} = \text{cost per pair} + \text{profit per pair}$$

Let p be the tag price per pair, in dollars. During the sale, the retailer actually receives $p - 0.2p$. This must equal the cost, 60, plus the profit, (0.15)(60). Hence,

$$\begin{aligned} \text{selling price} &= \text{cost} + \text{profit} \\ p - 0.2p &= 60 + (0.15)(60) \\ 0.8p &= 69 \\ p &= 86.25 \end{aligned}$$

Sportcraft should mark the price tag at \$86.25.

Now Work Problem 13 ◁

EXAMPLE 5 Investment

A total of \$10,000 was invested in two business ventures, A and B. At the end of the first year, A and B yielded returns of 6% and $5\frac{3}{4}\%$, respectively, on the original investments. How was the original amount allocated if the total amount earned was \$588.75?

Solution: Let x be the amount (in dollars) invested at 6%. Then $10{,}000 - x$ was invested at $5\frac{3}{4}\%$. The interest earned from A was $(0.06)(x)$ and that from B was $(0.0575)(10{,}000 - x)$ with a total of 588.75. Hence,

$$\begin{aligned} (0.06)x + (0.0575)(10{,}000 - x) &= 588.75 \\ 0.06x + 575 - 0.0575x &= 588.75 \\ 0.0025x &= 13.75 \\ x &= 5500 \end{aligned}$$

Thus, \$5500 was invested at 6%, and $\$10,000 - \$5500 = \$4500$ was invested at $5\frac{3}{4}\%$.

Now Work Problem 11 ◁

EXAMPLE 6 Bond Redemption

The board of directors of Maven Corporation agrees to redeem some of its bonds in two years. At that time, \$1,102,500 will be required. Suppose the firm presently sets aside \$1,000,000. At what annual rate of interest, compounded annually, will this money have to be invested in order that its future value be sufficient to redeem the bonds?

Solution: Let r be the required annual rate of interest. At the end of the first year, the accumulated amount will be \$1,000,000 plus the interest, $1{,}000{,}000r$, for a total of

$$1{,}000{,}000 + 1{,}000{,}000r = 1{,}000{,}000(1 + r)$$

Under compound interest, at the end of the second year the accumulated amount will be $1{,}000{,}000(1 + r)$ plus the interest on this, which is $1{,}000{,}000(1 + r)r$. Thus, the total value at the end of the second year will be

$$1{,}0000{,}000(1 + r) + 1{,}000{,}000(1 + r)r$$

This must equal \$1,102,500:

$$1{,}000{,}000(1 + r) + 1{,}000{,}000(1 + r)r = 1{,}102{,}500 \tag{1}$$

Since $1{,}000{,}000(1 + r)$ is a common factor of both terms on the left side, we have

$$\begin{aligned} 1{,}000{,}000(1 + r)(1 + r) &= 1{,}102{,}500 \\ 1{,}000{,}000(1 + r)^2 &= 1{,}102{,}500 \\ (1 + r)^2 &= \frac{1{,}102{,}500}{1{,}000{,}000} = \frac{11{,}025}{10{,}000} = \frac{441}{400} \\ 1 + r &= \pm\sqrt{\frac{441}{400}} = \pm\frac{21}{20} \\ r &= -1 \pm \frac{21}{20} \end{aligned}$$

Thus, $r = -1 + (21/20) = 0.05$, or $r = -1 - (21/20) = -2.05$. Although 0.05 and -2.05 are roots of Equation (1), we reject -2.05, since we require that r be positive. Hence, $r = 0.05 = 5\%$ is the desired rate.

Now Work Problem 15 ◁

At times there may be more than one way to model a word problem, as Example 7 shows.

EXAMPLE 7 Apartment Rent

A real-estate firm owns the Parklane Garden Apartments, which consist of 96 apartments. At \$550 per month, every apartment can be rented. However, for each \$25 per month increase, there will be three vacancies with no possibility of filling them. The firm wants to receive \$54,600 per month from rents. What rent should be charged for each apartment?

Solution:

Method I. Suppose r is the rent (in dollars) to be charged per apartment. Then the increase over the \$550 level is $r - 550$. Thus, the number of \$25 increases is $\dfrac{r-550}{25}$. Because each \$25 increase results in three vacancies, the total number of vacancies will be $3\left(\dfrac{r-550}{25}\right)$. Hence, the total number of apartments rented will be $96 - 3\left(\dfrac{r-550}{25}\right)$. Since

$$\text{total rent} = (\text{rent per apartment})(\text{number of apartments rented})$$

we have

$$54{,}600 = r\left(96 - \frac{3(r-550)}{25}\right)$$

$$54{,}600 = r\left(\frac{2400 - 3r + 1650}{25}\right)$$

$$54{,}600 = r\left(\frac{4050 - 3r}{25}\right)$$

$$1{,}365{,}000 = r(4050 - 3r)$$

Thus,

$$3r^2 - 4050r + 1{,}365{,}000 = 0$$

By the quadratic formula,

$$r = \frac{4050 \pm \sqrt{(-4050)^2 - 4(3)(1{,}365{,}000)}}{2(3)}$$

$$= \frac{4050 \pm \sqrt{22{,}500}}{6} = \frac{4050 \pm 150}{6} = 675 \pm 25$$

Hence, the rent for each apartment should be either \$650 or \$700.

Method II. Suppose n is the number of \$25 increases. Then the increase in rent per apartment will be $25n$ and there will be $3n$ vacancies. Since

$$\text{total rent} = (\text{rent per apartment})(\text{number of apartments rented})$$

we have

$$54{,}600 = (550 + 25n)(96 - 3n)$$

$$54{,}600 = 52{,}800 + 750n - 75n^2$$

$$75n^2 - 750n + 1800 = 0$$

$$n^2 - 10n + 24 = 0$$

$$(n-6)(n-4) = 0$$

Thus, $n = 6$ or $n = 4$. The rent charged should be either $550 + 25(6) = \$700$ or $550 + 25(4) = \$650$. However, it is easy to see that the real-estate firm can receive \$54,675 per month from rents by charging \$675 for each apartment and that \$54,675 is

the *maximum* amount from rents that it can receive, given to existing market conditions. In a sense, the firm posed the wrong question. A considerable amount of our work in this book focuses on a better question that the firm might have asked.

Now Work Problem 29 ◁

PROBLEMS 1.1

1. Fencing A fence is to be placed around a rectangular plot so that the enclosed area is 800 ft^2 and the length of the plot is twice the width. How many feet of fencing must be used?

2. Geometry The perimeter of a rectangle is 300 ft, and the length of the rectangle is 3 ft more than twice the width. Find the dimensions of the rectangle.

3. Tent Caterpillars One of the most damaging defoliating insects is the tent caterpillar, which feeds on foliage of shade, forest, and fruit trees. A homeowner lives in an area in which the tent caterpillar has become a problem. She wishes to spray the trees on her property before more defoliation occurs. She needs 145 oz of a solution made up of 4 parts of insecticide A and 5 parts of insecticide B. The solution is then mixed with water. How many ounces of each insecticide should be used?

4. Concrete Mix A builder makes a certain type of concrete by mixing together 1 part portland cement (made from lime and clay), 3 parts sand, and 5 parts crushed stone (by volume). If 765 ft^3 of concrete are needed, how many cubic feet of each ingredient does he need?

5. Furniture Finish According to *The Consumer's Handbook* [Paul Fargis, ed. (New York: Hawthorn, 1974)], a good oiled furniture finish contains two parts boiled linseed oil and one part turpentine. If you need a pint (16 fluid oz) of this furniture finish, how many fluid ounces of turpentine are needed?

6. Forest Management A lumber company owns a forest that is of rectangular shape, 1 mi by 2 mi. If the company cuts a uniform strip of trees along the outer edges of this forest, how wide should the strip be if $\frac{3}{4}$ sq mi of forest is to remain?

7. Garden Pavement A 10 m square plot is to have a circular flower bed of 60 m^2 centered at the center of the square. The other part of the plot is to be paved so that the owners can walk around the flower bed. What is the minimum "width" of the paved surface? In other words, what is the smallest distance from the flower bed to the edge of the plot?

8. Ventilating Duct The diameter of a circular ventilating duct is 140 mm. This duct is joined to a square duct system as shown in Figure 1.3. To ensure smooth airflow, the areas of the circle and square sections must be equal. To the nearest millimeter, what should the length x of a side of the square section be?

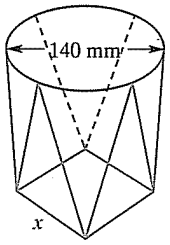


FIGURE 1.3 Ventilating duct (Problem 8).

9. Profit A corn refining company produces corn gluten cattle feed at a variable cost of \$82 per ton. If fixed costs are \$120,000 per month and the feed sells for \$134 per ton, how many tons must be sold each month for the company to have a monthly profit of \$560,000?

10. Sales The Smith Company management would like to know the total sales units that are required for the company to earn a profit of \$150,000. The following data are available: unit selling price of \$50; variable cost per unit of \$25; total fixed cost of \$500,000. From these data, determine the required sales units.

11. Investment A person wishes to invest \$20,000 in two enterprises so that the total income per year will be \$1440. One enterprise pays 6% annually; the other has more risk and pays $7\frac{1}{2}$% annually. How much must be invested in each?

12. Investment A person invested \$120,000, part at an interest rate of 4% annually and the remainder at 5% annually. The total interest at the end of 1 year was equivalent to an annual $4\frac{1}{2}$% rate on the entire \$120,000. How much was invested at each rate?

13. Pricing The cost of a product to a retailer is \$3.40. If the retailer wishes to make a profit of 20% on the selling price, at what price should the product be sold?

14. Bond Retirement In three years, a company will require \$1,125,800 in order to retire some bonds. If the company now invests \$1,000,000 for this purpose, what annual rate of interest, compounded annually, must it receive on that amount in order to retire the bonds?

15. Expansion Program In two years, a company will begin an expansion program. It has decided to invest \$3,000,000 now so that in two years the total value of the investment will be \$3,245,000, the amount required for the expansion. What is the annual rate of interest, compounded annually, that the company must receive to achieve its purpose?

16. Business A company finds that if it produces and sells q units of a product, its total sales revenue in dollars is $100\sqrt{q}$. If the variable cost per unit is \$2 and the fixed cost is \$1200, find the values of q for which

$$\text{total sales revenue} = \text{variable cost} + \text{fixed cost}$$

(That is, profit is zero.)

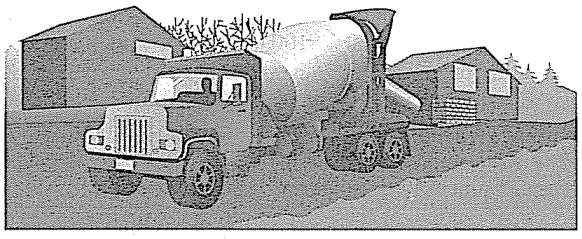
17. Overbooking A commuter airplane has 81 seats. On the average, 90% of those who book for a flight show up for it. How many seats should the airline book for a flight if it wants to fill the plane?

18. Poll A group of people were polled, and 20%, or 700, of them favored a new product over the best-selling brand. How many people were polled?

19. Prison Guard Salary It was reported that in a certain women's jail, female prison guards, called matrons, received 30% (or \$200) a month less than their male counterparts, deputy

sheriffs. Find the yearly salary of a deputy sheriff. Give your answer to the nearest dollar.

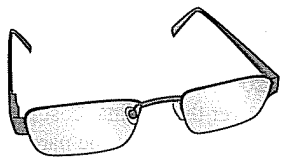
20. Striking Drivers A few years ago, cement drivers were on strike for 46 days. Before the strike, these drivers earned \$7.50 per hour and worked 260 eight-hour days a year. What percentage increase is needed in yearly income to make up for the lost time within 1 year?



21. Break Even A manufacturer of video-games sells each copy for \$21.95. The manufacturing cost of each copy is \$14.92. Monthly fixed costs are \$8500. During the first month of sales of a new game, how many copies must be sold in order for the manufacturer to break even (that is, in order that total revenue equal total cost)?

22. Investment Club An investment club bought a bond of an oil corporation for \$5000. The bond yields 4% per year. The club now wants to buy shares of stock in a windmill supply company. The stock sells at \$20 per share and earns a dividend of \$0.50 per share per year. How many shares should the club buy so that its total investment in stocks and bonds yields 3% per year?

23. Vision Care As a fringe benefit for its employees, a company established a vision-care plan. Under this plan, each year the company will pay the first \$35 of an employee's vision-care expenses and 80% of all additional vision-care expenses, up to a maximum *total* benefit payment of \$100. For an employee, find the total annual vision-care expenses covered by this program.



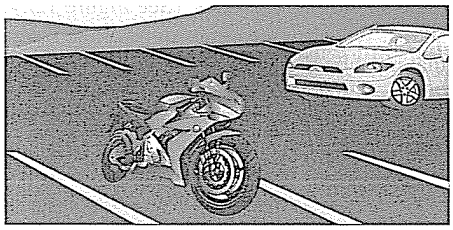
24. Quality Control Over a period of time, the manufacturer of a caramel-center candy bar found that 3.1% of the bars were rejected for imperfections.
(a) If c candy bars are made in a year, how many would the manufacturer expect to be rejected?
(b) This year, annual consumption of the candy is projected to be six hundred million bars. Approximately how many bars will have to be made if rejections are taken into consideration?

25. Business Suppose that consumers will purchase q units of a product when the price is $(80 - q)/4$ dollars *each*. How many units must be sold in order that sales revenue be \$400?

26. Investment How long would it take to triple an investment at simple interest with a rate of 4.5% per year? [*Hint*: See Example 6(a) of Section 0.7, and express 4.5% as 0.045.]

27. Business Alternatives The band Mongeese tried to sell its song Kobra Klub to a small label, Epsilon Records, for a lump-sum payment of \$50,000. After estimating that future sales possibilities of Kobra Klub beyond one year are nonexistent, Epsilon management is reviewing an alternative proposal to give a lump-sum payment of \$5000 to Mongeese plus a royalty of \$0.50 for each disc sold. How many units must be sold the first year to make this alternative as economically attractive to the band as their original request? [*Hint:* Determine when the incomes under both proposals are the same.]

28. Parking Lot A company parking lot is 120 ft long and 80 ft wide. Due to an increase in personnel, it is decided to double the area of the lot by adding strips of equal width to one end and one side. Find the width of one such strip.



29. Rentals You are the chief financial advisor to a corporation that owns an office complex consisting of 50 units. At \$400 per month, every unit can be rented. However, for each \$20 per month increase, there will be two vacancies with no possibility of filling them. The corporation wants to receive a total of \$20,240 per month from rents in the complex. You are asked to determine the rent that should be charged for each unit. What is your reply?

30. Investment Six months ago, an investment company had a \$3,100,000 portfolio consisting of blue-chip and glamour stocks. Since then, the value of the blue-chip investment increased by $\frac{1}{10}$, whereas the value of the glamour stocks decreased by $\frac{1}{10}$. The current value of the portfolio is \$3,240,000. What is the *current* value of the blue-chip investment?

31. Revenue The monthly revenue of a certain company is given by $R = 800p - 7p^2$, where p is the price in dollars of the product the company manufactures. At what price will the revenue be \$10,000 if the price must be greater than \$50?

32. Price-Earnings Ratio The *price-earnings ratio* P/E of a company is the ratio of the market value of one share of the company's outstanding common stock to the earnings per share. If P/E increases by 15% and the earnings per share decrease by 10%, determine the percentage change in the market value per share of the common stock.

33. Market Equilibrium When the price of a product is p dollars each, suppose that a manufacturer will supply $2p - 10$ units of the product to the market and that consumers will demand to buy $200 - 3p$ units. At the value of p for which supply equals demand, the market is said to be in equilibrium. Find this value of p.

34. Market Equilibrium Repeat Problem 33 for the following conditions: At a price of p dollars each, the supply is $2p^2 - 3p$ and the demand is $20 - p^2$.

35. Security Fence For security reasons, a company will enclose a rectangular area of 11,200 ft^2 in the rear of its plant. One side will be bounded by the building and the other three sides

by fencing. (See Figure 1.4.) If 300 ft of fencing will be used, what will be the dimensions of the rectangular area?

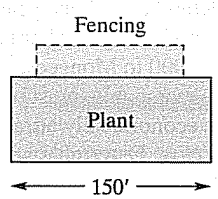


FIGURE 1.4 Security fencing (Problem 35).

36. Package Design A company is designing a package for its product. One part of the package is to be an open box made from a square piece of aluminum by cutting out a 2-in. square from each corner and folding up the sides. (See Figure 1.5.) The box is to contain 50 in^3. What are the dimensions of the square piece of aluminum that must be used?

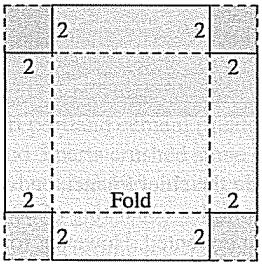


FIGURE 1.5 Box construction (Problem 36).

37. Product Design A candy company makes the popular Henney's, whose main ingredient is chocolate. The rectangular-shaped bar is 10 centimeters (cm) long, 5 cm wide, and 2 cm thick. (See Figure 1.6.) The spot price of chocolate has *decreased* by 60% and the company has decided to reward its loyal customers with a 50% increase in the volume of the bar! The thickness will remain the same, but the length and width will be increased by equal amounts. What will be the length and width of the new bar?

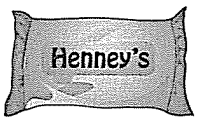


FIGURE 1.6 Candy bar (Problem 37).

38. Product Design A candy company makes a washer-shaped candy (a candy with a hole in it); see Figure 1.7. Because of increasing costs, the company will cut the volume of candy in each piece by 22%. To do this, the firm will keep the same thickness and outer radius but will make the inner radius larger. At present the thickness is 2.1 millimeters (mm), the inner radius is 2 mm, and the outer radius is 7.1 mm. Find the inner radius of the new-style candy. [*Hint*: The volume V of a solid disc is $\pi r^2 h$, where r is the radius and h is the thickness of the disc.]

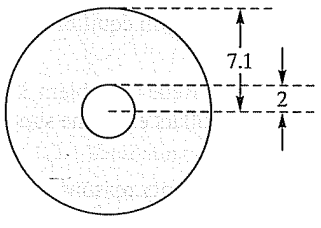
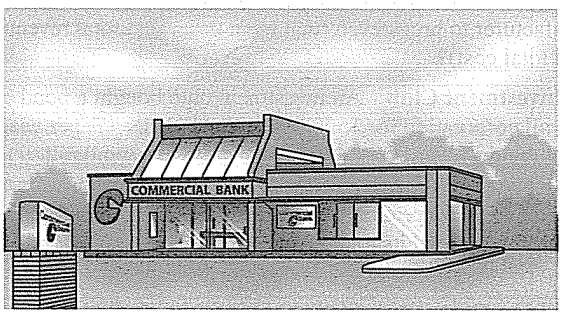


FIGURE 1.7 Washer-shaped candy (Problem 38).

39. Compensating Balance *Compensating balance* refers to that practice wherein a bank requires a borrower to maintain on deposit a certain portion of a loan during the term of the loan. For example, if a firm takes out a $100,000 loan that requires a compensating balance of 20%, it would have to leave $20,000 on deposit and would have the use of $80,000. To meet the expenses of retooling, the Barber Die Company needs $195,000. The Third National Bank, with whom the firm has had no prior association, requires a compensating balance of 16%. To the nearest thousand dollars, what amount of loan is required to obtain the needed funds? Now solve the general problem of determining the amount L of a loan that is needed to handle expenses E if the bank requires a compensating balance of p%.



40. Incentive Plan A machine company has an incentive plan for its salespeople. For each machine that a salesperson sells, the commission is $40. The commission for *every* machine sold will increase by $0.04 for each machine sold over 600. For example, the commission on each of 602 machines sold is $40.08. How many machines must a salesperson sell in order to earn $30,800?

41. Real Estate A land investment company purchased a parcel of land for $7200. After having sold all but 20 acres at a profit of $30 per acre over the original cost per acre, the company regained the entire cost of the parcel. How many acres were sold?

42. Margin of Profit The *margin of profit* of a company is the net income divided by the total sales. A company's margin of profit increased by 0.02 from last year. Last year the company sold its product at $3.00 each and had a net income of $4500. This year it increased the price of its product by $0.50 each, sold 2000 more, and had a net income of $7140. The company never has had a margin of profit greater than 0.15. How many of its product were sold last year and how many were sold this year?

43. Business A company manufactures products A and B. The cost of producing each unit of A is $2 more than that of B. The costs of production of A and B are $1500 and $1000, respectively, and 25 more units of A are produced than of B. How many of each are produced?

Objective

To solve linear inequalities in one variable and to introduce interval notation.

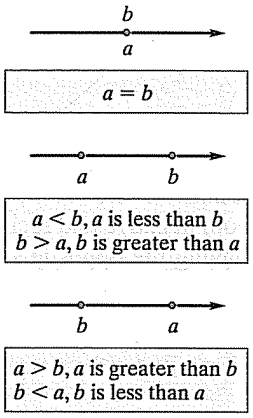


FIGURE 1.8 Relative positions of two points.

1.2 Linear Inequalities

Suppose a and b are two points on the real-number line. Then either a and b coincide, or a lies to the left of b, or a lies to the right of b. (See Figure 1.8.)

If a and b coincide, then $a = b$. If a lies to the left of b, we say that a is less than b and write $a < b$, where the *inequality symbol* "$<$" is read "is less than." On the other hand, if a lies to the right of b, we say that a is greater than b, written $a > b$. The statements $a > b$ and $b < a$ are equivalent. (If you have trouble keeping these symbols straight, it may help to notice that $<$ looks somewhat like the letter L for *left* and that we have $a < b$ precisely when a lies to the *left* of b.)

Another inequality symbol "$\leq$" is read "is less than or equal to" and is defined as follows: $a \leq b$ if and only if $a < b$ or $a = b$. Similarly, the symbol "$\geq$" is defined as follows: $a \geq b$ if and only if $a > b$ or $a = b$. In this case, we say that a is greater than or equal to b.

We often use the words *real numbers* and *points* interchangeably, since there is a one-to-one correspondence between real numbers and points on a line. Thus, we can speak of the points -5, -2, 0, 7, and 9 and can write $7 < 9$, $-2 > -5$, $7 \leq 7$, and $7 \geq 0$. (See Figure 1.9.) Clearly, if $a > 0$, then a is positive; if $a < 0$, then a is negative.

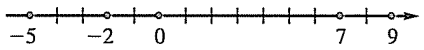


FIGURE 1.9 Points on a number line.

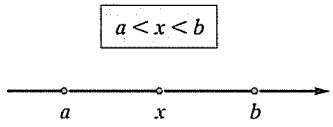


FIGURE 1.10 $a < x$ and $x < b$.

Suppose that $a < b$ and x is between a and b. (See Figure 1.10.) Then not only is $a < x$, but also, $x < b$. We indicate this by writing $a < x < b$. For example, $0 < 7 < 9$. (Refer back to Figure 1.9.)

Definition

An ***inequality*** is a statement that one quantity is less than, or greater than, or less than or equal to, or greater than or equal to, another quantity.

Of course, we represent inequalities by means of inequality symbols. If two inequalities have their inequality symbols pointing in the same direction, then the inequalities are said to have the *same sense*. If not, they are said to be *opposite in sense*. Hence, $a < b$ and $c < d$ have the same sense, but $a < b$ has the opposite sense of $c > d$.

Solving an inequality, such as $2(x-3) < 4$, means finding all values of the variable for which the inequality is true. This involves the application of certain rules, which we now state.

Rules for Inequalities

1. If the same number is added to or subtracted from both sides of an inequality, then another inequality results, having the same sense as the original inequality. Symbolically,

$$\text{If } a < b, \text{ then } a + c < b + c \textbf{ and } a - c < b - c.$$

Keep in mind that the rules also apply to $\leq$, $>$, and $\geq$.

For example, $7 < 10$ so $7 + 3 < 10 + 3$.

2. If both sides of an inequality are multiplied or divided by the same ***positive*** number, then another inequality results, having the same sense as the original inequality. Symbolically,

$$\text{If } a < b \text{ and } c > 0, \text{ then } ac < bc \text{ and } \frac{a}{c} < \frac{b}{c}.$$

For example, $3 < 7$ and $2 > 0$ so $3(2) < 7(2)$ and $\frac{3}{2} < \frac{7}{2}$.

CAUTION!

Multiplying or dividing an inequality by a negative number gives an inequality of the opposite sense.

3. If both sides of an inequality are multiplied or divided by the same ***negative*** number, then another inequality results, having the opposite sense of the original inequality. Symbolically,

$$\text{If } a < b \text{ and } c < 0, \text{ then } a(c) > b(c) \text{ and } \frac{a}{c} > \frac{b}{c}.$$

For example, $4 < 7$ and $-2 < 0$ so $4(-2) > 7(-2)$ and $\frac{4}{-2} > \frac{7}{-2}$.

4. Any side of an inequality can be replaced by an expression equal to it. Symbolically,

$$\text{If } a < b \text{ and } a = c, \text{ then } c < b.$$

For example, if $x < 2$ and $x = y + 4$, then $y + 4 < 2$.

5. If the sides of an inequality are either both positive or both negative and reciprocals are taken on both sides, then another inequality results, having the opposite sense of the original inequality. Symbolically,

$$\text{If } 0 < a < b \text{ or } a < b < 0, \text{ then } \frac{1}{a} > \frac{1}{b}.$$

For example, $2 < 4$ so $\frac{1}{2} > \frac{1}{4}$ and $-4 < -2$ so $\frac{1}{-4} > \frac{1}{-2}$.

6. If both sides of an inequality are positive and each side is raised to the same positive power, then another inequality results, having the same sense as the original inequality. Symbolically,

$$\text{If } 0 < a < b \text{ and } n > 0, \text{ then } a^n < b^n.$$

For n a positive integer, this rule further provides

$$\text{If } 0 < a < b, \text{ then } \sqrt[n]{a} < \sqrt[n]{b}.$$

For example, $4 < 9$ so $4^2 < 9^2$ and $\sqrt{4} < \sqrt{9}$.

A pair of inequalities will be said to be *equivalent inequalities* if when either is true then the other is true. When any of Rules 1–6 are applied to an inequality, it is easy to show that the result is an equivalent inequality.

Expanding on the terminology in Section 0.1, a number a is *positive* if $0 < a$ and *negative* if $a < 0$. It is often useful to say that a is *nonnegative* if $0 \leq a$. (One could also say that a is *nonpositive* if $a \leq 0$ but this terminology is not often used.)

Observe from Rule 1 that $a \leq b$ is equivalent to "$b - a$ is nonnegative." Another simple observation is that $a \leq b$ is equivalent to "there exists a nonnegative number s such that $a + s = b$." The s which does the job is just $b - a$ but the idea is useful when one side of $a \leq b$ contains an unknown.

This idea allows us to replace an inequality with an equality—at the expense of introducing a variable. In Chapter 7, the powerful simplex method builds on replacement of inequalities $a \leq b$ with equations $a + s = b$, for nonnegative s. In this context, s is called a *slack variable* because it takes up the "slack" between a and b.

We will now apply Rules 1–4 to a *linear inequality*.

The definition also applies to $\leq$, $>$, and $\geq$.

Definition

A ***linear inequality*** in the variable x is an inequality that can be written in the form

$$ax + b < 0$$

where a and b are constants and $a \neq 0$.

We should expect that the inequality will be true for some values of x and false for others. To ***solve*** an inequality involving a variable is to find all values of the variable for which the inequality is true.

APPLY IT ▷

1. A salesman has a monthly income given by $I = 200 + 0.8S$, where S is the number of products sold in a month. How many products must he sell to make at least \$4500 a month?

EXAMPLE 1 **Solving a Linear Inequality**

Solve $2(x - 3) < 4$.

Solution:

Strategy We will replace the given inequality by equivalent inequalities until the solution is evident.

$$\begin{aligned} 2(x-3) &< 4 \\ 2x - 6 &< 4 && \text{Rule 4} \\ 2x - 6 + 6 &< 4 + 6 && \text{Rule 1} \\ 2x &< 10 && \text{Rule 4} \\ \frac{2x}{2} &< \frac{10}{2} && \text{Rule 2} \\ x &< 5 && \text{Rule 4} \end{aligned}$$

All of the foregoing inequalities are equivalent. Thus, the original inequality is true for *all* real numbers x such that $x < 5$. For example, the inequality is true for $x = -10, -0.1, 0, \frac{1}{2}$, and 4.9. We can write our solution simply as $x < 5$ and present it geometrically by the colored half-line in Figure 1.11. The parenthesis indicates that 5 *is not included* in the solution.

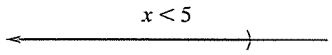


FIGURE 1.11 All real numbers less than 5.

Now Work Problem 9 ◁

APPLY IT ▷

2. A zoo veterinarian can purchase four different animal foods with various nutrient values for the zoo's grazing animals. Let x_1 represent the number of bags of food 1, x_2 represent the number of bags of food 2, and so on. The number of bags of each food needed can be described by the following equations:

$$x_1 = 150 - x_4$$

$$x_2 = 3x_4 - 210$$

$$x_3 = x_4 + 60$$

Assuming that each variable must be nonnegative, write four inequalities involving x_4 that follow from these equations.

In Example 1 the solution consisted of a set of numbers, namely, all real numbers less than 5. It is common to use the term **interval** to describe such a set. In the case of Example 1, the set of all x such that $x < 5$ can be denoted by the *interval notation* $(-\infty, 5)$. The symbol $-\infty$ is not a number, but is merely a convenience for indicating that the interval includes all numbers less than 5.

There are other types of intervals. For example, the set of all real numbers x for which $a \le x \le b$ is called a **closed interval** and includes the numbers a and b, which are called *endpoints* of the interval. This interval is denoted by $[a, b]$ and is shown in Figure 1.12(a). The square brackets indicate that a and b *are included* in the interval. On the other hand, the set of all x for which $a < x < b$ is called an **open interval** and is denoted by (a, b). The endpoints *are not included* in this set. [See Figure 1.12(b).] Extending these concepts and notations, we have the intervals shown in Figure 1.13. Just as $-\infty$ is not a number, so ∞ is not a number but (a, ∞) is a convenient notation for the set of all real numbers x for which $a < x$. Similarly, $[a, \infty)$ denotes all real x for which $a \le x$. It is a natural extension of this notation to *write* $(-\infty, \infty)$ *for the set of* all *real numbers and we will do so throughout this book.*

Closed interval $[a, b]$ (a) Open interval (a, b) (b)

FIGURE 1.12 Closed and open intervals.

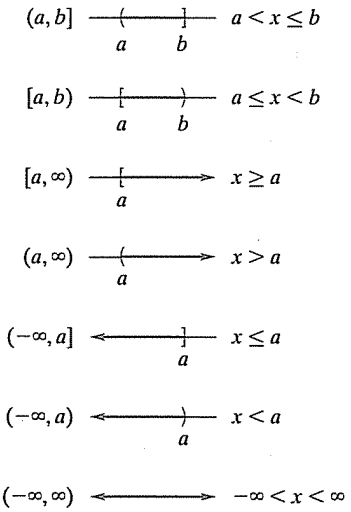


FIGURE 1.13 Intervals.

CAUTION!

Dividing both sides by -2 reverses the sense of the inequality.

EXAMPLE 2 **Solving a Linear Inequality**

Solve $3 - 2x \le 6$.

Solution:

$$\begin{aligned} 3 - 2x &\le 6 \\ -2x &\le 3 && \text{Rule 1} \\ x &\ge -\frac{3}{2} && \text{Rule 3} \end{aligned}$$

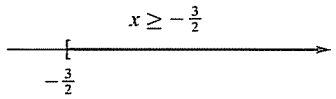


FIGURE 1.14 The interval $[-\frac{3}{2}, \infty)$.

The solution is $x \geq -\frac{3}{2}$, or, in interval notation, $[-\frac{3}{2}, \infty)$. This is represented geometrically in Figure 1.14.

Now Work Problem 7 ◁

EXAMPLE 3 **Solving a Linear Inequality**

Solve $\frac{3}{2}(s-2)+1 > -2(s-4)$.

Solution:

$$\frac{3}{2}(s-2)+1 > -2(s-4)$$

$$2\left[\frac{3}{2}(s-2)+1\right] > 2[-2(s-4)] \qquad \text{Rule 2}$$

$$3(s-2)+2 > -4(s-4)$$

$$3s-4 > -4s+16$$

$$7s > 20 \qquad \text{Rule 1}$$

$$s > \frac{20}{7} \qquad \text{Rule 2}$$

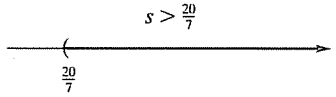


FIGURE 1.15 The interval $(\frac{20}{7}, \infty)$.

The solution is $(\frac{20}{7}, \infty)$; see Figure 1.15.

Now Work Problem 19 ◁

EXAMPLE 4 **Solving Linear Inequalities**

a. Solve $2(x-4)-3 > 2x-1$.

Solution:

$$2(x-4)-3 > 2x-1$$

$$2x-8-3 > 2x-1$$

$$-11 > -1$$

Since it is never true that $-11 > -1$, there is no solution, and the solution set is $\emptyset$ (the set with no elements).

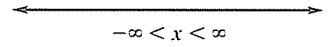


FIGURE 1.16 The interval $(-\infty, \infty)$.

b. Solve $2(x-4)-3 < 2x-1$.

Solution: Proceeding as in part (a), we obtain $-11 < -1$. This is true for all real numbers x, so the solution is $(-\infty, \infty)$; see Figure 1.16.

Now Work Problem 15 ◁

PROBLEMS 1.2

In Problems 1–34, solve the inequalities. Give your answer in interval notation, and indicate the answer geometrically on the real-number line.

1. $5x > 15$

2. $4x < -2$

3. $5x - 11 \leq 9$

4. $5x \leq 0$

5. $-4x \geq 2$

6. $3z + 2 > 0$

7. $5 - 7s > 3$

8. $4s - 1 < -5$

9. $3 < 2y + 3$

10. $4 \leq 3 - 2y$

11. $t + 6 \leq 2 + 3t$

12. $-3 \geq 8(2 - x)$

13. $3(2 - 3x) > 4(1 - 4x)$

14. $8(x + 1) + 1 < 3(2x) + 1$

15. $2(4x - 2) > 4(2x + 1)$

16. $5 - (x + 2) \leq 2(2 - x)$

17. $x + 2 < \sqrt{3} - x$

18. $\sqrt{2}(x + 2) > \sqrt{8}(3 - x)$

19. $\frac{5}{6}x < 40$

20. $-\frac{2}{3}x > 6$

21. $\frac{5y+2}{4} \leq 2y - 1$

22. $\frac{3y-2}{3} \geq \frac{1}{4}$

23. $-3x + 1 \leq -3(x - 2) + 1$

24. $0x \leq 0$

25. $\frac{1-t}{2} < \frac{3t-7}{3}$

26. $\frac{5(3t+1)}{3} > \frac{2t-4}{6} + \frac{t}{2}$

27. $2x + 13 \geq \frac{1}{3}x - 7$

28. $3x - \frac{1}{3} \leq \frac{5}{2}x$

29. $\frac{2}{3}r < \frac{5}{6}r$

30. $\frac{7}{4}t > -\frac{8}{3}t$

31. $y + \frac{y}{2} < \frac{y}{3} + \frac{y}{5}$

32. $9 - 0.1x \leq \frac{2 - 0.01x}{0.2}$

33. $0.1(0.03x + 4) \geq 0.02x + 0.434$

34. $\frac{3y-1}{-3} < \frac{5(y+1)}{-3}$

35. Savings Each month last year, Brittany saved more than \$50, but less than \$150. If S represents her total savings for the year, describe S by using inequalities.

36. Labor Using inequalities, symbolize the following statement: The number of labor hours t to produce a product is not less than 3 and not more than 5.

37. Geometry In a right triangle, one of the acute angles x is less than 3 times the other acute angle plus 10 degrees. Solve for x.

38. Spending A student has \$360 to spend on a stereo system and some compact discs. If she buys a stereo that costs \$219 and the discs cost \$18.95 each, find the greatest number of discs she can buy.

1.3 Applications of Inequalities

Objective

To model real-life situations in terms of inequalities.

Solving word problems may sometimes involve inequalities, as the following examples illustrate.

EXAMPLE 1 **Profit**

For a company that manufactures aquarium heaters, the combined cost for labor and material is \$21 per heater. Fixed costs (costs incurred in a given period, regardless of output) are \$70,000. If the selling price of a heater is \$35, how many must be sold for the company to earn a profit?

Solution:

Strategy Recall that

$$\text{profit} = \text{total revenue} - \text{total cost}$$

We will find total revenue and total cost and then determine when their difference is positive.

Let q be the number of heaters that must be sold. Then their cost is $21q$. The total cost for the company is therefore $21q + 70{,}000$. The total revenue from the sale of q heaters will be $35q$. Now,

$$\text{profit} = \text{total revenue} - \text{total cost}$$

and we want profit > 0. Thus,

$$\begin{aligned} \text{total revenue} - \text{total cost} &> 0 \\ 35q - (21q + 70{,}000) &> 0 \\ 14q &> 70{,}000 \\ q &> 5000 \end{aligned}$$

Since the number of heaters must be a nonnegative integer, we see that at least 5001 heaters must be sold for the company to earn a profit.

Now Work Problem 1 ◁

EXAMPLE 2 **Renting versus Purchasing**

A builder must decide whether to rent or buy an excavating machine. If he were to rent the machine, the rental fee would be \$3000 per month (on a yearly basis), and the daily cost (gas, oil, and driver) would be \$180 for each day the machine is used. If he were to buy it, his fixed annual cost would be \$20,000, and daily operating and maintenance costs would be \$230 for each day the machine is used. What is the least number of days each year that the builder would have to use the machine to justify renting it rather than buying it?

Solution:

Strategy We will determine expressions for the annual cost of renting and the annual cost of purchasing. We then find when the cost of renting is less than that of purchasing.

Let d be the number of days each year that the machine is used. If the machine is rented, the total yearly cost consists of rental fees, which are (12)(3000), and daily charges of $180d$. If the machine is purchased, the cost per year is $20{,}000 + 230d$. We want

$$\begin{aligned} \text{cost}_{\text{rent}} &< \text{cost}_{\text{purchase}} \\ 12(3000) + 180d &< 20{,}000 + 230d \\ 36{,}000 + 180d &< 20{,}000 + 230d \\ 16{,}000 &< 50d \\ 320 &< d \end{aligned}$$

Thus, the builder must use the machine at least 321 days to justify renting it.

Now Work Problem 3 ◁

EXAMPLE 3 Current Ratio

The *current ratio* of a business is the ratio of its current assets (such as cash, merchandise inventory, and accounts receivable) to its current liabilities (such as short-term loans and taxes payable).

After consulting with the comptroller, the president of the Ace Sports Equipment Company decides to take out a short-term loan to build up inventory. The company has current assets of $350,000 and current liabilities of $80,000. How much can the company borrow if the current ratio is to be no less than 2.5? (Note: The funds received are considered as current assets and the loan as a current liability.)

Solution: Let x denote the amount the company can borrow. Then current assets will be $350{,}000 + x$, and current liabilities will be $80{,}000 + x$. Thus,

$$\text{current ratio} = \frac{\text{current assets}}{\text{current liabilities}} = \frac{350{,}000 + x}{80{,}000 + x}$$

We want

$$\frac{350{,}000 + x}{80{,}000 + x} \geq 2.5$$

Although the inequality that must be solved is not linear, it is equivalent to a linear inequality.

Since x is positive, so is $80{,}000 + x$. Hence, we can multiply both sides of the inequality by $80{,}000 + x$ and the sense of the inequality will remain the same. We have

$$\begin{aligned} 350{,}000 + x &\geq 2.5(80{,}000 + x) \\ 150{,}000 &\geq 1.5x \\ 100{,}000 &\geq x \end{aligned}$$

Consequently, the company may borrow as much as $100,000 and still maintain a current ratio greater than or equal to 2.5.

◁

EXAMPLE 4 Publishing

A publishing company finds that the cost of publishing each copy of a certain magazine is $1.50. The revenue from dealers is $1.40 per copy. The advertising revenue is 10% of

the revenue received from dealers for all copies sold beyond 10,000. What is the least number of copies that must be sold so as to have a profit for the company?

Solution:

Strategy We have

$$\text{profit} = \text{total revenue} - \text{total cost}$$

so we find an expression for profit and then set it greater than 0.

Let q be the number of copies that are sold. The revenue from dealers is $1.40q$, and the revenue from advertising is $(0.10)[(1.40)(q - 10{,}000)]$. The total cost of publication is $1.50q$. Thus,

$$\begin{aligned}
\text{total revenue} - \text{total cost} &> 0 \\
1.40q + (0.10)[(1.40)(q - 10{,}000)] - 1.50q &> 0 \\
1.4q + 0.14q - 1400 - 1.5q &> 0 \\
0.04q - 1400 &> 0 \\
0.04q &> 1400 \\
q &> 35{,}000
\end{aligned}$$

Therefore, the total number of copies sold must be greater than 35,000. That is, at least 35,001 copies must be sold to guarantee a profit.

Now Work Problem 5 ◁

PROBLEMS 1.3

1. Profit The Davis Company manufactures a product that has a unit selling price of \$20 and a unit cost of \$15. If fixed costs are \$600,000, determine the least number of units that must be sold for the company to have a profit.

2. Profit To produce 1 unit of a new product, a company determines that the cost for material is \$2.50 and the cost of labor is \$4. The fixed cost, regardless of sales volume, is \$5000. If the cost to a wholesaler is \$7.40 per unit, determine the least number of units that must be sold by the company to realize a profit.

3. Leasing versus Purchasing A businesswoman wants to determine the difference between the costs of owning and leasing an automobile. She can lease a car for \$420 per month (on an annual basis). Under this plan, the cost per mile (gas and oil) is \$0.06. If she were to purchase the car, the fixed annual expense would be \$4700, and other costs would amount to \$0.08 per mile. What is the least number of miles she would have to drive per year to make leasing no more expensive than purchasing?

4. Shirt Manufacturer A T-shirt manufacturer produces N shirts at a total labor cost (in dollars) of $1.3N$ and a total material cost of $0.4N$. The fixed cost for the plant is \$6500. If each shirt sells for \$3.50, how many must be sold by the company to realize a profit?

5. Publishing The cost of publication of each copy of a magazine is \$1.30. It is sold to dealers for \$1.50 per copy. The amount received for advertising is 20% of the amount received for all magazines sold beyond 100,000. Find the least number of magazines that can be published profitably, if 80% of the issues published are sold.

6. Production Allocation A company produces alarm clocks. During the regular workweek, the labor cost for producing one clock is \$2.00. However, if a clock is produced on overtime, the labor cost is \$3.00. Management has decided to spend no more than a total of \$25,000 per week for labor. The company must produce 11,000 clocks this week. What is the minimum number of clocks that must be produced during the regular workweek?

7. Investment A company invests a total of \$30,000 of surplus funds at two annual rates of interest: 5% and $6\frac{3}{4}\%$. It wishes an annual yield of no less than $6\frac{1}{2}\%$. What is the least amount of money that the company must invest at the $6\frac{3}{4}\%$ rate?

8. Current Ratio The current ratio of Precision Machine Products is 3.8. If the firm's current assets are \$570,000, what are its current liabilities? To raise additional funds, what is the maximum amount the company can borrow on a short-term basis if the current ratio is to be no less than 2.6? (See Example 3 for an explanation of current ratio.)

9. Sales Allocation At present, a manufacturer has 2500 units of product in stock. The product is now selling at \$4 per unit. Next month the unit price will increase by \$0.50. The manufacturer wants the total revenue received from the sale of the 2500 units to be no less than \$10,750. What is the maximum number of units that can be sold this month?

10. Revenue Suppose consumers will purchase q units of a product at a price of $\frac{200}{q} + 3$ dollars per unit. What is the minimum number of units that must be sold in order that sales revenue be greater than \$9000?

11. Hourly Rate Painters are often paid either by the hour or on a per-job basis. The rate they receive can affect their working speed. For example, suppose they can work either for \$9.00 per hour or for \$320 plus \$3 for each hour less than 40 if they complete the job in less than 40 hours. Suppose the job will take t hours. If $t \geq 40$, clearly the hourly rate is better. If $t < 40$, for what values of t is the hourly rate the better pay scale?

12. Compensation Suppose a company offers you a sales position with your choice of two methods of determining your yearly salary. One method pays \$35,000 plus a bonus of 3% of your yearly sales. The other method pays a straight 5% commission on your sales. For what yearly sales amount is it better to choose the first method?

13. Acid Test Ratio The *acid test ratio* (or *quick ratio*) of a business is the ratio of its liquid assets—cash and securities plus accounts receivable—to its current liabilities. The minimum acid test ratio for a financially healthy company is around 1.0, but the standard varies somewhat from industry to industry. If a company has \$450,000 in cash and securities and has \$398,000 in current liabilities, how much does it need to be carrying as accounts receivable in order to keep its acid test ratio at or above 1.3?

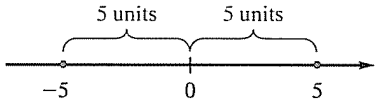
Objective

To solve equations and inequalities involving absolute values.

1.4 Absolute Value

Absolute-Value Equations

The absolute value of a real number is the number obtained when its sign is ignored.



$|5| = |-5| = 5$

FIGURE 1.17 Absolute value.

On the real-number line, the distance of a number x from 0 is called the **absolute value** of x and is denoted by $|x|$. For example, $|5| = 5$ and $|-5| = 5$ because both 5 and -5 are 5 units from 0. (See Figure 1.17.) Similarly, $|0| = 0$. Notice that $|x|$ can never be negative; that is, $|x| \geq 0$.

If x is positive or zero, then $|x|$ is simply x itself, so we can omit the vertical bars and write $|x| = x$. On the other hand, consider the absolute value of a negative number, like $x = -5$.

$$|x| = |-5| = 5 = -(-5) = -x$$

Thus, if x is negative, then $|x|$ is the positive number $-x$. The minus sign indicates that we have changed the sign of x. The geometric definition of absolute value as a distance is equivalent to the following:

Definition

The ***absolute value*** of a real number x, written $|x|$, is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

CAUTION!

$\sqrt{x^2}$ is not necessarily x but $\sqrt{x^2} = |x|$. For example, $\sqrt{(-2)^2} = |-2| = 2$, not -2. This agrees with the fact that $\sqrt{(-2)^2} = \sqrt{4} = 2$.

Observe that $|-x| = |x|$ follows from the definition.

Applying the definition, we have $|3| = 3, |-8| = -(-8) = 8$, and $|\frac{1}{2}| = \frac{1}{2}$. Also, $-|2| = -2$ and $-|-2| = -2$.

Also, $|-x|$ is not necessarily x and, thus, $|-x - 1|$ is not necessarily $x + 1$.

For example, if we let $x = -3$, then $|-(-3)| \neq -3$, and

$$|-(-3) - 1| \neq -3 + 1$$

EXAMPLE 1 **Solving Absolute-Value Equations**

a. Solve $|x-3|=2$.

Solution: This equation states that $x-3$ is a number 2 units from 0. Thus, either

$$x-3=2 \quad \text{or} \quad x-3=-2$$

Solving these equations gives $x=5$ or $x=1$.

b. Solve $|7-3x|=5$.

Solution: The equation is true if $7-3x=5$ or if $7-3x=-5$. Solving these equations gives $x=\frac{2}{3}$ or $x=4$.

c. Solve $|x-4|=-3$.

Solution: The absolute value of a number is never negative, so the solution set is $\emptyset$.

Now Work Problem 19 ◁

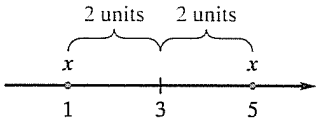


FIGURE 1.18 The solution of $|x-3|=2$ is 1 or 5.

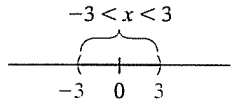
We can interpret $|a-b|=|-(b-a)|=|b-a|$ as the distance between a and b. For example, the distance between 5 and 9 can be calculated via

$$\begin{array}{ll} \text{either} & |9-5|=|4|=4 \\ \text{or} & |5-9|=|-4|=4 \end{array}$$

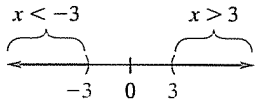
Similarly, the equation $|x-3|=2$ states that the distance between x and 3 is 2 units. Thus, x can be 1 or 5, as shown in Example 1(a) and Figure 1.18.

Absolute-Value Inequalities

Let us turn now to inequalities involving absolute values. If $|x|<3$, then x is less than 3 units from 0. Hence, x must lie between -3 and 3, that is, on the interval $-3<x<3$. [See Figure 1.19(a).] On the other hand, if $|x|>3$, then x must be greater than 3 units from 0. Hence, there are two intervals in the solution: Either $x<-3$ or $x>3$. [See Figure 1.19(b).] We can extend these ideas as follows: If $|x|\le 3$, then $-3\le x\le 3$; if $|x|\ge 3$, then $x\le -3$ or $x\ge 3$. Table 1.1 gives a summary of the solutions to absolute-value inequalities.



(a) Solution of $|x|<3$



(b) Solution of $|x|>3$

FIGURE 1.19 Solutions of $|x|<3$ and $|x|>3$.

Table 1.1

Inequality ($d>0$)	Solution
$\lvert x\rvert < d$	$-d<x<d$
$\lvert x\rvert \le d$	$-d\le x\le d$
$\lvert x\rvert > d$	$x<-d$ or $x>d$
$\lvert x\rvert \ge d$	$x\le -d$ or $x\ge d$

EXAMPLE 2 **Solving Absolute-Value Inequalities**

a. Solve $|x-2|<4$.

Solution: The number $x-2$ must be less than 4 units from 0. From the preceding discussion, this means that $-4<x-2<4$. We can set up the procedure for solving this inequality as follows:

$$\begin{aligned} &-4<x-2<4 \\ &-4+2<x<4+2 \qquad \text{adding 2 to each member} \\ &-2<x<6 \end{aligned}$$

Thus, the solution is the open interval $(-2,6)$. This means that all numbers between -2 and 6 satisfy the original inequality. (See Figure 1.20.)

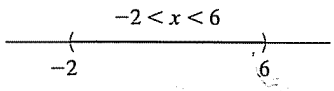


FIGURE 1.20 The solution of $|x-2|<4$ is the interval $(-2,6)$.

b. Solve $|3 - 2x| \leq 5$.

Solution:

$$\begin{aligned} -5 \leq 3 - 2x &\leq 5 \\ -5 - 3 \leq -2x &\leq 5 - 3 \quad \text{subtracting 3 throughout} \\ -8 \leq -2x &\leq 2 \\ 4 \geq x &\geq -1 \quad \text{dividing throughout by } -2 \\ -1 \leq x &\leq 4 \quad \text{rewriting} \end{aligned}$$

Note that the sense of the original inequality was *reversed* when we divided by a negative number. The solution is the closed interval $[-1, 4]$.

Now Work Problem 29 ◁

EXAMPLE 3 Solving Absolute-Value Inequalities

a. Solve $|x + 5| \geq 7$.

Solution: Here $x + 5$ must be *at least* 7 units from 0. Thus, either $x + 5 \leq -7$ *or* $x + 5 \geq 7$. This means that either $x \leq -12$ *or* $x \geq 2$. Thus, the solution consists of two intervals: $(-\infty, -12]$ and $[2, \infty)$. We can abbreviate this collection of numbers by writing

$$(-\infty, -12] \cup [2, \infty)$$

where the connecting symbol $\cup$ is called the *union* symbol. (See Figure 1.21.) More formally, the **union** of sets A and B is the set consisting of all elements that are in either A or B (or in both A and B).

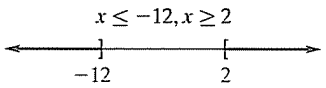


FIGURE 1.21 The union $(-\infty, -12] \cup [2, \infty)$.

b. Solve $|3x - 4| > 1$.

Solution: Either $3x - 4 < -1$ *or* $3x - 4 > 1$. Thus, either $3x < 3$ *or* $3x > 5$. Therefore, $x < 1$ *or* $x > \frac{5}{3}$, so the solution consists of all numbers in the set $(-\infty, 1) \cup (\frac{5}{3}, \infty)$.

Now Work Problem 31 ◁

CAUTION!

The inequalities $x \leq -12$ or $x \geq 2$ in (a) and $x < 1$ *or* $x > \frac{5}{3}$ in (b) do not give rise to a single interval as in Examples 2a and 2b.

APPLY IT ▷

3. Express the following statement using absolute-value notation: The actual weight w of a box of cereal must be within 0.3 oz of the weight stated on the box, which is 22 oz.

EXAMPLE 4 Absolute-Value Notation

Using absolute-value notation, express the following statements:

a. x is less than 3 units from 5.

Solution: $|x - 5| < 3$

b. x differs from 6 by at least 7.

Solution: $|x - 6| \geq 7$

c. $x < 3$ and $x > -3$ simultaneously.

Solution: $|x| < 3$

d. x is more than 1 unit from -2.

Solution: $|x - (-2)| > 1$

$|x + 2| > 1$

e. x is less than σ (a Greek letter read "sigma") units from μ (a Greek letter read "mu").

Solution: $|x - \mu| < \sigma$

Now Work Problem 11 ◁

Properties of the Absolute Value

Five basic properties of the absolute value are as follows:

1. $|ab| = |a| \cdot |b|$
2. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
3. $|a - b| = |b - a|$
4. $-|a| \le a \le |a|$
5. $|a + b| \le |a| + |b|$

For example, Property 1 states that the absolute value of the product of two numbers is equal to the product of the absolute values of the numbers. Property 5 is known as *the triangle inequality*.

EXAMPLE 5 **Properties of Absolute Value**

a. $|(-7) \cdot 3| = |-7| \cdot |3| = 21$

b. $|4 - 2| = |2 - 4| = 2$

c. $|7 - x| = |x - 7|$

d. $\left|\frac{-7}{3}\right| = \frac{|-7|}{|3|} = \frac{7}{3}; \left|\frac{-7}{-3}\right| = \frac{|-7|}{|-3|} = \frac{7}{3}$

e. $\left|\frac{x-3}{-5}\right| = \frac{|x-3|}{|-5|} = \frac{|x-3|}{5}$

f. $-|2| \le 2 \le |2|$

g. $|(-2) + 3| = |1| = 1 \le 5 = 2 + 3 = |-2| + |3|$

Now Work Problem 5 ◁

PROBLEMS 1.4

In Problems 1–10, evaluate the absolute value expression.

1. $|-13|$ **2.** $|2^{-1}|$ **3.** $|8 - 2|$

4. $|(-3 - 5)/2|$ **5.** $|2(-\frac{7}{2})|$ **6.** $|3 - 5| - |5 - 3|$

7. $|x| < 4$ **8.** $|x| < 10$ **9.** $|3 - \sqrt{10}|$

10. $|\sqrt{5} - 2|$

11. Using the absolute-value symbol, express each fact.
(a) x is less than 3 units from 7.
(b) x differs from 2 by less than 3.
(c) x is no more than 5 units from 7.
(d) The distance between 7 and x is 4.
(e) $x + 4$ is less than 2 units from 0.
(f) x is between -3 and 3, but is not equal to 3 or -3.
(g) $x < -6$ or $x > 6$.
(h) The number x of hours that a machine will operate efficiently differs from 105 by less than 3.
(i) The average monthly income x (in dollars) of a family differs from 850 by less than 100.

12. Use absolute-value notation to indicate that $f(x)$ and L differ by less than ϵ.

13. Use absolute-value notation to indicate that the prices p_1 and p_2 of two products may differ by no more than 9 (dollars).

14. Find all values of x such that $|x - \mu| < 3\sigma$.

In Problems 15–36, solve the given equation or inequality.

15. $|x| = 7$ **16.** $|-x| = 2$ **17.** $\left|\frac{x}{5}\right| = 7$

18. $\left|\frac{5}{x}\right| = 12$ **19.** $|x - 5| = 9$ **20.** $|4 + 3x| = 6$

21. $|5x - 2| = 0$ **22.** $|7x + 3| = x$ **23.** $|7 - 4x| = 5$

24. $|5 - 3x| = 7$ **25.** $|x| < M$ for $M > 0$

26. $|-x| < 3$ **27.** $\left|\frac{x}{4}\right| > 2$ **28.** $\left|\frac{x}{3}\right| > \frac{1}{2}$

29. $|x + 7| < 3$ **30.** $|2x - 17| < -4$ **31.** $\left|x - \frac{1}{2}\right| > \frac{1}{2}$

32. $|1 - 3x| > 2$ **33.** $|5 - 8x| \le 1$ **34.** $|3x - 2| \ge 0$

35. $\left|\frac{3x-8}{2}\right| \ge 4$ **36.** $\left|\frac{x-7}{3}\right| \le 5$

In Problems 37–38, express the statement using absolute-value notation.

37. In a science experiment, the measurement of a distance d is 35.2 m, and is accurate to ± 20 cm.

38. The difference in temperature between two chemicals that are to be mixed must be no less than 5 degrees and no more than 10 degrees.

39. Statistics In statistical analysis, the Chebyshev inequality asserts that if x is a random variable, μ is its mean, and σ is its standard deviation, then

$$(\text{probability that } |x-\mu| > h\sigma) \geq \frac{1}{h^2}$$

Find those values of x such that $|x-\mu| > h\sigma$.

40. Manufacturing Tolerance In the manufacture of widgets, the average dimension of a part is 0.01 cm. Using the absolute-value symbol, express the fact that an individual measurement x of a part does not differ from the average by more than 0.005 cm.

1.5 Summation Notation

Objective

To write sums in summation notation and evaluate such sums.

There was a time when school teachers made their students add up all the positive integers from 1 to 105 (say), perhaps as punishment for unruly behaviour while the teacher was out of the classroom. In other words, the students were to find

$$1+2+3+4+5+6+7+\cdots+104+105 \tag{1}$$

A related exercise was to find

$$1+4+9+16+\cdots+81+100+121 \tag{2}$$

The three dots notation is supposed to convey the idea of continuing the task, using the same pattern, until the last explicitly given terms have been added too. With this notation there are no hard and fast rules about how many terms at the beginning and end are to be given explicitly. The custom is to provide as many as are needed to ensure that the intended reader will find the expression unambiguous. This is too imprecise for many mathematical applications.

Suppose that for any positive integer i we define $a_i = i^2$. Then, for example, $a_6 = 36$ and $a_8 = 64$. The instruction, "Add together the numbers a_i, for i taking on the integer values 1 through 11 inclusive" is a precise statement of Equation (2). It would be precise regardless of the formula defining the values a_i, and this leads to the following:

Definition

If, for each positive integer i, there is given a unique number a_i, and m and n are positive integers with $m \leq n$, then ***the sum of the numbers a_i, with i successively taking on all the integer values in the interval $[m, n]$, is denoted***

$$\sum_{i=m}^{n} a_i$$

Thus

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + a_{m+2} + \cdots + a_n \tag{3}$$

The $\sum$ is the Greek capital letter sigma, from which we get the letter S. It stands for "sum" and the expression $\sum_{i=m}^{n} a_i$, can be read as the the sum of all numbers a_i, i ranging from m to n (through positive integers being understood). The description of a_i may be very simple. For example, in Equation (1) we have $a_i = i$ and

$$\sum_{i=1}^{105} i = 1+2+3+\cdots+105 \tag{4}$$

while Equation (2) is

$$\sum_{i=1}^{11} i^2 = 1 + 4 + 9 + \cdots + 121 \tag{5}$$

We have merely defined a notation, which is called ***summation notation.*** In Equation (3), i is the *index of summation* and m and n are called the *bounds of summation.* It is important to understand from the outset that the name of the index of summation can be replaced by any other so that we have

$$\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j = \sum_{\alpha=m}^{n} a_\alpha = \sum_{N=m}^{n} a_N$$

for example. In each case, replacing the index of summation by the positive integers m through n successively and adding gives

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

We now illustrate with some concrete examples.

EXAMPLE 1 **Evaluating Sums**

Evaluate the given sums.

a. $\sum_{n=3}^{7} (5n - 2)$

Solution:

$$\begin{aligned}\sum_{n=3}^{7} (5n - 2) &= [5(3) - 2] + [5(4) - 2] + [5(5) - 2] + [5(6) - 2] + [5(7) - 2] \\ &= 13 + 18 + 23 + 28 + 33 \\ &= 115\end{aligned}$$

b. $\sum_{j=1}^{6} (j^2 + 1)$

Solution:

$$\begin{aligned}\sum_{j=1}^{6} (j^2 + 1) &= (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) + (6^2 + 1) \\ &= 2 + 5 + 10 + 17 + 26 + 37 \\ &= 97\end{aligned}$$

Now Work Problem 5 ◁

EXAMPLE 2 **Writing a Sum Using Summation Notation**

Write the sum $14 + 16 + 18 + 20 + 22 + \cdots + 100$ in summation notation.

Solution: There are many ways to express this sum in summation notation. One method is to notice that the values being added are $2n$, for $n = 7$ to 50. The sum can thus be written as

$$\sum_{n=7}^{50} 2n$$

Another method is to notice that the values being added are $2k + 12$, for $k = 1$ to 44. The sum can thus also be written as

$$\sum_{k=1}^{44} (2k + 12)$$

Now Work Problem 9 ◁

Since summation notation is used to express the addition of terms, we can use the properties of addition when performing operations on sums written in summation notation. By applying these properties, we can create a list of properties, and formulas for summation notation.

By the distributive property of addition,

$$ca_1 + ca_2 + \cdots + ca_n = c(a_1 + a_2 + \cdots + a_n)$$

So, in summation notation,

$$\sum_{i=m}^{n} ca_i = c \sum_{i=m}^{n} a_i \tag{6}$$

Note that c must be constant with respect to i for Equation (6) to be used.

By the commutative property of addition,

$$a_1 + b_1 + a_2 + b_2 + \cdots + a_n + b_n = a_1 + a_2 + \cdots + a_n + b_1 + b_2 + \cdots + b_n$$

So we have

$$\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i \tag{7}$$

Sometimes we want to change the bounds of summation.

$$\sum_{i=m}^{n} a_i = \sum_{i=p}^{p+n-m} a_{i+m-p} \tag{8}$$

A sum of 37 terms can be regarded as the sum of the first 17 terms plus the sum of the next 20 terms. The next rule generalzes this observation.

$$\sum_{i=m}^{p-1} a_i + \sum_{i=p}^{n} a_i = \sum_{i=m}^{n} a_i \tag{9}$$

In addition to these four basic rules, there are some other rules worth noting.

$$\sum_{i=1}^{n} 1 = n \tag{10}$$

This is because $\sum_{i=1}^{n} 1$ is a sum of n terms, each of which is 1. The next follows from combining Equation (6) and Equation (10).

$$\sum_{i=1}^{n} c = cn \tag{11}$$

Similarly, from Equations (6) and (7) we have

$$\sum_{i=m}^{n} (a_i - b_i) = \sum_{i=m}^{n} a_i - \sum_{i=m}^{n} b_i \tag{12}$$

Establishing the next three formulas is best done by a proof method called mathematical induction, which we will not demonstrate here.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{13}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{14}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{15}$$

However, we can deduce Equation (13). If we add the following equations, "vertically," term by term,

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n$$

$$\sum_{i=1}^{n} i = n + (n-1) + (n-2) + \cdots + 1$$

we get

$$2\sum_{i=1}^{n} i = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

and since there are n terms on the right, we conclude

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Observe that if a teacher assigns the task of finding

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + \cdots + 104 + 105$$

as a *punishment* and if he or she knows the formula given by Equation (13), then a student's work can be checked quickly by

$$\sum_{i=1}^{105} i = \frac{105(106)}{2} = 105 \cdot 53 = 5300 + 265 = 5565$$

EXAMPLE 3 **Applying the Properties of Summation Notation**

Evaluate the given sums.

a. $\sum_{j=30}^{100} 4$ **b.** $\sum_{k=1}^{100} (5k+3)$ **c.** $\sum_{k=1}^{200} 9k^2$

Solutions:

a.

$$\sum_{j=30}^{100} 4 = \sum_{j=1}^{71} 4 \qquad \text{by Equation (8)}$$

$$= 4 \cdot 71 \qquad \text{by Equation (11)}$$

$$= 284$$

b.

$$\sum_{k=1}^{100} (5k+3) = \sum_{k=1}^{100} 5k + \sum_{k=1}^{100} 3 \qquad \text{by Equation (7)}$$

$$= 5\left(\sum_{k=1}^{100} k\right) + 3\left(\sum_{k=1}^{100} 1\right) \qquad \text{by Equation (6)}$$

$$= 5\left(\frac{100 \cdot 101}{2}\right) + 3(100) \qquad \text{by Equations (13) and (10)}$$
$$= 25{,}250 + 300$$
$$= 25{,}550$$

c.

$$\sum_{k=1}^{200} 9k^2 = 9\sum_{k=1}^{200} k^2 \qquad \text{by Equation (6)}$$
$$= 9\left(\frac{200 \cdot 201 \cdot 401}{6}\right) \qquad \text{by Equation (14)}$$
$$= 24{,}180{,}300$$

Now Work Problem 19 ◁

PROBLEMS 1.5

In Problems 1 and 2, give the bounds of summation and the index of summation for each expression.

1. $\sum_{t=12}^{17}(8t^2 - 5t + 3)$ **2.** $\sum_{m=3}^{450}(8m - 4)$

In Problems 3–6, evaluate the given sums.

3. $\sum_{i=1}^{5} 3i$ **4.** $\sum_{p=0}^{4} 10p$

5. $\sum_{k=3}^{9}(10k + 16)$ **6.** $\sum_{n=7}^{11}(2n - 3)$

In Problems 7–12, express the given sums in summation notation.

7. $36 + 37 + 38 + 39 + \cdots + 60$

8. $1 + 8 + 27 + 64 + 125$

9. $5^3 + 5^4 + 5^5 + 5^6 + 5^7 + 5^8$

10. $11 + 15 + 19 + 23 + \cdots + 71$

11. $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$

12. $10 + 100 + 1000 + \cdots + 100{,}000{,}000$

In Problems 13–26, evaluate the given sums.

13. $\sum_{k=1}^{875} 10$ **14.** $\sum_{k=35}^{135} 2$

15. $\sum_{k=1}^{n}\left(5 \cdot \frac{1}{n}\right)$ **16.** $\sum_{k=1}^{200}(k - 100)$

17. $\sum_{k=51}^{100} 10k$ **18.** $\sum_{k=1}^{n} \frac{n}{n+1}k^3$

19. $\sum_{k=1}^{20}(5k^2 + 3k)$ **20.** $\sum_{k=1}^{100} \frac{3k^2 - 200k}{101}$

21. $\sum_{k=51}^{100} k^2$ **22.** $\sum_{k=1}^{50}(k + 50)^2$

23. $\sum_{k=1}^{9}\left\{\left[3 - \left(\frac{k}{10}\right)^2\right]\left(\frac{1}{10}\right)\right\}$

24. $\sum_{k=1}^{100}\left\{\left[4 - \left(\frac{2}{100}k\right)^2\right]\left(\frac{2}{100}\right)\right\}$

25. $\sum_{k=1}^{n}\left\{\left[5 - \left(\frac{3}{n} \cdot k\right)^2\right]\frac{3}{n}\right\}$

26. $\sum_{k=1}^{n} \frac{k^2}{(n+1)(2n+1)}$

Objective

To introduce sequences, particularly arithmetic and geometric sequences, and their sums.

1.6 Sequences

Introduction

Consider the following list of five numbers:

$$2,\ 2+\sqrt{3},\ 2+2\sqrt{3},\ 2+3\sqrt{3},\ 2+4\sqrt{3} \tag{1}$$

If it is understood that the ordering of the numbers is to be taken into account, then such a list is called a **sequence of length** 5 and it is considered to be different from

$$2,\ 2+3\sqrt{3},\ 2+\sqrt{3},\ 2+4\sqrt{3},\ 2+2\sqrt{3} \tag{2}$$

which is also a sequence of length 5. Both of these sequences are different again from

$$2,\ 2,\ 2+\sqrt{3},\ 2+2\sqrt{3},\ 2+3\sqrt{3},\ 2+4\sqrt{3} \tag{3}$$

which is a sequence of length 6. However, each of the sequences (1), (2), and (3) takes on all the numbers in the 5-element *set*

$$\{2,\ 2+\sqrt{3},\ 2+2\sqrt{3},\ 2+3\sqrt{3},\ 2+4\sqrt{3}\}$$

CAUTION!

Both rearrangements and repetitions *do* affect a sequence.

In Section 0.1 we emphasized that "*a set is determined by its elements*, and neither repetitions nor rearrangements in a listing affect the set". Since both repetitions and rearrangements do affect a sequence, it follows that sequences are not the same as sets.

We will also consider listings such as

$$2,\ 2+\sqrt{3},\ 2+2\sqrt{3},\ 2+3\sqrt{3},\ 2+4\sqrt{3},\ \cdots,2+k\sqrt{3},\ \cdots \tag{4}$$

and

$$1,\ -1,\ 1,\ -1,\ 1,\ \cdots,\ (-1)^{k+1},\ \cdots \tag{5}$$

Both are examples of what is called an **infinite sequence**. However, note that the infinite sequence (4) involves the infinitely many different numbers in the set

$$\{2+k\sqrt{3} | k \text{ a nonnegative integer}\}$$

while the infinite sequence (5) involves only the numbers in the finite set

$$\{-1, 1\}$$

For n a positive integer, taking the first n numbers of an infinite sequence results in a sequence of length n. For example, taking the first five numbers of the infinite sequence (4) gives the sequence (1). The following more formal definitions are helpful in understanding the somewhat subtle idea of a sequence.

Definition

For n a positive integer, a ***sequence of length*** n is a rule which assigns to each element of the set $\{1, 2, 3, \cdots, n\}$ exactly one real number. The set $\{1, 2, 3, \cdots, n\}$ is called the ***domain*** of the sequence of length n. A ***finite sequence*** is a sequence of length n for some positive integer n.

Definition

An ***infinite sequence*** is a rule which assigns to each element of the set of all positive integers $\{1, 2, 3, \cdots\}$ exactly one real number. The set $\{1, 2, 3, \cdots\}$ is called the ***domain*** of the infinite sequence.

The word *rule* in both definitions may appear vague but the point is that for any sequence there must be a definite way of specifying exactly one number for each of the elements in its domain. For a finite sequence the rule can be given by simply listing the numbers in the sequence. There is no requirement that there be a discernible pattern (although in practice there often is). For example,

$$99,\ -\pi,\ \frac{3}{5},\ 102.7$$

is a perfectly good sequence of length 4. For an infinite sequence there should be some sort of procedure for generating its numbers, one after the other. However, the procedure may fail to be given by a simple formula. The infinite sequence

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ \cdots$$

is very important in number theory but its rule is not given by a mere formula. (What is *apparently* the rule which gives rise to this sequence? In that event, what is the next number in this sequence after those displayed?)

We often use letters like a, b, c, and so on, for the names of sequences. If the sequence is called a, we write a_1 for the number assigned to 1, a_2 for the number assigned to 2, a_3 for the number assigned to 3, and so on. In general, for k in the domain of the sequence, we write a_k for the number assigned to k and call it the kth ***term*** of the sequence. (If you have studied Section 1.5 on summation, you will already be familiar with this notation.) In fact, rather than listing all the numbers of a sequence by

$$a_1, a_2, a_3, \ldots, a_n$$

or an indication of all the numbers such as

$$a_1, a_2, a_3, \ldots, a_k, \ldots$$

a sequence is often denoted by (a_k). Sometimes $(a_k)_{k=1}^{n}$ is used to indicate that the sequence is finite, of length n, or $(a_k)_{k=1}^{\infty}$ is used to emphasize that the sequence is infinite. The ***range*** of a sequence (a_k) is the *set*

$$\{a_k | k \text{ is in the domain of } a\}$$

Notice that

$$\{(-1)^{k+1} | k \text{ is a positive integer}\} = \{-1, 1\}$$

so an infinite sequence may have a finite range. If a and b are sequences, then, by definition, $a = b$ if and only if a and b have the same domain and, for all k in the common domain, $a_k = b_k$.

APPLY IT ▷

4. A chain of hip coffee shops has 183 outlets in 2009. Starting in 2010 it plans to expand its number of outlets by 18 each year for five years. Writing c_k for the number of coffee shops in year k, measured from 2008, list the terms in the sequence $(c_k)_{k=1}^{6}$.

EXAMPLE 1 Listing the Terms in a Sequence

a. List the first four terms of the infinite sequence $(a_k)_{k=1}^{\infty}$ whose kth term is given by $a_k = 2k^2 + 3k + 1$.

Solution: We have $a_1 = 2(1^2) + 3(1) + 1 = 6$, $a_2 = 2(2^2) + 3(2) + 1 = 15$, $a_3 = 2(3^2) + 3(3) + 1 = 28$, and $a_4 = 2(4^2) + 3(4) + 1 = 45$. So the first four terms are

$$6, \quad 15, \quad 28, \quad 45$$

b. List the first four terms of the infinite sequence (e_k), where $e_k = \left(\frac{k+1}{k}\right)^k$.

Solution: We have $e_1 = \left(\frac{1+1}{1}\right)^1 = \left(\frac{2}{1}\right)^1 = 2$, $e_2 = \left(\frac{2+1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$, $e_3 = \left(\frac{3+1}{3}\right)^3 = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$, $e_4 = \left(\frac{4+1}{4}\right)^4 = \left(\frac{5}{4}\right)^4 = \frac{625}{256}$.

c. Display the sequence $\left(\frac{3}{2^{k-1}}\right)_{k=1}^{6}$.

Solution: Noting that $2^0 = 1$, we have

$$3, \quad \frac{3}{2}, \quad \frac{3}{4}, \quad \frac{3}{8}, \quad \frac{3}{16}, \quad \frac{3}{32}$$

Now Work Problem 3 ◁

APPLY IT ▷

5. A certain inactive bank account which bears interest at the rate of 6% compounded yearly shows year-end balances, for four consecutive years, of \$9.57, \$10.14, \$10.75, \$11.40. Write the sequence of amounts in the form $(a_k)_{k=1}^{4}$.

EXAMPLE 2 Giving a Formula for a Sequence

a. Write 41, 44, 47, 50, 53 in the form $(a_k)_{k=1}^{5}$.

Solution: Each term of the sequence is obtained by adding three to the previous term. Since the first term is 41, we can write the sequence as $(41 + (k-1)3)_{k=1}^{5}$. Observe that this formula is not unique. The sequence is also described by $(38 + 3k)_{k=1}^{5}$ and by $(32 + (k+2)3)_{k=1}^{5}$, to give just two more possibilities.

b. Write the sequence 1, 4, 9, 16, ... in the form (a_k).

Solution: The sequence is *apparently* the sequence of squares of positive integers so that (k^2) or $(k^2)_{k=1}^{\infty}$ would be regarded as the correct answer by most people. But the sequence described by $(k^4 - 10k^3 + 36k^2 - 50k + 24)$ also has its first four

terms given by 1, 4, 9, 16 and yet its fifth term is 49. The sixth and seventh terms are 156 and 409, respectively. The point we are making is that an infinite sequence cannot be determined by finitely many values alone.

On the other hand, it is correct to write

$$1, 4, 9, 16, \ldots, k^2, \ldots = (k^2)$$

because the display on the left side of the equation makes it clear that the ***general term*** is k^2.

Now Work Problem 9 ◁

EXAMPLE 3 **Demonstrating Equality of Sequences**

Show that the sequences $((i+3)^2)_{i=1}^{\infty}$ and $(j^2+6j+9)_{j=1}^{\infty}$ are equal.

Solution: Both $((i+3)^2)_{i=1}^{\infty}$ and $(j^2+6j+9)_{j=1}^{\infty}$ are explicitly given to have the same domain, namely $\{1, 2, 3, \ldots\}$, the infinite set of all positive integers The names i and j being used to name a typical element of the domain are unimportant. The first sequence is the same as $((k+3)^2)_{k=1}^{\infty}$ and the second sequence is the same as $(k^2+6k+9)_{k=1}^{\infty}$. The first rule assigns, to any positive integer k, the number $(k+3)^2$ and the second assigns, to any positive integer k, the number k^2+6k+9. However, for all k, $(k+3)^2 = k^2+6k+9$, so by the definition of equality of sequences the sequences are equal.

Now Work Problem 13 ◁

Recursively Defined Sequences

Suppose that a is a sequence with

$$a_1 = 1 \text{ and, for each positive integer } k,\ a_{k+1} = (k+1)a_k \tag{6}$$

Taking $k = 1$, we see that $a_2 = (2)a_1 = (2)1 = 2$, while with $k = 2$ we have $a_3 = (3)a_2 = (3)2 = 6$. A sequence whose rule is defined in terms of itself evaluated at smaller values, and some explicitly given small values, is said to be ***recursively defined***. Thus we can say that there is a sequence a recursively defined by (6) above.

Another famous example of a recursively defined sequence is the Fibonacci sequence:

$$F_1 = 1 \text{ and } F_2 = 1 \text{ and, for each positive integer } k,\ F_{k+2} = F_{k+1} + F_k \tag{7}$$

Taking $k = 1$, we see that $F_3 = F_2 + F_1 = 1 + 1 = 2$, $F_4 = F_3 + F_2 = 2 + 1 = 3$, $F_5 = F_4 + F_3 = 3 + 2 = 5$. In fact, the first ten terms of (F_k) are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55$$

EXAMPLE 4 **Applying a Recursive Definition**

a. Use the recursive definition (6) to determine a_5 (without referring to the earlier calculations).

Solution: We have

$$\begin{aligned} a_5 &= (5)a_4 \\ &= (5)(4)a_3 \\ &= (5)(4)(3)a_2 \\ &= (5)(4)(3)(2)a_1 \\ &= (5)(4)(3)(2)(1) \\ &= 120 \end{aligned}$$

The standard notation for a_k as defined by (6) is $k!$ and it is read "k factorial". We also define $0! = 1$.

b. Use the recursive definition (7) to determine F_6.
Solution:

$$\begin{aligned} F_6 &= F_5 + F_4 \\ &= (F_4 + F_3) + (F_3 + F_2) \\ &= F_4 + 2F_3 + F_2 \\ &= (F_3 + F_2) + 2(F_2 + F_1) + F_2 \\ &= F_3 + 4F_2 + 2F_1 \\ &= (F_2 + F_1) + 4F_2 + 2F_1 \\ &= 5F_2 + 3F_1 \\ &= 5(1) + 3(1) \\ &= 8 \end{aligned}$$

Now Work Problem 17 ◁

In Example 4 we deliberately avoided making any numerical evaluations until *all* terms had been expressed using only those terms whose values were given explicitly in the recursive definition. This helps to illustrate the structure of the recursive definition in each case.

While recursive definitions are very useful in applications, the computations above underscore that, for large values of k, the computation of the kth term may be time-consuming. It is desirable to have a simple formula for a_k that does not refer to a_l, for $l < k$. Sometimes it is possible to find such a *closed* formula. In the case of (6) it is easy to see that $a_k = k \cdot (k-1) \cdot (k-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$. On the other hand, in the case of (7), it is not so easy to derive

$$F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

Arithmetic Sequences and Geometric Sequences

Definition

An ***arithmetic sequence*** is a sequence (b_k) defined recursively by

$$b_1 = a \text{ and, for each positive integer } k,\ b_{k+1} = d + b_k \tag{8}$$

for fixed real numbers a and d.

In words, the definition tells us to start the sequence at a and get the *next* term by adding d (no matter which term is currently under consideration). The number a is simply the first term of the arithmetic sequence. Since the recursive definition gives $b_{k+1} - b_k = d$, for every positive integer k, we see that the number d is the difference between any pair of successive terms. It is, accordingly, called the ***common difference*** of the arithmetic sequence. Any pair of real numbers a and d determines an infinite arithmetic sequence. By restricting to a finite number of terms, we can speak of finite arithmetic sequences.

APPLY IT ▷

6. In 2009 the enrollment at Springfield High was 1237, and demographic studies suggest that it will decline by 12 students a year for the next seven years. List the projected enrollments of Springfield High.

EXAMPLE 5 **Listing an Arithmetic Sequence**

Write explicitly the terms of an arithmetic sequence of length 6 with first term $a = 1.5$ and common difference $d = 0.7$.

Solution: Let us write (b_k) for the arithmetic sequence. Then

$$\begin{aligned} b_1 &= 1.5 \\ b_2 &= 0.7 + b_1 = 0.7 + 1.5 = 2.2 \\ b_3 &= 0.7 + b_2 = 0.7 + 2.2 = 2.9 \\ b_4 &= 0.7 + b_3 = 0.7 + 2.9 = 3.6 \\ b_5 &= 0.7 + b_4 = 0.7 + 3.6 = 4.3 \\ b_6 &= 0.7 + b_5 = 0.7 + 4.3 = 5.0 \end{aligned}$$

Thus the required sequence is

$$1.5,\ 2.2,\ 2.9,\ 3.6,\ 4.3,\ 5.0$$

Now Work Problem 21 ◁

Definition

A ***geometric sequence*** is a sequence (c_k) defined recursively by

$$c_1 = a \text{ and, for each positive integer } k,\ c_{k+1} = c_k \cdot r \tag{9}$$

for fixed real numbers a and r.

In words, the definition tells us to start the sequence at a and get the *next* term by multiplying by r (no matter which term is currently under consideration). The number a is simply the first term of the geometric sequence. Since the recursive definition gives $c_{k+1}/c_k = r$, for every positive integer k with $c_k \neq 0$, we see that the number r is the ratio between any pair of successive terms, with the first of these not 0. It is, accordingly, called the ***common ratio*** of the geometric sequence. Any pair of real numbers a and r determines an infinite geometric sequence. By restricting to a finite number of terms, we can speak of finite geometric sequences.

APPLY IT ▷

7. The population of the rural area surrounding Springfield is declining as a result of movement to the urban core. In 2009 it was 23,500 and each year, for the next four years, it is expected to be only 92% of the previous year's population. List the anticipated annual population numbers for the rural area.

EXAMPLE 6 **Listing a Geometric Sequence**

Write explicitly the terms of a geometric sequence of length 5 with first term $a = \sqrt{2}$ and common ratio $r = 1/2$.

Solution: Let us write (c_k) for the geometric sequence. Then

$$\begin{aligned}
c_1 &= \sqrt{2}\\
c_2 = (c_1) \cdot 1/2 &= (\sqrt{2})1/2 = \sqrt{2}/2\\
c_3 = (c_2) \cdot 1/2 &= (\sqrt{2}/2)1/2 = \sqrt{2}/4\\
c_4 = (c_3) \cdot 1/2 &= (\sqrt{2}/4)1/2 = \sqrt{2}/8\\
c_5 = (c_4) \cdot 1/2 &= (\sqrt{2}/8)1/2 = \sqrt{2}/16
\end{aligned}$$

Thus the required sequence is

$$\sqrt{2},\ \sqrt{2}/2,\ \sqrt{2}/4,\ \sqrt{2}/8,\ \sqrt{2}/16$$

Now Work Problem 25 ◁

We have remarked that sometimes it is possible to determine an explicit formula for the kth term of a recursively-defined sequence. This is certainly the case for arithmetic and geometric sequences.

EXAMPLE 7 **Finding the kth term of an Arithmetic Sequence**

Find an explicit formula for the kth term of an arithmetic sequence (b_k) with first term a and common difference d.

Solution: We have

$$\begin{aligned}
b_1 &= a = 0d + a\\
b_2 = d + (b_1) &= d + a = 1d + a\\
b_3 = d + (b_2) &= d + (1d + a) = 2d + a\\
b_4 = d + (b_3) &= d + (2d + a) = 3d + a\\
b_5 = d + (b_4) &= d + (3d + a) = 4d + a
\end{aligned}$$

It *appears* that, for each positive integer k, the kth term of an arithmetic sequence (b_k) with first term a and common difference d is given by

$$b_k = (k-1)d + a \tag{10}$$

This is true and follows easily via the proof method called mathematical induction, which we will not demonstrate here.

Now Work Problem 29 ◁

EXAMPLE 8 **Finding the *k*th term of a Geometric Sequence**

Find an explicit formula for the kth term of a geometric sequence (c_k) with first term a and common ratio r.

Solution: We have

$$\begin{aligned} c_1 &= a = ar^0 \\ c_2 &= (c_1) \cdot r = ar = ar^1 \\ c_3 &= (c_2) \cdot r = ar^1 r = ar^2 \\ c_4 &= (c_3) \cdot r = ar^2 r = ar^3 \\ c_5 &= (c_4) \cdot r = ar^3 r = ar^4 \end{aligned}$$

It *appears* that, for each positive integer k, the kth term of a geometric sequence (c_k) with first term a and common difference r is given by

$$c_k = ar^{k-1} \tag{11}$$

This is true and also follows easily via mathematical induction.

Now Work Problem 31 ◁

It is clear that any arithmetic sequence has a unique first term a and a unique common difference d. For a geometric sequence we have to be a little more careful. From (11) we see that if any term c_k is 0, then either $a = 0$ or $r = 0$. If $a = 0$, then every term in the geometric sequence is 0. In this event, there is not a uniquely determined r because $r \cdot 0 = 0$, for *any* r. If $a \neq 0$ but $r = 0$, then every term except the first is 0.

Sums of Sequences

For any sequence (c_k) we can speak of the sum of the first k terms. Let us call this sum s_k. Using the summation notation introduced in Section 1.5, we can write

$$s_k = \sum_{i=1}^{k} c_i = c_1 + c_2 + \cdots + c_k \tag{12}$$

We can regard the s_k as terms of a new sequence (s_k), of sums, associated to the original sequence (s_k). If a sequence (c_k) is finite, of length n then s_n can be regarded as ***the sum of the sequence***.

APPLY IT ▷

8. If a company has an annual revenue of 27M$ in 2009 and revenue grows by 1.5M$ each year, find the total revenue through 2009–2015 inclusive.

EXAMPLE 9 **Finding the Sum of an Arithmetic Sequence**

Find a formula for the sum s_n of the first n terms of an arithmetic sequence (b_k) with first term a and common difference d.

Solution: Since the arithmetic sequence (b_k) in question has, by Example 7, $b_k = (k-1)d + a$, the required sum is given by

$$s_n = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} ((k-1)d + a) = \sum_{k=1}^{n} (dk - (d-a)) = \sum_{k=1}^{n} dk - \sum_{k=1}^{n} (d-a)$$

$$= d\sum_{k=1}^{n} k - (d-a)\sum_{k=1}^{n} 1 \overset{\star}{=} d\frac{n(n+1)}{2} - (d-a)n = \frac{n}{2}((n-1)d + 2a)$$

Notice that the equality labeled $\star$ uses both (13) and (10) of Section 1.5. We remark that the last term under consideration in the sum is $b_n = (n-1)d + a$ so that in our formula for s_n the factor $((n-1)d + 2a)$ is the first term a plus the last term $(n-1)d + a$. If we

write $z = (n-1)d + a$ for the last term, then we can summarize with

$$s_n = \frac{n}{2}((n-1)d + 2a) = \frac{n}{2}(a+z) \tag{13}$$

Note that we could also have found (13) by the same technique used to find (13) of Section 1.5. We preferred to calculate using summation notation here. Finally, we should remark that the sum (13) in Section 1.5 is the sum of the first n terms of the special arithmetic sequence with $a = 1$ and $d = 1$.

Now Work Problem 33 ◁

APPLY IT ▷

9. Mrs. Simpson put \$1000 in a special account for Bart on each of his first 21 birthdays. The account earned interest at the rate of 7% compounded annually. It follows (see Chapter 5) that the amount deposited on Bart's $(22-k)$th birthday is worth $\$1000(1.07)^{k-1}$ on Bart's 21st birthday. Find the total amount in the special account on Bart's 21st birthday.

EXAMPLE 10 Finding the Sum of a Geometric Sequence

Find a formula for the sum s_n of the first n terms of a geometric sequence (c_k) with first term a and common ratio r.

Solution: Since the geometric sequence (c_k) in question has, by Example 8, $c_k = ar^{k-1}$, the required sum is given by

$$s_n = \sum_{k=1}^{n} c_k = \sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1} \tag{14}$$

It follows that if we multiply (14) by r we have

$$rs_n = r\sum_{k=1}^{n} c_k = r\sum_{k=1}^{n} ar^{k-1} = \sum_{k=1}^{n} ar^k = ar + ar^2 + \cdots + ar^{n-1} + ar^n \tag{15}$$

If we subtract (15) from (14) we get

$$s_n - rs_n = a - ar^n \text{ so that } (1-r)s_n = a(1-r^n)$$

Thus we have

$$s_n = \frac{a(1-r^n)}{1-r} \quad \text{for } r \neq 1 \tag{16}$$

(Note that if $r = 1$, then each term in the sum is a and, since there are n terms, the answer in this easy case is $s_n = na$)

Now Work Problem 37 ◁

For *some* infinite sequences $(c_k)_{k=1}^{\infty}$ the sequence of sums $(s_k)_{k=1}^{\infty} = \left(\sum_{i=1}^{k} c_i\right)_{k=1}^{\infty}$ appears to approach a definite number. When this is indeed the case we write the number as $\sum_{i=1}^{\infty} c_i$. Here we consider only the case of a geometric sequence. As we see from (16), if $c_k = ar^{k-1}$ then, for $r \neq 1$, $s_k = \frac{a(1-r^k)}{1-r}$. Observe that only the factor $1 - r^k$ depends on k. If $|r| > 1$, then for large values of k, $|r^k|$ will become large, as will $|1 - r^k|$. In fact, for $|r| > 1$ we can make the values $|1 - r^k|$ as large as we like by taking k to be sufficiently large. It follows that, for $|r| > 1$, the sums $\frac{a(1-r^k)}{1-r}$ *do not* approach a definite number. If $r = 1$, then $s_k = ka$ and, again, the sums do not approach a definite *number*.

However, for $|r| < 1$ (that is for $-1 < r < 1$), we can make the values r^k as close to 0 as we like, by taking k to be sufficiently large. (Be sure to convince yourself that this is true before reading further because the rest of the argument hinges on this point.) Thus, for $|r| < 1$, we can make the values $1 - r^k$ as close to 1 as we like by taking k to be sufficiently large. Finally, for $|r| < 1$, we can make the values $\frac{a(1-r^k)}{1-r}$ as close to $\frac{a}{1-r}$ as we like by taking k to be sufficiently large. In precisely this sense, an infinite geometric sequence with $|r| < 1$ has a sum and we have

$$\text{for } |r| < 1, \quad \sum_{i=1}^{\infty} ar^{i-1} = \frac{a}{1-r} \tag{17}$$

EXAMPLE 11 Finding the Sum of an Infinite Geometric Sequence

A rich woman would like to leave \$100,000 a year, starting now, to be divided equally among all her direct descendants. She puts no time limit on this bequeathment and is able to invest for this long-term outlay of funds at 2% compounded annually. How much must she invest now to meet such a long-term commitment?

Solution: Let us write $R = 100,000$, set the clock to 0 now, and measure time in years from now. With these conventions we are to account for payments of R at times 0, 1, 2, 3, ..., k, ... by making a single investment now. (Such a sequence of payments is called a *perpetuity*.) The payment now simply costs her R. The payment at time 1 has a *present value* of $R(1.02)^{-1}$. (See Chapter 5.) The payment at time 2 has a present value of $R(1.02)^{-2}$. The payment at time 3 has a present value of $R(1.02)^{-3}$, and, quite generally, the payment at time k has a present value of $R(1.02)^{-k}$. Her investment *now* must exactly cover the present value of *all* these future payments. In other words, the investment must equal the sum

$$R + R(1.02)^{-1} + R(1.02)^{-2} + R(1.02)^{-3} + \cdots + R(1.02)^{-k} + \cdots$$

We recognize the infinite sum as that of a geometric series, with first term $a = R = 100,000$ and common ratio $r = (1.02)^{-1}$. Since $|r| = (1.02)^{-1} < 1$, we can evaluate the required investment as

$$\frac{a}{1-r} = \frac{100,000}{1 - \frac{1}{1.02}} = \frac{100,000}{\frac{0.02}{1.02}} = \frac{100,000(1.02)}{0.02} = 5,100,000$$

In other words, an investment of a mere \$5,100,000 now will allow her to leave \$100,000 per year to her descendants *forever*!

Now Work Problem 57 ◁

PROBLEMS 1.6

In Problems 1–8, write the indicated term of the given sequence.

1. $a = \sqrt{2},\ -\frac{3}{7},\ 2.3,\ 57;\quad a_3$

2. $b = 1,\ 13,\ -0.9,\ \frac{5}{2},\ 100,\ 39;\quad b_6$

3. $(a_k)_{k=1}^{7} = (3^k);\quad a_4$

4. $(c_k)_{k=1}^{9} = (3^k + k);\quad c_4$

5. $(a_k) = (2 + (k-1)3);\quad a_{24}$

6. $(b_k) = (5 \cdot 2^{k-1});\quad b_6$

7. $(a_k) = (k^4 - 2k^2 + 1);\quad a_2$

8. $(a_k) = (k^3 + k^2 - 2k + 7);\quad a_3$

In Problems 9–12, find a general term, (a_k), description that fits the displayed terms of the given sequence.

9. $-1,\ 2,\ 5,\ 8$

10. $5,\ 3,\ 1,\ -1,\ \ldots$

11. $2,\ -4,\ 8,\ -16$

12. $5,\ \frac{5}{3},\ \frac{5}{9},\ \frac{5}{27},\ \cdots$

In Problems 13–16, determine whether the given sequences are equal to each other.

13. $((i+3)^3)$ and $(j^3 - 9j^2 + 9j - 27)$

14. $(k^2 - 4)$ and $((k+2)(k-2))$

15. $\left(\pi\frac{1}{2^{k-1}}\right)_{k=1}^{\infty}$ and $\left(\frac{\pi}{2^k}\right)_{k=1}^{\infty}$

16. $(j^3 - 9j^2 + 27j - 27)_{j=1}^{\infty}$ and $((k-3)^3)_{k=1}^{\infty}$

In Problems 17–20, determine the indicated term of the given recursively defined sequence.

17. $a_1 = 1,\ a_2 = 2,\ a_{k+2} = a_{k+1} \cdot a_k;\quad a_7$

18. $a_1 = 1,\ a_{k+1} = a_{a_k};\quad a_{17}$

19. $b_1 = 1,\ b_{k+1} = \frac{b_k}{k};\quad b_6$

20. $a_1 = 1,\ a_{k+1} = (k+1) + a_k;\quad a_8$

In Problems 21–24, write the first five terms of the arithmetic sequence with the given first term a and common difference d.

21. $a = 22.5,\ d = 0.9$

22. $a = 0,\ d = 1$

23. $a = 96,\ d = -1.5$

24. $a = A,\ d = D$

In Problems 25–28, write the first five terms of the geometric sequence with the given first term a and common ratio r.

25. $a = -2,\ r = -0.5$

26. $a = 50,\ r = (1.06)^{-1}$

27. $a = 100,\ r = 1.05$

28. $a = 3,\ r = \frac{1}{3}$

In Problems 29–32, write the indicated term of the arithmetic sequence with given parameters a and d or of the geometric sequence with given parameters a and r.

29. 27th term, $a = 3,\ d = 2$

30. 9th term, $a = 2.7,\ d = -0.3$

31. 11th term, $a = 1,\ r = 2$

32. 7th term, $a = 2,\ r = 10$

In Problems 33–40, find the required sums.

33. $\sum_{k=1}^{7}((k-1)3 + 5)$

34. $\sum_{k=1}^{9}(k \cdot 2 + 9)$

35. $\sum_{k=1}^{6}((k-1)0.5+2.3)$ **36.** $\sum_{k=1}^{34}((k-1)10+5)$

37. $\sum_{k=1}^{10}100(1/2)^{k-1}$ **38.** $\sum_{k=1}^{10}50(1.07)^{k-1}$

39. $\sum_{k=1}^{10}50(1.07)^{1-k}$ **40.** $\sum_{k=1}^{7}5\cdot 2^{k}$

In Problems 41–46, find the infinite sums, if possible, or state why this cannot be done.

41. $\sum_{k=1}^{\infty}3\left(\frac{1}{2}\right)^{k-1}$ **42.** $\sum_{i=0}^{\infty}\left(\frac{1}{3}\right)^{i}$

43. $\sum_{k=1}^{\infty}\frac{1}{2}(17)^{k-1}$ **44.** $\sum_{k=1}^{\infty}\frac{2}{3}(1.5)^{k-1}$

45. $\sum_{k=1}^{\infty}50(1.05)^{1-k}$ **46.** $\sum_{j=1}^{\infty}75(1.09)^{1-j}$

47. Inventory Every thirty days a grocery store stocks 90 cans of elephant noodle soup and, rather surprisingly, sells 3 cans each day. Describe the inventory levels of elephant noodle soup at the end of each day, as a sequence, and determine the inventory level 19 days after restocking.

48. Inventory If a corner store has 95 previously viewed DVD movies for sale today and manages to sell 6 each day, write the first seven terms of the store's daily inventory sequence for the DVDs. How many DVDs will the store have on hand after 10 days?

49. Checking Account A checking account, which earns no interest, contains \$125.00 and is forgotten. It is nevertheless subject to a \$5.00 per month service charge. The account is remembered after 9 months. How much does it then contain?

50. Savings Account A savings account, which earns interest at a rate of 5% compounded annually, contains \$125.00 and is forgotten. It is remembered 9 years later. How much does it then contain?

51. Population Change A town with a population of 50,000 in 2009 is growing at the rate of 8% per year. In other words, at the end of each year the population is 1.08 times the population at the end of the preceding year. Describe the population sequence and determine what the population will be at the end of 2020, if this rate of growth is maintained.

52. Population Change Each year 5% of the inhabitants of a rural area move to the city. If the current population is 24,000, and this rate of decrease continues, give a formula for the population k years from now.

53. Revenue Current daily revenue at a campus burger restaurant is \$12,000. Over the next seven days revenue is expected to increase by \$1000 each day as students return for the fall semester. What is the projected total revenue for the eight days for which we have projected data?

54. Revenue A car dealership's finance department is going to receive payments of \$300 per month for the next 60 months to pay for Bart's car. The kth such payment has a present value of $\$300(1.01)^{-k}$. The sum of the present values of all 60 payments must equal the selling price of the car. Write an expression for the selling price of the car and evaluate it using your calculator.

55. Future Value Six years from now, Nicole will need a new tractor for her farm. Starting next month, she is going to put \$100 in the bank each month to save for the inevitable purchase. Six years from now the kth bank deposit will be worth $\$100(1.005)^{72-k}$ (due to compounded interest). Write a formula for the accumulated amount of money from her 72 bank deposits. Use your calculator to determine how much Nicole will have available towards her tractor purchase.

56. Future Value Lisa has just turned seven years old. She would like to save some money each month, starting next month, so that on her 21st birthday she will have \$1000 in her bank account. Marge told her that with current interest rates her kth deposit will be worth, on her 21st birthday, $(1.004)^{168-k}$ times the deposited amount. Lisa wants to deposit the same amount each month. Write a formula for the amount Lisa needs to deposit each month to meet her goal. Use your calculator to evaluate the required amount.

57. Perpetuity Brad's will includes an endowment to Dalhousie University that is to provide each year after his death, forever, a \$500 prize for the top student in the business mathematics class, MATH 1115. Brad's estate can make an investment at 5% compounded annually to pay for this endowment. Adapt the solution of Example 11 to determine how much this endowment will cost Brad's estate.

58. Perpetuity Rework Problem 57 under the assumption that Brad's estate can make an investment at 10% compounded annually.

59. The Fibonacci sequence given in (7) is defined recursively using addition. Is it an arithmetic sequence? Explain.

60. The factorial sequence given in (6) is defined recursively using multiplication. Is it a geometric sequence? Explain.

61. The recursive definition for an arithmetic sequence (b_k) called for starting with a number a and adding a fixed number d to each term to get the next term. Similarly, the recursive definition for a geometric sequence (c_k) called for starting with a number a and multiplying each term by a fixed number r to get the next term. If instead of addition or multiplication we use *exponentiation*, we get two other classes of recursively defined sequences:

$$d_1 = a \text{ and, for each positive integer } k,\ d_{k+1} = (d_k)^p$$

for fixed real numbers a and p and

$$e_1 = a \text{ and, for each positive integer } k,\ e_{k+1} = b^{e_k}$$

for fixed real numbers a and b. To get an idea of how sequences can grow in size, take each of the parameters a, d, r, p, and b that have appeared in these definitions to be the number 2 and write the first five terms of each of the arithmetic sequence (b_k), the geometric sequence (c_k), and the sequences (d_k) and (e_k) defined above.

Chapter 1 Review

Important Terms and Symbols

Examples

Section 1.1 Applications of Equations

fixed cost variable cost total cost total revenue profit — Ex. 3, p. 48

Section 1.2 Linear Inequalities

$a < b$ $a \le b$ $a > b$ $a \ge b$ $a < x < b$ — Ex. 1, p. 53

inequality sense of an inequality — Ex. 2, p. 53

equivalent inequalities linear inequality — Ex. 1, p. 53

interval open interval closed interval endpoint

(a, b) $[a, b]$ $(-\infty, b)$ $(-\infty, b]$ (a, ∞) $[a, \infty)$ $(-\infty, \infty)$ — Ex. 3, p. 54

Section 1.3 Applications of Inequalities

renting versus purchasing — Ex. 2, p. 55

current assets current liabilities current ratio — Ex. 3, p. 56

Section 1.4 Absolute Value

distance absolute value, $|x|$ union, $\cup$ — Ex. 3, p. 60

Section 1.5 Summation Notation

$\sum$ notation index bounds — Ex. 1, p. 63

Section 1.6 Sequences

arithmetic sequence — Ex. 5, p. 70

geometric sequence — Ex. 6, p. 71

sum of an arithmetic sequence — Ex. 9, p. 72

sum of a geometric sequence — Ex. 10, p. 73

Summary

With word problems, you may not be given any equations. You may have to construct equations and inequalities (often more than one) by translating natural language statements of the word problem into mathematical statements. This process is *mathematical modeling*. First, read the problem more than once so that you understand what facts are given and what you are to find. Next, choose variables to represent the unknown quantities you need to find. Translate each relationship or fact given in the problem into equations or inequalities involving the variables. Finally, solve the equations (respecting any inequalities) and check that your solution answers what was asked. Sometimes solutions to the *equations* will not be answers to the *problem* (but they may help in obtaining the final answers).

Some basic relationships that are used in solving business problems are as follows:

$$\text{total cost} = \text{variable cost} + \text{fixed cost}$$

$$\text{total revenue} = (\text{price per unit})(\text{number of units sold})$$

$$\text{profit} = \text{total revenue} - \text{total cost}$$

The inequality symbols $<$, $\le$, $>$, and $\ge$ are used to represent an inequality, which is a statement that one number is, for example, less than another number. Three basic operations that, when applied to an inequality, guarantee an equivalent inequality, are as follows:

1. Adding (or subtracting) the same number to (or from) both sides.

2. Multiplying (or dividing) both sides by the same positive number.

3. Multiplying (or dividing) both sides by the same negative number and reversing the sense of the inequality.

The algebraic definition of absolute value is

$$|x| = x \text{ if } x \ge 0 \quad \text{and} \quad |x| = -x \text{ if } x < 0$$

We interpret $|a - b|$ or $|b - a|$ as the distance between a and b. If $d > 0$, then the solution to the inequality $|x| < d$ is the interval $(-d, d)$. The solution to $|x| > d$ consists of the union of two intervals and is given by $(-\infty, -d) \cup (d, \infty)$. Some basic properties of the absolute value are as follows:

1. $|ab| = |a| \cdot |b|$ **2.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

3. $|a - b| = |b - a|$ **4.** $-|a| \le a \le |a|$

5. $|a + b| \le |a| + |b|$

Summation notation provides a compact and precise way of writing sums that have many terms. The basic equations of summation notation are just restatements of the properties of addition. Certain particular sums, such as $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$, are memorable and useful.

Both arithmetic sequences and geometric sequences arise in applications, particularly in business applications. Sums of sequences, particularly those of geometric sequences, will be important in our study of the mathematics of finance in Chapter 5.

Review Problems

In Problems 1–15, solve the equation or inequality.

1. $3x - 1 \geq 2(x - 3)$

2. $2x - (7 + x) \leq x$

3. $-(5x + 2) < -(2x + 4)$

4. $-2(x + 6) > x + 4$

5. $3p(1 - p) > 3(2 + p) - 3p^2$

6. $2\left(6 - \frac{5}{2}p\right) < 7$

7. $\frac{x+5}{3} - \frac{1}{2} \leq 2$

8. $\frac{x}{3} - \frac{x}{4} > \frac{x}{5}$

9. $\frac{1}{4}s - 3 \leq \frac{1}{8}(3 + 2s)$

10. $\frac{1}{3}(t + 2) \geq \frac{1}{4}t + 4$

11. $|4 - 3x| = 9$

12. $\left|\frac{5x - 6}{13}\right| = 0$

13. $|2z - 3| < 5$

14. $4 < \left|\frac{2}{3}x + 5\right|$

15. $|3 - 2x| \geq 4$

16. Evaluate $\sum_{k=1}^{8} (k + 3)^3$ by first cubing the binomial and then using Equations (10), (13), (14), and (15) of Section 1.5.

17. Evaluate $\sum_{i=4}^{11} i^3$ by using $\sum_{i=1}^{11} i^3 - \sum_{i=1}^{3} i^3$. Explain why this works quoting any equations from Section 1.5 that are used. Explain why the answer is necessarily the same as that in Problem 16.

18. Profit A profit of 40% on the selling price of a product is equivalent to what percent profit on the cost?

19. Stock Exchange On a certain day, there were 1132 different issues traded on the New York Stock Exchange. There were 48 more issues showing an increase than showing a decline, and no issues remained the same. How many issues suffered a decline?

20. Sales Tax The sales tax in a certain state is 6.5%. If a total of $3039.29 in purchases, including tax, is made in the course of a year, how much of it is tax?

21. Production Allocation A company will manufacture a total of 10,000 units of its product at plants A and B. Available data are as follows:

	Plant A	Plant B
Unit cost for labor and material	$5	$5.50
Fixed cost	$30,000	$35,000

Between the two plants the company has decided to allot no more than $117,000 for total costs. What is the minimum number of units that must be produced at plant A?

22. Propane Tanks A company is replacing two propane tanks with one new tank. The old tanks are cylindrical, each 25 ft high. One has a radius of 10 ft and the other a radius of 20 ft. The new tank is essentially spherical and it will have the same volume as the old tanks combined. Find the radius of the new tank. [*Hint*: The volume V of a cylindrical tank is $V = \pi r^2 h$, where r is the radius of the circular base and h is the height of the tank. The volume of a spherical tank is $W = \frac{4}{3}\pi R^3$, where R is the radius of the tank.]

23. Operating Ratio The *operating ratio* of a retail business is the ratio, expressed as a percentage, of operating costs (everything from advertising expenses to equipment depreciation) to net sales (i.e., gross sales minus returns and allowances). An operating ratio less than 100% indicates a profitable operation, while an operating ratio in the 80%–90% range is extremely good. If a company has net sales of $236,460 in one period, write an inequality describing the operating costs that would keep the operating ratio below 90%.

24. Write the first five terms of the arithmetic sequence with first term 32 and common difference 3.

25. Write the first five terms of the geometric sequence with first term 100 and common ratio 1.02.

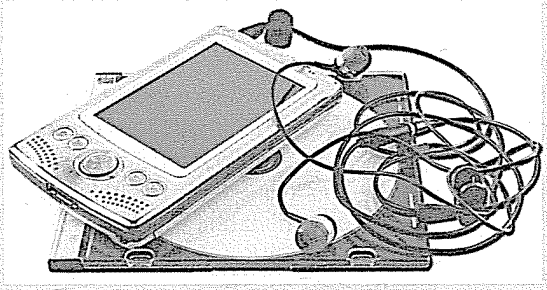
26. Find the sum of the first five terms of the arithmetic sequence with first term 32 and common difference 3.

27. Find the sum of the first five terms of the geometric sequence with first term 100 and common ratio 1.02.

EXPLORE & EXTEND Variable-Quality Recording[1]

There is a bewildering array of technological equipment for a consumer to use to record movies, television programs, computer programs, games, and songs. Whether the equipment be iPod, DVD, CD, or even VHS, it is nearly always possible to record at variable compression ratios with variable quality.

(If you are using an older tape device with different *speeds*, then the fastest speed is ratio 1 to 1 while a lower speed that allows you to store r times as much has a



[1]Adapted from Gregory N. Fiore, "An Application of Linear Equations to the VCR," *Mathematics Teacher*, 81 (October 1988), 570–72. By permission of the National Council of Teachers of Mathematics.

compression ratio of r to 1. For example, the VHS standard, SP, is 1 to 1, while LP is 2 to 1 and EP is 3 to 1.)

The storage medium might be a disc or a tape (or something not yet marketed), but there is always a quantity versus quality trade-off that is inherent in any recording device imaginable. For a given medium, the more information that is stored on it by increased compression, the poorer the quality becomes.

Suppose, for argument's sake, that you want to record a movie that is 210 minutes long on a DVD. To get it all to fit on a single disc at a fixed compression ratio, you will need the ratio that allows between 3 and 4 hours of recording time. The ratio that is judged to be movie quality will only allow about 2 hours of recording time and so will not suffice alone. However, you might want to record as much as you can at the better quality by switching from one ratio to another at a precalculated time.

We will solve the problem of finding the time *to switch* in a general way that will allow for all possible applications of this kind. We want to store M minutes of entertainment on a device that with a compression of 1 to 1 will store m minutes. We have available recording compression ratios of r to 1 and R to 1, say with $1 < r < R$, so that R corresponds to packing in more at lower quality. The number $\frac{M}{r}$ gives the number of 1 to 1 minutes that will be needed to store M minutes at ratio r to 1. If the number $\frac{M}{r}$ is greater than m, then we cannot store all M minutes on our device at ratio r. Assuming that $\frac{M}{R}$ *is* less than m, we want to find the time t when we need to switch from r to R in order to record all M minutes of the entertainment.

If t minutes are recorded at ratio r, then they will consume $\frac{t}{r}$ of the available m 1 to 1 minutes of recording time. The remaining $M - t$ minutes of entertainment will consume a further $\frac{M-t}{R}$ of the the available m 1 to 1 minutes at ratio R. Thus in order to use up *all* the available recording space, we should find t so that

$$\frac{t}{r} + \frac{M-t}{R} = m$$

While this equation, being completetly literal, might appear complicated, it is very simple with respect to t, the variable for which we want to solve. Indeed it is *linear* in t, and there are only a few steps needed to a get a general solution.

$$\frac{t}{r} + \frac{M}{R} - \frac{t}{R} = m$$

$$\left(\frac{1}{r} - \frac{1}{R}\right)t = m - \frac{M}{R}$$

$$\left(\frac{R-r}{rR}\right)t = \frac{mR - M}{R}$$

$$t = \frac{mR - M}{R} \cdot \frac{rR}{R-r}$$

$$t = \frac{r(mR - M)}{R - r}$$

Notice that the formula is not symmetric with respect to r and R. It tells us how many minutes after recording at high quality we should switch to the poorer quality in order to complete the recording in the available space. If you want to save the higher-quality component for the *end* of the recording, you will have to adjust the formula. See Problems 1, 2, 3, 4, and 7. It should be emphasized that the formula need not be memorized (unless you plan to use it a lot). The method is far more important. The existence of the general solution ensures that the method will always work.

In the Problems, try to set up and solve specific problems using the method rather than substituting into the formula.

To learn more about compression schemes, visit `wikipedia.org` and look up "data compression" and related terms.

Problems

A VCR using standard T-120 tape will record for 2 hours in SP mode. Thus $m = 120$ for standard VCR equipment. Use this value in Problems 1–4.

1. If LP and SP modes on a VCR are used, in that order, to record a $2\frac{1}{2}$-hour movie, how long after the start of the movie should the switch *from* LP *to* SP be made?

2. If EP and SP modes are used, in that order, to record a $2\frac{1}{2}$-hour show, how many minutes after the start of the show should the switch from EP to SP be made?

3. If LP and SP modes are used, in that order, to record a movie of length M minutes, how long after the start of the movie should the switch from LP to SP be made?

4. If EP and SP modes are used, in that order, to record a movie of length M minutes, how long after the start of the movie should the switch from EP to SP be made?

5. For a standard CD, the value of m is about 74. Use the Solver function on a graphing calculator to solve the equation

$$\frac{x}{12} + \frac{1080 - x}{20} = 74$$

Then, in a similar manner, solve the equation

$$\frac{x}{15} + \frac{1590 - x}{24} = 74$$

6. What, in the context of recording compressed audio on CDs, do each of the equations in Problem 5 represent?

7. Derive the general formula for finding the time to switch recording ratios if the higher quality (ratio r) is to be saved for the end of the recording.

2 Functions and Graphs

2.1 Functions

2.2 Special Functions

2.3 Combinations of Functions

2.4 Inverse Functions

2.5 Graphs in Rectangular Coordinates

2.6 Symmetry

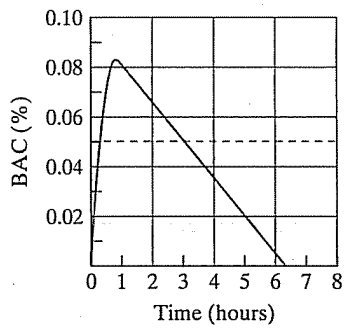
2.7 Translations and Reflections

2.8 Functions of Several Variables

Chapter 2 Review

EXPLORE & EXTEND

A Taxing Experience!



Suppose a 180-pound man drinks four beers in quick succession. We know that his blood alcohol concentration, or BAC, will first rise, then gradually fall back to zero. But what is the best way to describe how quickly the BAC rises, where it peaks, and how fast it falls again?

If we obtain measured BAC values for this particular drinker, we can display them in a table, as follows:

Time (h)	1	2	3	4	5	6
BAC(%)	0.0820	0.0668	0.0516	0.0364	0.0212	0.0060

However, a table can show only a limited number of values and so does not really give the overall picture.

We might instead relate the BAC to time t using a combination of linear and quadratic equations (recall Chapter 0):

$$\text{BAC} = -0.1025t^2 + 0.1844t \qquad \text{if } t \le 0.97$$
$$\text{BAC} = -0.0152t + 0.0972 \qquad \text{if } t > 0.97$$

As with the table, however, it is hard to look at the equations and quickly understand what is happening with BAC over time.

Probably the best description of changes in the BAC over time is a graph, like the one on the left. Here we easily see what happens. The blood alcohol concentration climbs rapidly, peaks at 0.083% after about an hour, and then gradually tapers off over the next five-and-a-half hours. Note that for almost three hours, this male drinker's BAC is above 0.05%, the point at which one's driving skills typically begin to decline. The curve will vary from one drinker to the next, but women are generally affected more severely than men, not only because of weight differences but also because of the different water contents in men's and women's bodies.

The relationship between time and blood alcohol content is an example of a function. This chapter deals in depth with functions and their graphs.

Objective

To understand what a function is and to determine domains and function values.

2.1 Functions

In the 17th century, Gottfried Wilhelm Leibniz, one of the inventors of calculus, introduced the term *function* into the mathematical vocabulary. The concept of a function is one of the most basic in all of mathematics. In particular, it is essential to the study of calculus.

In everyday speech we often hear educated people say things like "Interest rates are a function of oil prices." or "Pension income is a function of years worked." or "Blood alcohol concentration after drinking beer is a function of time." Sometimes such usage agrees with mathematical usage—but not always. We have to be careful with our meaning of the word *function* in order to make it a good mathematical tool. Neverthess, everyday examples can help our understanding. We build the definition in the next three paragraphs.

A key idea is to realize that a **set**, as first mentioned in Section 0.1, need not have numbers as its **elements**. We can speak of a set of interest rates, a set of oil prices, a set of incomes, and so on. If X and Y are sets, in that generality, and x is an element of X and y is an an element of Y, then we write (x, y) for what we call the **ordered pair** consisting of x and y in the order displayed. Thus (y, x) is in general different from (x, y). In fact, given two ordered pairs (x, y) and (a, b), we have $(x, y) = (a, b)$ if and only if both $x = a$ and $y = b$. We will write $X \times Y$ for the set of all ordered pairs (x, y), where x is an element of X and y is an element of Y. For example, if X is the set of oil prices and Y is the set of interest rates, then an element of $X \times Y$ is a pair (p, r), where p is an oil price and r is an interest rate.

A **relation** R from a set X to a set Y is a subset of $X \times Y$. We recall from Section 0.1 that this means any element of R is also an element of $X \times Y$. If it happens that (x, y) is an element of R, then we say that x is R-related to y and write xRy. Each of $<$, $>$, $\leq$, and $\geq$ are relations from the set $(-\infty, \infty)$ of all real numbers to itself. For example, we can define $<$ as that subset of $(-\infty, \infty) \times (-\infty, \infty)$ consisting of all (a, b) such that $a < b$ is true. The use of xRy for "x is R-related to y" is inspired by the notation for inequalities. To give another example, let P and L denote, respectively, the set of all points and the set of all lines in a given plane. For an ordered pair (p, l) in $P \times L$, it is either the case that "p is on l" or "p is not on l". If we write $p \circ l$ for "p is on l", then $\circ$ is a relation from P to L in the sense of this paragraph. Returning to prices and rates, we might say that oil price p is R-related to interest rate r, and write pRr, if "there has been a time at which both the price of oil has been p and the (Fed's) interest rate has been r".

A **function** f from a set X to a set Y is a relation from X to Y with the special property that if both xfy and xfz are true, then $y = z$. (In many books it is also required that, for each x in X, there exists a y in Y, such that xfy. We will not impose this further condition.) The point is that if x is f-related to anything, then that thing is uniquely determined by x. After all, the definition says that if two things, y and z, are both f-related to x, then they are in fact the same thing, $y = z$. We write $y = f(x)$ for the unique y, if there is one, such that x is f-related to y.

With this definition we see that the notion of function is not symmetric in x and y. The notation $f\colon X \longrightarrow Y$ is often used for "f is a function from X to Y" because it underscores the directedness of the concept.

We now examine the putative examples from everyday speech. The relation R defined by pRr if "there has been a time at which both the price of oil has been p and the (Fed's) interest rate has been r" does *not* define a function from oil prices to interest rates. Many people will be able to recall a time when oil was \$30 a barrel and the interest rate was 6% and another time when oil was \$30 a barrel and the interest rate was 1%. In other words, both $(30, 6)$ and $(30, 1)$ are ordered pairs belonging to the R relation, and since $6 \neq 1$, R is not a function. Lest you think that we may be trying to do it the wrong way around, let us write R° for the relation from the set of interest rates to the set of oil prices given by $rR^\circ p$ if and only if pRr. If you can remember a time when the interest rate was 6% with oil at \$30 a barrel and another time when the

interest rate was 6% with oil at \$40 a barrel, then you will have both (6, 30) and (6, 40) in the relation $R°$. The fact that $30 \neq 40$ shows that $R°$ is also not a function.

On the other hand, suppose we bring into a testing facility a person who has just drunk five beers and test her blood alcohol concentration then and each hour thereafter for six hours. For each of the time values {0, 1, 2, 3, 4, 5, 6}, the measurement of blood alcohol concentration will produce *exactly one value*. If we write T for the set of all times beginning with that of the first test and B for the set of all blood alcohol concentration values, then testing the woman in question will determine a function $b : T \longrightarrow B$, where, for any time t in T, $b(t)$ is *the* blood alcohol concentration of the woman at time t.

It is not true that "Pension income is a function of years worked." If the value of "years worked" is 25, then the value of "pension income" is not yet determined. In most organizations, a CEO and a systems manager will retire with quite different pensions after 25 years of service. However, in this example we might be able to say that, *for each job description in a particular organization,* pension income is a function of years worked.

If \$100 is invested at, say, 6% simple interest, then the interest earned I is a function of the length of time t that the money is invested. These quantities are related by

$$I = 100(0.06)t \tag{1}$$

Here, for each value of t, there is exactly one value of I given by Equation (1). In a situation like this we will often write $I(t) = 100(0.06)t$ to reinforce the idea that the I-value is determined by the t-value. Sometimes we write $I = I(t)$ to make the claim that I is a function of t even if we do not know a formula for it. Formula (1) assigns the output 3 to the input $\frac{1}{2}$ and the output 12 to the input 2. We can think of Formula (1) as defining a *rule*: Multiply t by 100(0.06). The rule assigns to each input number t exactly one output number I, which is often symbolized by the following arrow notation:

$$t \longmapsto 100(0.06)t$$

A formula provides a way of describing a rule to cover potentially infinitely many cases, but if there are only finitely many values of the input variable, as in the chapter opening paragraph, then the *rule*, as provided by the observations recorded in the table there, may not be part of any recognizable *formula*. We use the word *rule* rather than *formula* below to allow us to capture this useful generality. The following definition is sometimes easier to keep in mind than our description of a function as a special kind of relation:

Definition

A ***function*** $f : X \longrightarrow Y$ is a rule that assigns, to each element x of X, at most one element of Y. If an element is assigned to x in X, it is denoted by $f(x)$. The subset of X consisting of all the x for which $f(x)$ is defined is called the ***domain*** of f. The set of all elements in Y of the form $f(x)$, for some x in X, is called the ***range*** of f.

For the interest function defined by Equation (1), the input number t cannot be negative, because negative time makes no sense in this example. Thus, the domain consists of all nonnegative numbers—that is, all $t \geq 0$, where the variable gives the time elapsed from when the investment was made.

A variable that takes on values in the domain of a function $f: X \longrightarrow Y$ is sometimes called an **input** or an **independent variable** for f. A variable that takes on values in the range of f is sometimes called an **ouput** or a **dependent variable** of f. Thus, for the interest formula $I = 100(0.06)t$, the independent variable is t, the dependent variable is I, and I is a function of t.

As another example, the equation

$$y = x + 2 \tag{2}$$

defines y as a function of x. The equation gives the rule, "Add 2 to x." This rule assigns to each input x exactly one output $x + 2$, which is y. If $x = 1$, then $y = 3$; if $x = -4$, then $y = -2$. The independent variable is x and the dependent variable is y.

CAUTION!

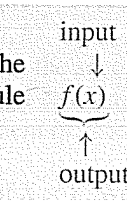
In $y^2 = x$, x and y are related, but the relationship does not give y as a function of x.

Not all equations in x and y define y as a function of x. For example, let $y^2 = x$. If x is 9, then $y^2 = 9$, so $y = \pm 3$. Hence, to the input 9, there are assigned not one, but *two*, output numbers: 3 and -3. This violates the definition of a function, so y is **not** a function of x.

On the other hand, some equations in two variables define either variable as a function of the other variable. For example, if $y = 2x$, then for each input x, there is exactly one output, $2x$. Thus, y is a function of x. However, solving the equation for x gives $x = y/2$. For each input y, there is exactly one output, $y/2$. Consequently, x is a function of y.

Usually, the letters f, g, h, F, G, and so on are used to name functions. For example, Equation (2), $y = x + 2$, defines y as a function of x, where the rule is "Add 2 to the input." Suppose we let f represent this rule. Then we say that f is the function. To indicate that f assigns the output 3 to the input 1, we write $f(1) = 3$, which is read "f of 1 equals 3." Similarly, $f(-4) = -2$. More generally, if x is any input, we have the following notation:

$f(x)$, which is read "f of x," and which means the output, in the range of f, that results when the rule f is applied to the input x, from the domain of f.



CAUTION!

$f(x)$ does **not** mean f times x. $f(x)$ is the output that corresponds to the input x.

Thus, the output $f(x)$ is the same as y. But since $y = x + 2$, we can also write $f(x) = y = x + 2$ or more simply

$$f(x) = x + 2$$

For example, to find $f(3)$, which is the output corresponding to the input 3, we replace each x in $f(x) = x + 2$ by 3:

$$f(3) = 3 + 2 = 5$$

Outputs are also called **function values.**

For another example, the equation $g(x) = x^3 + x^2$ defines the function g that assigns the output $x^3 + x^2$ to an input x:

$$g: x \mapsto x^3 + x^2$$

In other words, g adds the cube of the input to the square of the input. Some function values are

$$g(2) = 2^3 + 2^2 = 12$$
$$g(-1) = (-1)^3 + (-1)^2 = -1 + 1 = 0$$
$$g(t) = t^3 + t^2$$
$$g(x + 1) = (x + 1)^3 + (x + 1)^2$$

The idea of *replacement* is very important in determining function values.

Note that $g(x + 1)$ was found by replacing each x in $x^3 + x^2$ by the input $x + 1$. When we refer to the function g defined by $g(x) = x^3 + x^2$, we are free to say simply "the function $g(x) = x^3 + x^2$" and similarly "the function $y = x + 2$."

Unless otherwise stated, the domain of a function $f: X \longrightarrow Y$ is the set of all x in X for which $f(x)$ makes sense, as an element of Y. When X and Y are both $(-\infty, \infty)$, this convention often refers to arithmetical restrictions. For example, suppose

$$h(x) = \frac{1}{x - 6}$$

Here any real number can be used for x except 6, because the denominator is 0 when x is 6. So the domain of h is understood to be all real numbers except 6. A useful notation

for this set is $(-\infty, \infty) - \{6\}$. More generally, if A and B are subsets of a set X, then we write $A - B$ for the set of all x in X such that x is in A and x is *not* in B. We note too that the range of h is the set of all real numbers except 0. Each output of h is a fraction and the only way that a fraction can be 0 is for its numerator to be 0. While we do have

$$\frac{1}{x-6} = \frac{c}{c(x-6)} \quad \text{for all } c \neq 0$$

by the *fundamental principle of fractions* of Section 0.2, we see that 0 is not a function value for h. But if y is any nonzero real number, we can solve $\dfrac{1}{x-6} = y$ for x and get $x = 6 + \dfrac{1}{y}$ as the (unique) input for which $h(x)$ is the given y. Thus the range is $(-\infty, \infty) - \{0\}$, the set of all real numbers other than 0.

Equality of Functions

To say that two functions $f, g : X \longrightarrow Y$ are equal, denoted $f = g$, is to say that

1. The domain of f is equal to the domain of g;
2. For every x in the domain of f and g, $f(x) = g(x)$.

Requirement 1 says that an element x is in the domain of f if and only if x is in the domain of g. Thus, if we have $f(x) = x^2$, with no explicit mention of domain, and $g(x) = x^2$ for $x \geq 0$, then $f \neq g$. For here the domain of f is the whole real line $(-\infty, \infty)$ and the domain of g is $[0, \infty)$. On the other hand, if we have $f(x) = (x+1)^2$ and $g(x) = x^2 + 2x + 1$, then, for both f and g, the domain is understood to be $(-\infty, \infty)$ and the issue for deciding if $f = g$ is whether, for each real number x, we have $(x+1)^2 = x^2 + 2x + 1$. But this is true; it is a special case of item 4 in the Special Products of Section 0.4. In fact, older textbooks refer to statements like $(x+1)^2 = x^2 + 2x + 1$ as "identities," to indicate that they are true for any admissible value of the variable and to distinguish them from statements like $(x+1)^2 = 0$, which are true for some values of x.

Given functions f and g, it follows that we have $f \neq g$ if *either* the domain of f is different from the domain of g *or* there is some x for which $f(x) \neq g(x)$.

EXAMPLE 1 Determining Equality of Functions

Determine which of the following functions are equal.

a. $f(x) = \dfrac{(x+2)(x-1)}{(x-1)}$

b. $g(x) = x + 2$

c. $h(x) = \begin{cases} x+2 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$

d. $k(x) = \begin{cases} x+2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$

Solution: The domain of f is the set of all real numbers other than 1, while that of g is the set of all real numbers. (For these we are following the convention that the domain is the set of all real numbers for which the rule makes sense.) We will have more to say about functions like h and k that are defined by *cases* in Example 4 of Section 2.2. Here we observe that the domain of h and the domain of k are both $(-\infty, \infty)$, since for both we have a rule that makes sense for each real number. The domains of g, h, and k are equal to each other, but that of f is different. So by requirement 1 for equality of functions, $f \neq g$, $f \neq h$ and $f \neq k$. By definition, $g(x) = h(x) = k(x)$ for all $x \neq 1$, so the matter of equality of g, h and k depends on their values at 1. Since $g(1) = 3$, $h(1) = 0$ and $k(1) = 3$, we conclude that $g = k$ and $g \neq h$ (and $h \neq k$).

While this example might appear to be contrived, it is typical of an issue that arises frequently in calculus.

Now Work Problem 3 ◁

APPLY IT ▷

1. The area of a circle depends on the length of the radius of the circle.

a. Write a function $a(r)$ for the area of a circle when the length of the radius is r.
b. What is the domain of this function out of context?
c. What is the domain of this function in the given context?

EXAMPLE 2 **Finding Domains**

Find the domain of each function.

a. $f(x) = \dfrac{x}{x^2 - x - 2}$

Solution: We cannot divide by zero, so we must find any values of x that make the denominator 0. These *cannot* be inputs. Thus, we set the denominator equal to 0 and solve for x:

$$\begin{aligned} x^2 - x - 2 &= 0 && \text{quadratic equation} \\ (x-2)(x+1) &= 0 && \text{factoring} \\ x &= 2, -1 \end{aligned}$$

Therefore, the domain of f is all real numbers *except* 2 and -1.

b. $g(t) = \sqrt{2t-1}$ as a function g: $(-\infty, \infty) \longrightarrow (-\infty, \infty)$

Solution: $\sqrt{2t-1}$ is a real number if $2t - 1$ is greater than or equal to 0. If $2t - 1$ is negative, then $\sqrt{2t-1}$ is not a real number, so we must assume that

$$\begin{aligned} 2t - 1 &\geq 0 \\ 2t &\geq 1 && \text{adding 1 to both sides} \\ t &\geq \frac{1}{2} && \text{dividing both sides by 2} \end{aligned}$$

Thus, the domain is the interval $[\frac{1}{2}, \infty)$.

Now Work Problem 7 ◁

APPLY IT ▷

2. The time it takes to go a given distance depends on the speed at which one is traveling.

a. Write a function $t(r)$ for the time it takes if the distance is 300 miles and the speed is r.
b. What is the domain of this function out of context?
c. What is the domain of this function in the given context?
d. Find $t(x)$, $t\left(\frac{x}{2}\right)$, and $t\left(\frac{x}{4}\right)$.
e. What happens to the time if the speed is reduced (divided) by a constant c? Describe this situation using an equation.

EXAMPLE 3 **Finding Domain and Function Values**

Let $g(x) = 3x^2 - x + 5$. Any real number can be used for x, so the domain of g is all real numbers.

a. Find g(z).

Solution: Replacing each x in $g(x) = 3x^2 - x + 5$ by z gives

$$g(z) = 3(z)^2 - z + 5 = 3z^2 - z + 5$$

b. Find $g(r^2)$.

Solution: Replacing each x in $g(x) = 3x^2 - x + 5$ by r^2 gives

$$g(r^2) = 3(r^2)^2 - r^2 + 5 = 3r^4 - r^2 + 5$$

c. Find $g(x + h)$.

Solution:

$$\begin{aligned} g(x+h) &= 3(x+h)^2 - (x+h) + 5 \\ &= 3(x^2 + 2hx + h^2) - x - h + 5 \\ &= 3x^2 + 6hx + 3h^2 - x - h + 5 \end{aligned}$$

Now Work Problem 31(a) ◁

CAUTION!

Don't be confused by notation. In Example 3(c), we find $g(x + h)$ by replacing each x in $g(x) = 3x^2 - x + 5$ by the input $x + h$. $g(x + h)$, $g(x) + h$ and $g(x) + g(h)$ are all quite different quantities.

EXAMPLE 4 **Finding a Difference Quotient**

If $f(x) = x^2$, find $\dfrac{f(x+h)-f(x)}{h}$.

Solution: The expression $\dfrac{f(x+h)-f(x)}{h}$ is referred to as a **difference quotient**. Here the numerator is a difference of function values. We have

The difference quotient of a function is an important mathematical concept.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2-x^2}{h}\\ &= \frac{x^2+2hx+h^2-x^2}{h} = \frac{2hx+h^2}{h}\\ &= \frac{h(2x+h)}{h}\\ &= 2x+h \quad \text{for } h \neq 0\end{aligned}$$

If we consider the original difference quotient as a function of h, then it is different from $2x + h$ because 0 is not in the domain of the original difference quotient but it *is* in the default domain of $2x + h$. For this reason, we had to restrict the final equality.

Now Work Problem 35 ◁

In some cases, the domain of a function is restricted for physical or economic reasons. For example, the previous interest function $I = 100(0.06)t$ has $t \geq 0$ because t represents time elapsed since the investment was made. Example 5 will give another illustration.

EXAMPLE 5 **Demand Function**

APPLY IT ▶

3. Suppose the weekly demand function for large pizzas at a local pizza parlor is $p = 26 - \dfrac{q}{40}$.
a. If the current price is \$18.50 per pizza, how many pizzas are sold each week?
b. If 200 pizzas are sold each week, what is the current price?
c. If the owner wants to double the number of large pizzas sold each week (to 400), what should the price be?

Suppose that the equation $p = 100/q$ describes the relationship between the price per unit p of a certain product and the number of units q of the product that consumers will buy (that is, demand) per week at the stated price. This equation is called a *demand equation* for the product. If q is an input, then to each value of q there is assigned at most one output p:

$$q \mapsto \frac{100}{q} = p$$

For example,

$$20 \mapsto \frac{100}{20} = 5$$

that is, when q is 20, p is 5. Thus, price p is a function of quantity demanded, q. This function is called a **demand function**. The independent variable is q, and p is the dependent variable. Since q cannot be 0 (division by 0 is not defined) and cannot be negative (q represents quantity), the domain is all $q > 0$.

Now Work Problem 43 ◁

We have seen that a function is a rule which assigns to each input in the domain exactly one output in the range. For the rule given by $f(x) = x^2$, some sample assignments are shown by the arrows in Figure 2.1. The next example discusses a rule given by a finite listing rather than an an algebraic formula.

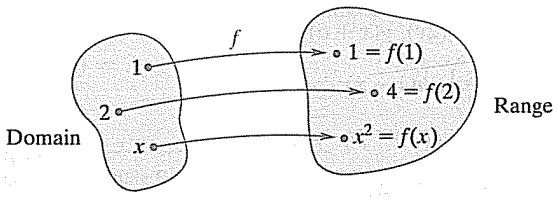


FIGURE 2.1 Some function values for $f(x) = x^2$.

APPLY IT ▷

4. For the supply function given by the following table, determine the weekly revenue function, assuming that all units supplied are sold.

p Price per Unit in Dollars	q Quanity Supplied per Week
500	11
600	14
700	17
800	20

EXAMPLE 6 Supply Schedule

The table in Apply It 4 on this page is a *supply schedule*. Such a table lists for each of certain prices p of a certain product the quantity q that producers will supply per week at that price. For each price, the table provides exactly one quantity so that it exhibits q as a function of p.

But also, for each quantity, the table provides exactly one price so that it also exhibits p as a function of q. If we write $q = f(p)$, then the table provides

$$f(500) = 11 \quad f(600) = 14 \quad f(700) = 17 \quad f(800) = 20$$

If we write $p = g(q)$, then the table also provides

$$g(11) = 500 \quad g(14) = 600 \quad g(17) = 700 \quad g(20) = 800$$

Observe that we have $g(f(p)) = p$, for all p, and $f(g(q)) = q$, for all q. We will have more to say about pairs of functions of this kind in Section 2.4. Both functions determined by this table are called **supply functions.**

Now Work Problem 53 ◁

PROBLEMS 2.1

In Problems 1–4, determine whether the given functions are equal.

1. $f(x) = \sqrt{x^2}$; $g(x) = x$

2. $G(x) = (\sqrt{x+1})^2$; $H(x) = x + 1$

3. $h(x) = \dfrac{|x|}{x}$; $k(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

4. $f(x) = \begin{cases} \dfrac{x^2 - 4x + 3}{x - 3} & \text{if } x \neq 3 \\ 2 & \text{if } x = 3 \end{cases}$;
$g(x) = x - 1$

In Problems 5–16, give the domain of each function.

5. $f(x) = \dfrac{6}{x-1}$

6. $g(x) = \dfrac{x}{5}$

7. $h(x) = \sqrt{x-3}$

8. $K(z) = \dfrac{1}{\sqrt{z-1}}$

9. $f(z) = 3z^2 + 2z - 4$

10. $H(x) = \dfrac{x^2}{x+3}$

11. $f(x) = \dfrac{9x-9}{2x+7}$

12. $g(x) = \sqrt{4x+3}$

13. $g(y) = \dfrac{4}{y^2 - 4y + 4}$

14. $\phi(x) = \dfrac{x+5}{x^2 + x - 6}$

15. $h(s) = \dfrac{3 - x^2}{3x^2 - 5x - 2}$

16. $G(r) = \dfrac{2}{r^2 + 1}$

In Problems 17–28, find the function values for each function.

17. $f(x) = 2x + 1$; $f(0)$, $f(3)$, $f(-4)$

18. $H(s) = 5s^2 - 3$; $H(4)$, $H(\sqrt{2})$, $H\left(\dfrac{2}{3}\right)$

19. $G(x) = 2 - x^2$; $G(-8)$, $G(u)$, $G(u^2)$

20. $F(x) = -7x + 1$; $F(s)$, $F(t+1)$, $F(x+3)$

21. $\gamma(u) = 2u^2 - u$; $\gamma(-2)$, $\gamma(2v)$, $\gamma(x+a)$

22. $h(v) = \dfrac{1}{\sqrt{v}}$; $h(16)$, $h\left(\dfrac{1}{4}\right)$, $h(1-x)$

23. $f(x) = x^2 + 2x + 1$; $f(1)$, $f(-1)$, $f(x+h)$

24. $H(x) = (x+4)^2$; $H(0)$, $H(2)$, $H(t-4)$

25. $k(x) = \dfrac{x-5}{x^2+1}$; $k(5)$, $k(2x)$, $k(x+h)$

26. $k(x) = \sqrt{x-3}$; $k(4)$, $k(3)$, $k(x+1) - k(x)$

27. $f(x) = x^{4/3}$; $f(0)$, $f(64)$, $f\left(\dfrac{1}{8}\right)$

28. $g(x) = x^{2/5}$; $g(32)$, $g(-64)$, $g(t^{10})$

In Problems 29–36, find (a) $f(x+h)$ and (b) $\dfrac{f(x+h) - f(x)}{h}$; simplify your answers.

29. $f(x) = 4x - 5$

30. $f(x) = \dfrac{x}{3}$

31. $f(x) = x^2 + 2x$

32. $f(x) = 3x^2 - 2x - 1$

33. $f(x) = 3 - 2x + 4x^2$

34. $f(x) = x^3$

35. $f(x) = \dfrac{1}{x-1}$

36. $f(x) = \dfrac{x+8}{x}$

37. If $f(x) = 5x + 3$, find $\dfrac{f(3+h) - f(3)}{h}$.

38. If $f(x) = 2x^2 - x + 1$, find $\dfrac{f(x) - f(2)}{x-2}$.

In Problems 39–42, is y a function of x? Is x a function of y?

39. $9y - 3x - 4 = 0$

40. $x^4 - 1 + y = 0$

41. $y = 7x^2$

42. $x^2 + y^2 = 1$

43. The formula for the area of a circle of radius r is $A = \pi r^2$. Is the area a function of the radius?

44. Suppose $f(b) = a^2b^3 + a^3b^2$. (a) Find $f(a)$. (b) Find $f(ab)$.

45. Value of Business A business with an original capital of \$50,000 has income and expenses each week of \$7200 and \$4900, respectively. If all profits are retained in the business, express the value V of the business at the end of t weeks as a function of t.

46. Depreciation If a \$30,000 machine depreciates 2% of its original value each year, find a function f that expresses the machine's value V after t years have elapsed.

47. Profit Function If q units of a certain product are sold (q is nonnegative), the profit P is given by the equation $P = 1.25q$. Is P a function of q? What is the dependent variable? the independent variable?

48. Demand Function Suppose the yearly demand function for a particular actor to star in a film is $p = \dfrac{1{,}200{,}000}{q}$, where q is the number of films he stars in during the year. If the actor currently charges \$600,000 per film, how many films does he star in each year? If he wants to star in four films per year, what should his price be?

49. Supply Function Suppose the weekly supply function for a pound of house-blend coffee at a local coffee shop is $p = \dfrac{q}{48}$, where q is the number of pounds of coffee supplied per week. How many pounds of coffee per week will be supplied if the price is \$8.39 a pound? How many pounds of coffee per week will be supplied if the price is \$19.49 a pound? How does the amount supplied change as the price increases?

50. Hospital Discharges An insurance company examined the records of a group of individuals hospitalized for a particular illness. It was found that the total proportion discharged at the end of t days of hospitalization is given by

$$f(t) = 1 - \left(\frac{200}{200 + t}\right)^3$$

Evaluate (a) $f(0)$, (b) $f(100)$, and (c) $f(800)$. (d) At the end of how many days was half of the group discharged?

51. Psychology A psychophysical experiment was conducted to analyze human response to electrical shocks.[1] The subjects received a shock of a certain intensity. They were told to assign a magnitude of 10 to this particular shock, called the standard stimulus. Then other shocks (stimuli) of various intensities were given. For each one, the response R was to be a number that indicated the perceived magnitude of the shock relative to that of the standard stimulus. It was found that R was a function of the intensity I of the shock (I in microamperes) and was estimated by

$$R = f(I) = \frac{I^{4/3}}{2500} \qquad 500 \le I \le 3500$$

Evaluate (a) $f(1000)$ and (b) $f(2000)$. (c) Suppose that I_0 and $2I_0$ are in the domain of f. Express $f(2I_0)$ in terms of $f(I_0)$. What effect does the doubling of intensity have on response?

52. Psychology In a paired-associate learning experiment,[2] the probability of a correct response as a function of the number n of trials has the form

$$P(n) = 1 - \frac{1}{2}(1 - c)^{n-1} \qquad n \ge 1$$

where the estimated value of c is 0.344. Find $P(1)$ and $P(2)$ by using this value of c.

53. Demand Schedule The following table is called a *demand schedule*. It gives a correspondence between the price p of a product and the quantity q that consumers will demand (that is, purchase) at that price. (a) If $p = f(q)$, list the numbers in the domain of f. Find $f(2900)$ and $f(3000)$. (b) If $q = g(p)$, list the numbers in the domain of g. Find $g(10)$ and $g(17)$.

Price per Unit, p	Quantity Demanded per Week, q
\$10	3000
12	2900
17	2300
20	2000

In Problems 54–57, use your calculator to find the indicated values for the given function. Round answers to two decimal places.

54. $f(x) = 2.03x^3 - 5.27x^2 - 13.71$; (a) $f(1.73)$, (b) $f(-5.78)$, (c) $f(\sqrt{2})$

55. $f(x) = \dfrac{14.7x^2 - 3.95x - 15.76}{24.3 - x^3}$; (a) $f(4)$, (b) $f(-17/4)$, (c) $f(\pi)$

56. $f(x) = (20.3 - 3.2x)(2.25x^2 - 7.1x - 16)^4$; (a) $f(0.3)$, (b) $f(-0.02)$, (c) $f(1.9)$

57. $f(x) = \sqrt{\dfrac{\sqrt{2}x^2 + 7.31(x + 1)}{5.03}}$; (a) $f(12.35)$, (b) $f(-123)$, (c) $f(0)$

Objective

To introduce constant functions, polynomial functions, rational functions, case-defined functions, the absolute-value function, and factorial notation.

2.2 Special Functions

In this section, we look at functions having special forms and representations. We begin with perhaps the simplest type of function there is: a *constant function*.

EXAMPLE 1 **Constant Functions**

Let $h : (-\infty, \infty) \longrightarrow (-\infty, \infty)$ be given by $h(x) = 2$. The domain of h is $(-\infty, \infty)$, the set of all real numbers. All function values are 2. For example,

$$h(10) = 2 \qquad h(-387) = 2 \qquad h(x + 3) = 2$$

[1] Adapted from H. Babkoff, "Magnitude Estimation of Short Electrocutaneous Pulses," *Psychological Research*, 39, no. 1 (1976), 39–49.

[2] D. Laming, *Mathematical Psychology* (New York: Academic Press, 1983).

APPLY IT ▷

5. Suppose the monthly health insurance premiums for an individual are $125.00.
a. Write the monthly health insurance premiums as a function of the number of visits the individual makes to the doctor.
b. How do the health insurance premiums change as the number of visits to the doctor increases?
c. What kind of function is this?

We call h a *constant function* because all the function values are the same. More generally, we have this definition:

A function of the form $h(x) = c$, where c is a *constant*, is called a **constant function.**

Now Work Problem 17 ◁

A constant function belongs to a broader class of functions, called *polynomial functions*. In general, a function of the form

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

Each term in a polynomial function is either a constant or a constant times a positive integral power of x.

where n is a nonnegative integer and $c_n, c_{n-1}, \ldots, c_0$ are constants with $c_n \neq 0$, is called a **polynomial function** (in x). The number n is called the **degree** of the polynomial, and c_n is the **leading coefficient.** Thus,

$$f(x) = 3x^2 - 8x + 9$$

is a polynomial function of degree 2 with leading coefficient 3. Likewise, $g(x) = 4 - 2x$ has degree 1 and leading coefficient -2. Polynomial functions of degree 1 or 2 are called **linear** or **quadratic functions,** respectively. For example, $g(x) = 4 - 2x$ is linear and $f(x) = 3x^2 - 8x + 9$ is quadratic. Note that a nonzero constant function, such as $f(x) = 5$ [which can be written as $f(x) = 5x^0$], is a polynomial function of degree 0. The constant function $f(x) = 0$ is also considered a polynomial function but has no degree assigned to it. The domain of any polynomial function is the set of all real numbers.

APPLY IT ▷

6. The function $d(t) = 3t^2$, for $t \geq 0$, represents the distance in meters a car will go in t seconds when it has a constant acceleration of 6 m per second.
a. What kind of function is this?
b. What is its degree?
c. What is its leading coefficient?

EXAMPLE 2 Polynomial Functions

a. $f(x) = x^3 - 6x^2 + 7$ is a polynomial (function) of degree 3 with leading coefficient 1.

b. $g(x) = \dfrac{2x}{3}$ is a linear function with leading coefficient $\dfrac{2}{3}$.

c. $f(x) = \dfrac{2}{x^3}$ is *not* a polynomial function. Because $f(x) = 2x^{-3}$ and the exponent for x is not a nonnegative integer, this function does not have the proper form for a polynomial. Similarly, $g(x) = \sqrt{x}$ is not a polynomial, because $g(x) = x^{1/2}$.

Now Work Problem 3 ◁

A function that is a quotient of polynomial functions is called a **rational function.**

EXAMPLE 3 Rational Functions

a. $f(x) = \dfrac{x^2 - 6x}{x + 5}$ is a rational function, since the numerator and denominator are each polynomials. Note that this rational function is not defined for $x = -5$.

b. $g(x) = 2x + 3$ is a rational function, since $2x + 3 = \dfrac{2x + 3}{1}$. In fact, every polynomial function is also a rational function.

Every polynomial function is a rational function.

Now Work Problem 5 ◁

Sometimes more than one expression is needed to define a function, as Example 4 shows.

EXAMPLE 4 **Case-Defined Function**

APPLY IT ▷

7. To reduce inventory, a department store charges three rates. If you buy 0–5 pairs of socks, the price is \$3.50 per pair. If you buy 6–10 pairs of socks, the price is \$3.00 per pair. If you buy more than 10 pairs, the price is \$2.75 per pair. Write a case-defined function to represent the cost of buying n pairs of socks.

Let

$$F(s) = \begin{cases} 1 & \text{if } -1 \le s < 1 \\ 0 & \text{if } 1 \le s \le 2 \\ s-3 & \text{if } 2 < s \le 8 \end{cases}$$

This is called a **case-defined function** because the rule for specifying it is given by rules for each of several disjoint cases. Here s is the independent variable, and the domain of F is all s such that $-1 \le s \le 8$. The value of s determines which expression to use.

Find $F(0)$: Since $-1 \le 0 < 1$, we have $F(0) = 1$.

Find $F(2)$: Since $1 \le 2 \le 2$, we have $F(2) = 0$.

Find $F(7)$: Since $2 < 7 \le 8$, we substitute 7 for s in $s - 3$.

$$F(7) = 7 - 3 = 4$$

Now Work Problem 19 ◁

EXAMPLE 5 **Absolute-Value Function**

The absolute-value function can be considered a case-defined function.

The function $|-|(x) = |x|$ is called the *absolute-value function*. Recall that the **absolute value,** of a real number x, is denoted $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$$

Thus, the domain of $|-|$ is all real numbers. Some function values are

$$|16| = 16$$
$$|-\tfrac{4}{3}| = -\left(-\tfrac{4}{3}\right) = \tfrac{4}{3}$$
$$|0| = 0$$

Now Work Problem 21 ◁

In our next examples, we make use of *factorial notation.*

The symbol $r!$, with r a positive integer, is read "r **factorial.**" It represents the product of the first r positive integers:

$$r! = 1 \cdot 2 \cdot 3 \cdots r$$

We also define

$$0! = 1$$

For each nonnegative integer n, $(-)!(n) = n!$ determines a unique number so it follows that $(-)!$ is a function whose domain is the set of nonnegative integers.

EXAMPLE 6 **Factorials**

APPLY IT ▷

8. Seven different books are to be placed on a shelf. How many ways can they be arranged? Represent the question as a factorial problem and give the solution.

a. $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b. $3!(6-5)! = 3! \cdot 1! = (3 \cdot 2 \cdot 1)(1) = (6)(1) = 6$

c. $\dfrac{4!}{0!} = \dfrac{1 \cdot 2 \cdot 3 \cdot 4}{1} = \dfrac{24}{1} = 24$

Now Work Problem 27 ◁

EXAMPLE 7 Genetics

Factorials occur frequently in probability theory.

Suppose two black guinea pigs are bred and produce exactly five offspring. Under certain conditions, it can be shown that the probability P that exactly r of the offspring will be brown and the others black is a function of r, $P = P(r)$, where

$$P(r) = \frac{5!\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{5-r}}{r!(5-r)!} \qquad r = 0, 1, 2, \ldots, 5$$

The letter P in $P = P(r)$ is used in two ways. On the right side, P represents the function rule. On the left side, P represents the dependent variable. The domain of P is all integers from 0 to 5, inclusive. Find the probability that exactly three guinea pigs will be brown.

Solution: We want to find $P(3)$. We have

$$P(3) = \frac{5!\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2}{3!2!} = \frac{120\left(\frac{1}{64}\right)\left(\frac{9}{16}\right)}{6(2)} = \frac{45}{512}$$

Now Work Problem 35 ◁

PROBLEMS 2.2

In Problems 1–4, determine whether the given function is a polynomial function.

1. $f(x) = x^2 - x^4 + 4$

2. $f(x) = \dfrac{x^3 + 7x - 3}{3}$

3. $g(x) = \dfrac{1}{x^2 + 2x + 1}$

4. $g(x) = 2^{-3}x^3$

In Problems 5–8, determine whether the given function is a rational function.

5. $f(x) = \dfrac{x^2 + x}{x^3 + 4}$

6. $f(x) = \dfrac{3}{2x + 1}$

7. $g(x) = \begin{cases} 1 & \text{if } x < 5 \\ 4 & \text{if } x \geq 5 \end{cases}$

8. $g(x) = 4x^{-4}$

In Problems 9–12, find the domain of each function.

9. $k(z) = 26$

10. $f(x) = \sqrt{\pi}$

11. $f(x) = \begin{cases} 5x & \text{if } x > 1 \\ 4 & \text{if } x \leq 1 \end{cases}$

12. $f(x) = \begin{cases} 4 & \text{if } x = 3 \\ x^2 & \text{if } 1 \leq x < 3 \end{cases}$

In Problems 13–16, state (a) the degree and (b) the leading coefficient of the given polynomial function.

13. $F(x) = 7x^3 - 2x^2 + 6$

14. $g(x) = 9x^2 + 2x + 1$

15. $f(x) = \dfrac{1}{\pi} - 3x^5 + 2x^6 + x^7$

16. $f(x) = 9$

In Problems 17–22, find the function values for each function.

17. $f(x) = 8$; $f(2)$, $f(t + 8)$, $f(-\sqrt{17})$

18. $g(x) = |x - 3|$; $g(10)$, $g(3)$, $g(-3)$

19. $F(t) = \begin{cases} 2 & \text{if } t > 1 \\ 0 & \text{if } t = 1 \\ -1 & \text{if } t < 1 \end{cases}$;

$F(12)$, $F(-\sqrt{3})$, $F(1)$, $F\left(\dfrac{18}{5}\right)$

20. $f(x) = \begin{cases} 4 & \text{if } x \geq 0 \\ 3 & \text{if } x < 0 \end{cases}$;

$f(3), f(-4), f(0)$

21. $G(x) = \begin{cases} x - 1 & \text{if } x \geq 3 \\ 3 - x^2 & \text{if } x < 3 \end{cases}$;

$G(8)$, $G(3)$, $G(-1)$, $G(1)$

22. $F(\theta) = \begin{cases} 2\theta - 5 & \text{if } \theta < 2 \\ \theta^2 - 3\theta + 1 & \text{if } \theta > 2 \end{cases}$;

$F(3)$, $F(-3)$, $F(2)$

In Problems 23–28, determine the value of each expression.

23. $6!$

24. $(3 - 3)!$

25. $(4 - 2)!$

26. $6! \cdot 2!$

27. $\dfrac{n!}{(n-1)!}$

28. $\dfrac{8!}{5!(8-5)!}$

29. Subway Ride A return subway ride ticket within the city costs \$2.50. Write the cost of a return ticket as a function of a passenger's income. What kind of function is this?

30. Geometry A rectangular prism has length three more than its width and height one less than twice the width. Write the volume of the rectangular prism as a function of the width. What kind of function is this?

31. Cost Function In manufacturing a component for a machine, the initial cost of a die is \$850 and all other additional costs are \$3 per unit produced. (a) Express the total cost C (in dollars) as a linear function of the number q of units produced. (b) How many units are produced if the total cost is \$1600?

32. Investment If a principal of P dollars is invested at a simple annual interest rate of r for t years, express the total accumulated amount of the principal and interest as a function of t. Is your result a linear function of t?

33. Sales To encourage large group sales, a theater charges two rates. If your group is less than 12, each ticket costs \$9.50. If your group is 12 or more, each ticket costs \$8.75. Write a case-defined function to represent the cost of buying n tickets.

34. Factorials The business mathematics class has elected a grievance committee of five to complain to the faculty about the introduction of factorial notation into the course. They decide that they will be more effective if they label themselves as members A, G, M, N, and S, where member A will lobby faculty with

surnames A through F, member G will lobby faculty with surnames G through L, and so on. In how many ways can the committee so label its members?

35. Genetics Under certain conditions, if two brown-eyed parents have exactly three children, the probability that there will be exactly r blue-eyed children is given by the function $P = P(r)$, where

$$P(r) = \frac{3!\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{3-r}}{r!(3-r)!}, \qquad r = 0, 1, 2, 3$$

Find the probability that exactly two of the children will be blue-eyed.

36. Genetics In Example 7, find the probability that all five offspring will be brown.

37. Bacteria Growth Bacteria are growing in a culture. The time t (in hours) for the bacteria to double in number (the generation time) is a function of the temperature T (in °C) of the culture. If this function is given by[3]

$$t = f(T) = \begin{cases} \frac{1}{24}T + \frac{11}{4} & \text{if } 30 \le T \le 36 \\ \frac{4}{3}T - \frac{175}{4} & \text{if } 36 < T \le 39 \end{cases}$$

(a) determine the domain of f and **(b)** find $f(30)$, $f(36)$, and $f(39)$.

In Problems 38–41, use your calculator to find the indicated function values for the given function. Round answers to two decimal places.

38. $f(x) = \begin{cases} 0.19x^4 - 27.99 & \text{if } x \ge 5.99 \\ 0.63x^5 - 57.42 & \text{if } x < 5.99 \end{cases}$

(a) $f(7.98)$ **(b)** $f(2.26)$ **(c)** $f(9)$

39. $f(x) = \begin{cases} 29.5x^4 + 30.4 & \text{if } x < 3 \\ 7.9x^3 - 2.1x & \text{if } x \ge 3 \end{cases}$

(a) $f(2.5)$ **(b)** $f(-3.6)$ **(c)** $f(3.2)$

40. $f(x) = \begin{cases} 4.07x - 2.3 & \text{if } x < -8 \\ 19.12 & \text{if } -8 \le x < -2 \\ x^2 - 4x^{-2} & \text{if } x \ge -2 \end{cases}$

(a) $f(-5.8)$ **(b)** $f(-14.9)$ **(c)** $f(7.6)$

41. $f(x) = \begin{cases} x/(x+3) & \text{if } x < -5 \\ x(x-4)^2 & \text{if } -5 \le x < 0 \\ \sqrt{2.1x+3} & \text{if } x \ge 0 \end{cases}$

(a) $f(-\sqrt{30})$ **(b)** $f(46)$ **(c)** $f(-2/3)$

Objective

To combine functions by means of addition, subtraction, multiplication, division, multiplication by a constant, and composition.

2.3 Combinations of Functions

There are several ways of combining two functions to create a new function. Suppose f and g are the functions given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = 3x$$

Adding $f(x)$ and $g(x)$ gives

$$f(x) + g(x) = x^2 + 3x$$

This operation defines a new function called the *sum* of f and g, denoted $f + g$. Its function value at x is $f(x) + g(x)$. That is,

$$(f + g)(x) = f(x) + g(x) = x^2 + 3x$$

For example,

$$(f + g)(2) = 2^2 + 3(2) = 10$$

In general, for any functions $f, g : X \longrightarrow (-\infty, \infty)$, we define the **sum** $f + g$, the **difference** $f - g$, the **product** fg, and the **quotient** $\frac{f}{g}$ as follows:

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (f-g)(x) &= f(x) - g(x) \\ (fg)(x) &= f(x) \cdot g(x) \\ \frac{f}{g}(x) &= \frac{f(x)}{g(x)} \quad \text{for } g(x) \neq 0 \end{aligned}$$

For each of the four new functions the domain is the set of all x which belong to both the domain of f *and* the domain of g. For the quotient, we further restrict the domain to

[3]Adapted from F. K. E. Imrie and A. J. Vlitos, "Production of Fungal Protein from Carob," in *Single-Cell Protein II*, ed. S. R. Tannenbaum and D. I. C. Wang (Cambridge, MA: MIT Press, 1975).

exclude any value of x for which $g(x) = 0$. In each of the four combinations we have a new function from X to $(-\infty, \infty)$. For example, we have $f + g : X \longrightarrow (-\infty, \infty)$. A special case of fg deserves separate mention. For any real number c and any function f, we define cf by

$$(cf)(x) = c \cdot f(x)$$

This restricted case of product is called **scalar product.** The scalar product tends to share properties with sums (and differences) that are not enjoyed by general products (and quotients).

For $f(x) = x^2$, $g(x) = 3x$, and $c = \sqrt{2}$ we have

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) = x^2 + 3x \\ (f-g)(x) &= f(x) - g(x) = x^2 - 3x \\ (fg)(x) &= f(x) \cdot g(x) = x^2(3x) = 3x^3 \\ \frac{f}{g}(x) &= \frac{f(x)}{g(x)} = \frac{x^2}{3x} = \frac{x}{3} \quad \text{for } x \neq 0 \\ (cf)(x) &= cf(x) = \sqrt{2}x^2\end{aligned}$$

EXAMPLE 1 Combining Functions

If $f(x) = 3x - 1$ and $g(x) = x^2 + 3x$, find

a. $(f + g)(x)$
b. $(f - g)(x)$
c. $(fg)(x)$
d. $\frac{f}{g}(x)$
e. $\left(\frac{1}{2}f\right)(x)$

Solution:

a. $(f+g)(x) = f(x) + g(x) = (3x-1) + (x^2+3x) = x^2 + 6x - 1$
b. $(f-g)(x) = f(x) - g(x) = (3x-1) - (x^2+3x) = -1 - x^2$
c. $(fg)(x) = f(x)g(x) = (3x-1)(x^2+3x) = 3x^3 + 8x^2 - 3x$
d. $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{3x-1}{x^2+3x}$
e. $\left(\frac{1}{2}f\right)(x) = \frac{1}{2}(f(x)) = \frac{1}{2}(3x-1) = \frac{3x-1}{2}$

Now Work Problem 3(a)–(f) ◁

Composition

We can also combine two functions by first applying one function to an input and then applying the other function to the output of the first. For example, suppose $g(x) = 3x$, $f(x) = x^2$, and $x = 2$. Then $g(2) = 3 \cdot 2 = 6$. Thus, g sends the input 2 to the output 6:

$$2 \overset{g}{\mapsto} 6$$

Next, we let the output 6 become the input for f:

$$f(6) = 6^2 = 36$$

So f sends 6 to 36:

$$6 \overset{f}{\mapsto} 36$$

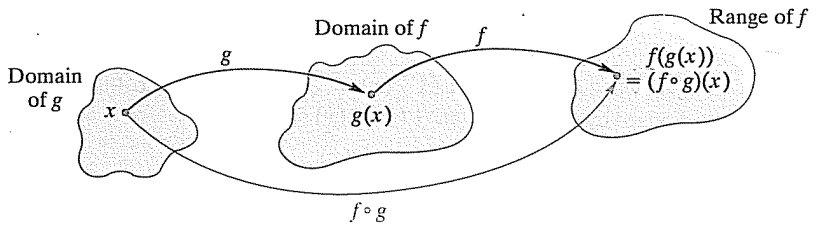


FIGURE 2.2 Composite of f with g.

By first applying g and then f, we send 2 to 36:

$$2 \overset{g}{\mapsto} 6 \overset{f}{\mapsto} 36$$

To be more general, replace the 2 by x, where x is in the domain of g. (See Figure 2.2.) Applying g to x, we get the number $g(x)$, which we will assume is in the domain of f. By applying f to $g(x)$, we get $f(g(x))$, read "f of g of x," which is in the range of f. The operation of applying g and then applying f to the result is called *composition* and the resulting function, denoted $f \circ g$, is called the *composite* of f with g. This function assigns the output $f(g(x))$ to the input x. (See the bottom arrow in Figure 2.2.) Thus, $(f \circ g)(x) = f(g(x))$.

Definition

For functions $g : X \longrightarrow Y$ and $f : Y \longrightarrow Z$, the ***composite of f with g is the function*** $f \circ g : X \longrightarrow Z$ ***defined by***

$$(f \circ g)(x) = f(g(x))$$

where the domain of $f \circ g$ is the set of all those x in the domain of g such that $g(x)$ is in the domain of f.

For $f(x) = x^2$ and $g(x) = 3x$, we can get a simple form for $f \circ g$:

$$(f \circ g)(x) = f(g(x)) = f(3x) = (3x)^2 = 9x^2$$

For example, $(f \circ g)(2) = 9(2)^2 = 36$, as we saw before.

When dealing with real numbers and the operation of addition, 0 is special in that, for any real number a, we have

$$a + 0 = a = 0 + a$$

The number 1 enjoys a similar property with respect to multiplication. For any real number a, we have

$$a1 = a = 1a$$

For reference in Section 2.4, we note that the function I defined by $I(x) = x$, satisfies, for any function f,

$$f \circ I = f = I \circ f$$

where here we mean equality of functions as defined in Section 2.1. Indeed, for any x,

$$(f \circ I)(x) = f(I(x)) = f(x) = I(f(x)) = (I \circ f)(x)$$

The function I is called the *identity* function.

APPLY IT ▶

9. A CD costs x dollars wholesale. The price the store pays is given by the function $s(x) = x + 3$. The price the customer pays is $c(x) = 2x$, where x is the price the store pays. Write a composite function to find the customer's price as a function of the wholesale price.

CAUTION!

Generally, $f \circ g$ and $g \circ f$ are quite different. In Example 2,

$$(f \circ g)(x) = \sqrt{x+1}$$

but we have

$$(g \circ f)(x) = \sqrt{x} + 1$$

Observe that $(f \circ g)(1) = \sqrt{2}$, while $(g \circ f)(1) = 2$. Also, do not confuse $f(g(x))$ with $(fg)(x)$, which is the product $f(x)g(x)$. Here

$$f(g(x)) = \sqrt{x+1}$$

but

$$f(x)g(x) = \sqrt{x}(x+1)$$

EXAMPLE 2 Composition

Let $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Find

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

Solution:

a. $(f \circ g)(x)$ is $f(g(x))$. Now g adds 1 to x, and f takes the square root of the result. Thus,

$$(f \circ g)(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$$

The domain of g is all real numbers x, and the domain of f is all nonnegative reals. Hence, the domain of the composite is all x for which $g(x) = x + 1$ is nonnegative. That is, the domain is all $x \geq -1$, which is the interval $[-1, \infty)$.

b. $(g \circ f)(x)$ is $g(f(x))$. Now f takes the square root of x, and g adds 1 to the result. Thus, g adds 1 to $\sqrt{x}$, and we have

$$(g \circ f)(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$$

The domain of f is all $x \geq 0$, and the domain of g is all reals. Hence, the domain of the composite is all $x \geq 0$ for which $f(x) = \sqrt{x}$ is real, namely, all $x \geq 0$.

Now Work Problem 7 ◁

Composition is *associative*, meaning that for any three functions f, g, and h,

$$(f \circ g) \circ h = f \circ (g \circ h)$$

EXAMPLE 3 Composition

If $F(p) = p^2 + 4p - 3$, $G(p) = 2p + 1$, and $H(p) = |p|$, find

a. $F(G(p))$

b. $F(G(H(p)))$

c. $G(F(1))$

Solution:

a. $F(G(p)) = F(2p+1) = (2p+1)^2 + 4(2p+1) - 3 = 4p^2 + 12p + 2 = (F \circ G)(p)$

b. $F(G(H(p))) = (F \circ (G \circ H))(p) = ((F \circ G) \circ H)(p) = (F \circ G)(H(p)) = (F \circ G)(|p|) = 4|p|^2 + 12|p| + 2 = 4p^2 + 12|p| + 2$

c. $G(F(1)) = G(1^2 + 4 \cdot 1 - 3) = G(2) = 2 \cdot 2 + 1 = 5$

Now Work Problem 9 ◁

In calculus, it is necessary at times to think of a particular function as a composite of two simpler functions, as the next example shows.

APPLY IT ▶

10. Suppose the area of a square garden is $g(x) = (x + 3)^2$. Express g as a composite of two functions, and explain what each function represents.

EXAMPLE 4 Expressing a Function as a Composite

Express $h(x) = (2x - 1)^3$ as a composite.

Solution:

We note that $h(x)$ is obtained by finding $2x - 1$ and cubing the result. Suppose we let $g(x) = 2x - 1$ and $f(x) = x^3$. Then

$$h(x) = (2x-1)^3 = (g(x))^3 = f(g(x)) = (f \circ g)(x)$$

which gives h as a composite of two functions.

Now Work Problem 13 ◁

TECHNOLOGY

Two functions can be combined by using a graphing calculator. Consider the functions

$$f(x) = 2x + 1 \quad \text{and} \quad g(x) = x^2$$

which we enter as Y_1 and Y_2, as shown in Figure 2.3. The sum of f and g is given by $Y_3 = Y_1 + Y_2$ and the composite $f \circ g$ by $Y_4 = Y_1(Y_2)$. For example, $f(g(3))$ is obtained by evaluating Y_4 at 3.

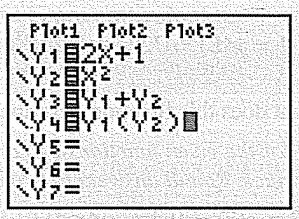


FIGURE 2.3 Y_3 and Y_4 are combinations of Y_1 and Y_2.

PROBLEMS 2.3

1. If $f(x) = x + 3$ and $g(x) = x + 5$, find the following.

(a) $(f+g)(x)$ **(b)** $(f+g)(0)$ **(c)** $(f-g)(x)$

(d) $(fg)(x)$ **(e)** $(fg)(-2)$ **(f)** $\frac{f}{g}(x)$

(g) $(f \circ g)(x)$ **(h)** $(f \circ g)(3)$ **(i)** $(g \circ f)(x)$

(j) $(g \circ f)(3)$

2. If $f(x) = 2x$ and $g(x) = 6 + x$, find the following.

(a) $(f+g)(x)$ **(b)** $(f-g)(x)$ **(c)** $(f-g)(4)$

(d) $(fg)(x)$ **(e)** $\frac{f}{g}(x)$ **(f)** $\frac{f}{g}(2)$

(g) $(f \circ g)(x)$ **(h)** $(g \circ f)(x)$ **(i)** $(g \circ f)(2)$

3. If $f(x) = x^2 - 1$ and $g(x) = x^2 + x$, find the following.

(a) $(f+g)(x)$ **(b)** $(f-g)(x)$ **(c)** $(f-g)\left(-\frac{1}{2}\right)$

(d) $(fg)(x)$ **(e)** $\frac{f}{g}(x)$ **(f)** $\frac{f}{g}\left(-\frac{1}{2}\right)$

(g) $(f \circ g)(x)$ **(h)** $(g \circ f)(x)$ **(i)** $(g \circ f)(-3)$

4. If $f(x) = x^2 + 1$ and $g(x) = 5$, find the following.

(a) $(f+g)(x)$ **(b)** $(f+g)\left(\frac{2}{3}\right)$ **(c)** $(f-g)(x)$

(d) $(fg)(x)$ **(e)** $(fg)(7)$ **(f)** $\frac{f}{g}(x)$

(g) $(f \circ g)(x)$ **(h)** $(f \circ g)(12{,}003)$ **(i)** $(g \circ f)(x)$

5. If $f(x) = 3x^2 + 6$ and $g(x) = 4 - 2x$, find $f(g(2))$ and $g(f(2))$.

6. If $f(p) = \dfrac{4}{p}$ and $g(p) = \dfrac{p-2}{3}$, find both $(f \circ g)(p)$ and $(g \circ f)(p)$.

7. If $F(t) = t^2 + 7t + 1$ and $G(t) = \dfrac{2}{t-1}$, find $(F \circ G)(t)$ and $(G \circ F)(t)$.

8. If $F(t) = \sqrt{t}$ and $G(t) = 2t^2 - 2t + 1$, find $(F \circ G)(t)$ and $(G \circ F)(t)$.

9. If $f(v) = \dfrac{1}{v^2+1}$ and $g(v) = \sqrt{v+2}$, find $(f \circ g)(v)$ and $(g \circ f)(v)$.

10. If $f(x) = x^2 + 2x - 1$, find $(f \circ f)(x)$.

In Problems 11–16, find functions f and g such that $h(x) = f(g(x))$.

11. $h(x) = 11x - 7$

12. $h(x) = \sqrt{x^2 - 2}$

13. $h(x) = \dfrac{3}{x^2 + x + 1}$

14. $h(x) = (9x^3 - 5x)^3 - (9x^3 - 5x)^2 + 11$

15. $h(x) = \sqrt[4]{\dfrac{x^2-1}{x+3}}$

16. $h(x) = \dfrac{2 - (3x-5)}{(3x-5)^2 + 2}$

17. Profit A coffeehouse sells a pound of coffee for \$9.75. Expenses are \$4500 each month, plus \$4.25 for each pound of coffee sold.
(a) Write a function $r(x)$ for the total monthly revenue as a function of the number of pounds of coffee sold.
(b) Write a function $e(x)$ for the total monthly expenses as a function of the number of pounds of coffee sold.
(c) Write a function $(r-e)(x)$ for the total monthly profit as a function of the number of pounds of coffee sold.

18. Geometry Suppose the volume of a sphere is $v(x) = \frac{4}{3}\pi(3x-1)^3$. Express v as a composite of two functions, and explain what each function represents.

19. Business A manufacturer determines that the total number of units of output per day, q, is a function of the number of employees, m, where

$$q = f(m) = \frac{(40m - m^2)}{4}$$

The total revenue r that is received for selling q units is given by the function g, where $r = g(q) = 40q$. Find $(g \circ f)(m)$. What does this composite function describe?

20. Sociology Studies have been conducted concerning the statistical relations among a person's status, education, and income.[4] Let S denote a numerical value of status based on annual income I. For a certain population, suppose

$$S = f(I) = 0.45(I - 1000)^{0.53}$$

[4] R. K. Leik and B. F. Meeker, *Mathematical Sociology* (Englewood Cliffs, NJ: Prentice Hall, 1975).

Furthermore, suppose a person's income I is a function of the number of years of education E, where

$$I = g(E) = 7202 + 0.29E^{3.68}$$

Find $(f \circ g)(E)$. What does this function describe?

In Problems 21–24, for the given functions f and g, find the indicated function values. Round answers to two decimal places.

21. $f(x) = (4x - 13)^2$, $g(x) = 0.2x^2 - 4x + 3$

(a) $(f + g)(4.5)$, **(b)** $(f \circ g)(-2)$

22. $f(x) = \sqrt{\dfrac{x-3}{x+1}}$, $g(x) = 11.2x + 5.39$

(a) $\dfrac{f}{g}(-2)$, **(b)** $(g \circ f)(-10)$

23. $f(x) = x^{4/5}$, $g(x) = x^2 - 8$

(a) $(fg)(7)$, **(b)** $(g \circ f)(3.75)$

24. $f(x) = \dfrac{5}{x+3}$, $g(x) = \dfrac{2}{x^2}$

(a) $(f - g)(7.3)$, **(b)** $(f \circ g)(-4.17)$

Objective

To introduce inverse functions, their properties, and their uses.

2.4 Inverse Functions

Just as $-a$ is the number for which

$$a + (-a) = 0 = (-a) + a$$

and, for $a \neq 0$, a^{-1} is the number for which

$$aa^{-1} = 1 = a^{-1}a$$

so, given a function $f : X \longrightarrow Y$, we can inquire about the existence of a function g satisfying

$$f \circ g = I = g \circ f \tag{1}$$

where I is the identity function, introduced in the subsection titled "Composition" of Section 2.3 and given by $I(x) = x$. Suppose that we have g as above and a function h that also satisfies the equations of (1) so that

$$f \circ h = I = h \circ f$$

Then

$$h = h \circ I = h \circ (f \circ g) = (h \circ f) \circ g = I \circ g = g$$

shows that there is at most one function satisfying the requirements of g in (1). In mathematical jargon, g is uniquely determined by f and is therefore given a name, $g = f^{-1}$, that reflects its dependence on f. The function f^{-1} is read as f **inverse** and called the **inverse** of f.

The additive inverse $-a$ exists for any number a; the multiplicative inverse a^{-1} exists precisely if $a \neq 0$. The existence of f^{-1} places a strong requirement on a function f. It can be shown that f^{-1} exists if and only if, for all a and b, whenever $f(a) = f(b)$, then $a = b$. It may be helpful to think that such an f can be *canceled (on the left)*.

A function f that satisfies

$$\text{for all } a \text{ and } b, \text{ if } \quad f(a) = f(b) \quad \text{then} \quad a = b$$

is called a **one-to-one** function.

Thus, we can say that a function has an inverse precisely if it is one-to-one. An equivalent way to express the one-to-one condition is

$$\text{for all } a \text{ and } b, \text{ if } \quad a \neq b \quad \text{then} \quad f(a) \neq f(b)$$

so that distinct inputs give rise to distinct outputs. Observe that this condition is not met for many simple functions. For example, if $f(x) = x^2$, then $f(-1) = (-1)^2 = 1 = (1)^2 = f(1)$ and $-1 \neq 1$ shows that the squaring function is not one-to-one. Similarly, $f(x) = |x|$ is not one-to-one.

CAUTION!

Do not confuse f^{-1}, the inverse of f, and $\dfrac{1}{f}$, the multiplicative reciprocal of f. Unfortunately, the nomenclature for inverse functions clashes with the numerical use of $(-)^{-1}$. Usually, $f^{-1}(x)$ is different from $\dfrac{1}{f}(x) = \dfrac{1}{f(x)}$. For example, $I^{-1} = I$ (since $I \circ I = I$) so $I^{-1}(x) = x$, but $\dfrac{1}{I}(x) = \dfrac{1}{I(x)} = \dfrac{1}{x}$.

In general, the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

Let us note here that the equations of (1) are equivalent to

$$f^{-1}(f(x)) = x \quad \text{for all } x \text{ in the domain of } f \tag{2}$$

and

$$f(f^{-1}(y)) = y \quad \text{for all } y \text{ in the range of } f \tag{3}$$

In general, the range of f, which is equal to the domain of f^{-1}, can be quite different from the domain of f.

EXAMPLE 1 Inverses of Linear Functions

According to Section 2.2, a function of the form $f(x) = ax + b$, where $a \neq 0$, is a linear function. Show that a linear function is one-to-one. Find the inverse of $f(x) = ax + b$ and show that it is also linear.

Solution: Assume that $f(u) = f(v)$; that is,

$$au + b = av + b \tag{4}$$

To show that f is one-to-one, we must show that $u = v$ follows from this assumption. Subtracting b from both sides of (4) gives $au = av$, from which $u = v$ follows by dividing both sides by a. (We assumed that $a \neq 0$.) Since f is given by first multiplying by a and then adding b, we might expect that the effect of f can be undone by first subtracting b and then dividing by a. So consider $g(x) = \dfrac{x-b}{a}$. We have

$$(f \circ g)(x) = f(g(x)) = a\frac{x-b}{a} + b = (x-b) + b = x$$

and

$$(g \circ f)(x) = g(f(x)) = \frac{(ax+b)-b}{a} = \frac{ax}{a} = x$$

Since g satisfies the two requirements of (1), it follows that g is the inverse of f. That is, $f^{-1}(x) = \dfrac{x-b}{a} = \dfrac{1}{a}x + \dfrac{-b}{a}$ and the last equality shows that f^{-1} is also a linear function.

Now Work Problem 1 ◁

EXAMPLE 2 Identities for Inverses

Show that

a. If f and g are one-to-one functions, the composite $f \circ g$ is also one-to-one and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

b. If f is one-to-one, then $(f^{-1})^{-1} = f$.

Solution:

a. Assume $(f \circ g)(a) = (f \circ g)(b)$; that is, $f(g(a)) = f(g(b))$. Since f is one-to-one, $g(a) = g(b)$. Since g is one-to-one, $a = b$ and this shows that $f \circ g$ is one-to-one. The equations

$$(f \circ g) \circ (g^{-1} \circ f^{-1}) = f \circ (g \circ g^{-1}) \circ f^{-1} = f \circ I \circ f^{-1} = f \circ f^{-1} = I$$

and

$$(g^{-1} \circ f^{-1}) \circ (f \circ g) = g^{-1} \circ (f^{-1} \circ f) \circ g = g^{-1} \circ I \circ g = g^{-1} \circ g = I$$

show that $g^{-1} \circ f^{-1}$ is the inverse of $f \circ g$, which, in symbols, is the statement $g^{-1} \circ f^{-1} = (f \circ g)^{-1}$.

b. In Equations (2) and (3), replace f by f^{-1}. Taking g to be f shows that Equations (1) are satisfied, and this gives $(f^{-1})^{-1} = f$.

◁

EXAMPLE 3 Inverses Used to Solve Equations

Many equations take the form $f(x) = 0$, where f is a function. If f is a one-to-one function, then the equation has $x = f^{-1}(0)$ as its unique solution.

Solution: Applying f^{-1} to both sides of $f(x) = 0$ gives $f^{-1}(f(x)) = f^{-1}(0)$, and $f^{-1}(f(x)) = x$ shows that $x = f^{-1}(0)$ is the only possible solution. Since $f(f^{-1}(0)) = 0$, $f^{-1}(0)$ is indeed a solution.

◁

EXAMPLE 4 Restricting the Domain of a Function

It may happen that a function f whose domain is the natural one, consisting of all elements for which the defining rule makes sense, is not one-to-one, and yet a one-to-one function g can be obtained by restricting the domain of f.

Solution: For example, we have shown that the function $f(x) = x^2$ is not one-to-one but the function $g(x) = x^2$ *with domain explicitly given as* $[0, \infty)$ is one-to-one. Since $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$, for $x \geq 0$, it follows that $\sqrt{\ }$ is the inverse of the restricted squaring function g. Here is a more contrived example. Let $f(x) = |x|$ (with its natural domain). Let $g(x) = |x|$ *with domain explicitly given as* $(-\infty, -1) \cup [0, 1]$. The function g is one-to-one and hence has an inverse.

◁

EXAMPLE 5 Finding the Inverse of a Function

To find the inverse of a one-to-one function f, solve the equation $y = f(x)$ for x in terms of y obtaining $x = g(y)$. Then $f^{-1}(x) = g(x)$. To illustrate, find $f^{-1}(x)$ if $f(x) = (x-1)^2$, for $x \geq 1$.

Solution: Let $y = (x-1)^2$, for $x \geq 1$. Then $x - 1 = \sqrt{y}$ and hence $x = \sqrt{y} + 1$. It follows that $f^{-1}(x) = \sqrt{x} + 1$.

Now Work Problem 5 ◁

PROBLEMS 2.4

In Problems 1–6, find the inverse of the given function.

1. $f(x) = 3x + 7$
2. $g(x) = 5x - 3$
3. $F(x) = \frac{1}{2}x - 7$
4. $f(x) = (4x - 5)^2$, for $x \geq \frac{5}{4}$
5. $A(r) = \pi r^2$, for $r \geq 0$
6. $V(r) = \frac{4}{3}\pi r^3$

In Problems 7–10, determine whether or not the function is one-to-one.

7. $f(x) = 5x + 12$
8. $g(x) = (3x + 4)^2$
9. $h(x) = (5x + 12)^2$, for $x \geq -\frac{12}{5}$
10. $F(x) = |x - 9|$

In Problems 11 and 12, solve each equation by finding an inverse function.

11. $(4x - 5)^2 = 23$, for $x \geq \frac{5}{4}$
12. $2x^3 + 1 = 129$

13. **Demand Function** The function

$$p = p(q) = \frac{1{,}200{,}000}{q} \qquad q > 0$$

expresses an actor's charge per film p as a function of the number of films q that she stars in. Express the number of films in which she stars in terms of her charge per film. Show that the expression is a function of p. Show that the resulting function is inverse to the function giving p in terms of q.

14. **Supply Function** The weekly supply function for a pound of house-blend coffee at a coffee shop is

$$p = p(q) = \frac{q}{48} \qquad q > 0$$

where q is the number of pounds of coffee supplied per week and p is the price per pound. Express q as a function of p and demonstrate the relationship between the two functions.

15. Does the function $f(x) = 2^x$ have an inverse?

Objective

To graph equations and functions in rectangular coordinates, to determine intercepts, to apply the vertical-line test and the horizontal-line test, and to determine the domain and range of a function from a graph.

2.5 Graphs in Rectangular Coordinates

A **rectangular coordinate system** allows us to specify and locate points in a plane. It also provides a geometric way to graph equations in two variables, in particular those arising from functions.

In a plane, two real-number lines, called *coordinate axes*, are constructed perpendicular to each other so that their origins coincide, as in Figure 2.4. Their point of intersection is called the *origin* of the coordinate system. For now, we will call the horizontal line the *x-axis* and the vertical line the *y-axis*. The unit distance on the x-axis need not be the same as on the y-axis.

The plane on which the coordinate axes are placed is called a *rectangular coordinate plane* or, more simply, an *x,y-plane*. Every point in the x,y-plane can be labeled to indicate its position. To label point P in Figure 2.5(a), we draw perpendiculars from P to the x-axis and y-axis. They meet these axes at 4 and 2, respectively. Thus, P determines two numbers, 4 and 2. We say that the **rectangular coordinates** of P are given by the **ordered pair** (4, 2). As we remarked in Section 2.1, the word *ordered* is important. In the terminology of Section 2.1, we are labeling the points of the plane by the elements of the set $(-\infty, \infty) \times (-\infty, \infty)$. In Figure 2.5(b), the point corresponding to (4, 2) is not the same as that corresponding to (2, 4):

$$(4, 2) \neq (2, 4)$$

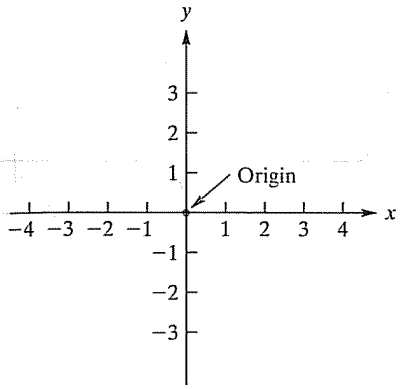


FIGURE 2.4 Coordinate axes.

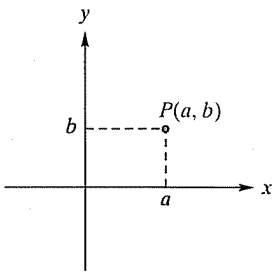


FIGURE 2.6 Coordinates of P.

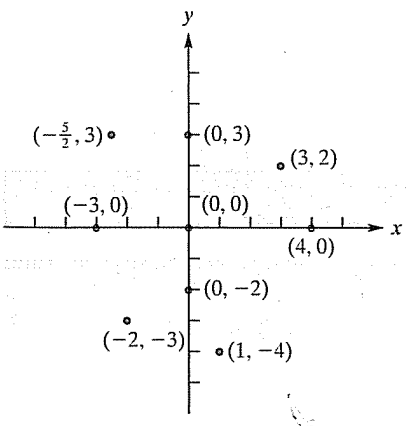


FIGURE 2.7 Coordinates of points.

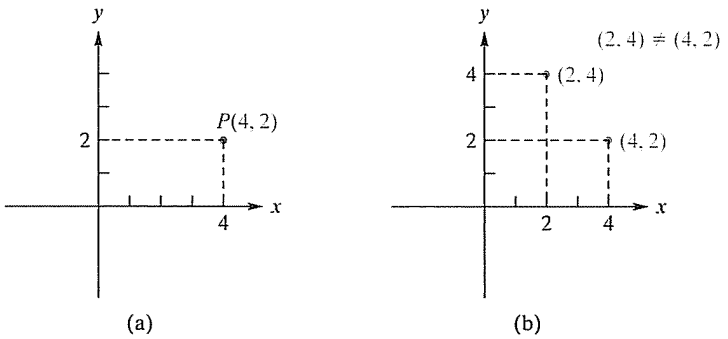


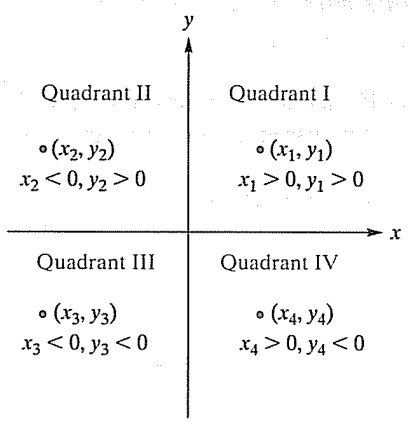
FIGURE 2.5 Rectangular coordinates.

In general, if P is any point, then its rectangular coordinates will be given by an ordered pair of the form (a, b). (See Figure 2.6.) We call a the *x-coordinate* of P, and b the *y-coordinate* of P. We accept that the notation for an ordered pair of real numbers is the same as that for an open interval but the practice is strongly entrenched and almost never causes any confusion.

Accordingly, with each point in a given coordinate plane, we can associate exactly one ordered pair (a, b) of real numbers. Also, it should be clear that with each ordered pair (a, b) of real numbers, we can associate exactly one point in that plane. Since there is a *one-to-one correspondence* between the points in the plane and all ordered pairs of real numbers, we refer to a point P with x-coordinate a and y-coordinate b simply as the point (a, b), or as $P(a, b)$. Moreover, we use the words *point* and *ordered pair of real numbers* interchangeably.

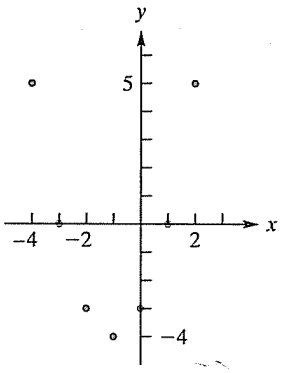
In Figure 2.7, the coordinates of various points are indicated. For example, the point $(1, -4)$ is located one unit to the right of the y-axis and four units below the x-axis. The origin is (0, 0). The x-coordinate of every point on the y-axis is 0, and the y-coordinate of every point on the x-axis is 0.

The coordinate axes divide the plane into four regions called *quadrants* (Figure 2.8). For example, quadrant I consists of all points (x_1, y_1) with $x_1 > 0$ and $y_1 > 0$. The points on the axes do not lie in any quadrant.

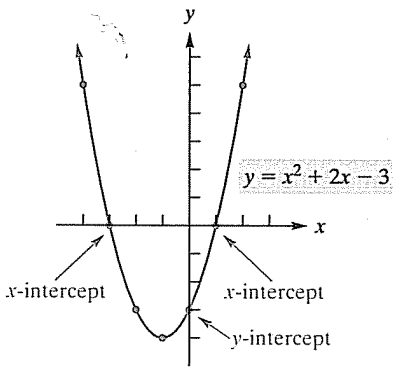


x	y
−4	5
−3	0
−2	−3
−1	−4
0	−3
1	0
2	5

(a)



(b)



(c)

FIGURE 2.8 Quadrants.

FIGURE 2.9 Graphing $y = x^2 + 2x - 3$.

Using a rectangular coordinate system, we can geometrically represent equations in two variables. For example, let us consider

$$y = x^2 + 2x - 3 \tag{1}$$

A solution of this equation is a value of x and a value of y that make the equation true. For example, if $x = 1$, substituting into Equation (1) gives

$$y = 1^2 + 2(1) - 3 = 0$$

Thus, $x = 1, y = 0$ is a solution of Equation (1). Similarly,

$$\text{if } x = -2 \quad \text{then} \quad y = (-2)^2 + 2(-2) - 3 = -3$$

and so $x = -2, y = -3$ is also a solution. By choosing other values for x, we can get more solutions. [See Figure 2.9(a).] It should be clear that there are infinitely many solutions of Equation (1).

Each solution gives rise to a point (x, y). For example, to $x = 1$ and $y = 0$ corresponds $(1, 0)$. The **graph** of $y = x^2 + 2x - 3$ is the geometric representation of all its solutions. In Figure 2.9(b), we have plotted the points corresponding to the solutions in the table.

Since the equation has infinitely many solutions, it seems impossible to determine its graph precisely. However, we are concerned only with the graph's general shape. For this reason, we plot enough points so that we can intelligently guess its proper shape. (The calculus techniques to be studied in Chapter 13 will make such "guesses" much more intelligent.) Then we join these points by a smooth curve wherever conditions permit. This gives the curve in Figure 2.9(c). Of course, the more points we plot, the better our graph is. Here we assume that the graph extends indefinitely upward, as indicated by arrows.

Often, we simply say that the y-intercept is -3 and the x-intercepts are -3 and 1.

The point $(0, -3)$ where the curve intersects the y-axis is called the *y-intercept*. The points $(-3, 0)$ and $(1, 0)$ where the curve intersects the x-axis are called the *x-intercepts*. In general, we have the following definition.

Definition

An ***x-intercept*** of the graph of an equation in x and y is a point where the graph intersects the x-axis. A ***y-intercept*** is a point where the graph intersects the y-axis.

To find the x-intercepts of the graph of an equation in x and y, we first set $y = 0$ and then solve the resulting equation for x. To find the y-intercepts, we first set $x = 0$ and then solve for y. For example, let us find the x-intercepts for the graph of $y = x^2 + 2x - 3$. Setting $y = 0$ and solving for x gives

$$0 = x^2 + 2x - 3$$
$$0 = (x + 3)(x - 1)$$
$$x = -3, \ 1$$

Thus, the x-intercepts are $(-3, 0)$ and $(1, 0)$, as we saw before. If $x = 0$, then

$$y = 0^2 + 2(0) - 3 = -3$$

So $(0, -3)$ is the y-intercept. Keep in mind that an x-intercept has its y-coordinate 0, and a y-intercept has its x-coordinate 0. Intercepts are useful because they indicate precisely where the graph intersects the axes.

EXAMPLE 1 **Intercepts and Graph**

APPLY IT ▷

11. Rachel has saved \$7250 for college expenses. She plans to spend \$600 a month from this account. Write an equation to represent the situation, and identify the intercepts.

Find the x- and y-intercepts of the graph of $y = 2x + 3$, and sketch the graph.

Solution: If $y = 0$, then

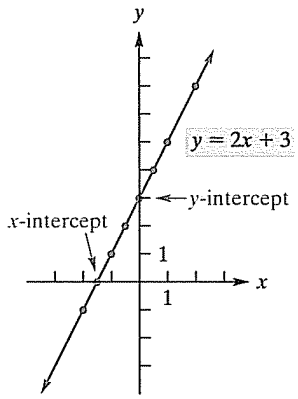
$$0 = 2x + 3 \quad \text{so that} \quad x = -\frac{3}{2}$$

Thus, the x-intercept is $(-\frac{3}{2}, 0)$. If $x = 0$, then

$$y = 2(0) + 3 = 3$$

So the y-intercept is $(0, 3)$. Figure 2.10 shows a table of some points on the graph and a sketch of the graph.

Now Work Problem 9 ◁



x	0	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	−1	2	−2
y	3	0	4	2	5	1	7	−1

FIGURE 2.10 Graph of $y = 2x + 3$.

EXAMPLE 2 **Intercepts and Graph**

APPLY IT ▷

12. The price of admission to an amusement park is \$24.95. This fee allows the customer to ride all the rides at the park as often as he or she likes. Write an equation that represents the relationship between the number of rides, x, that a customer takes and the cost per ride, y, to that customer. Describe the graph of this equation, and identify the intercepts. Assume $x > 0$.

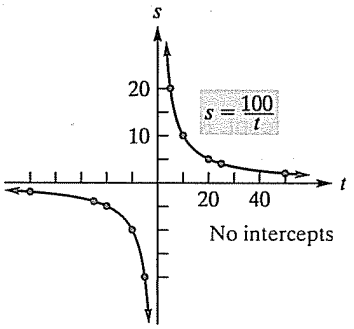
Determine the intercepts, if any, of the graph of $s = \dfrac{100}{t}$, and sketch the graph.

Solution: For the graph, we will label the horizontal axis t and the vertical axis s (Figure 2.11). Because t cannot equal 0 (division by 0 is not defined), there is no s-intercept. Thus, the graph has no point corresponding to $t = 0$. Moreover, there is no t-intercept, because if $s = 0$, then the equation

$$0 = \frac{100}{t}$$

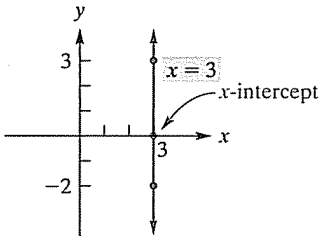
has no solution. Remember, the only way that a fraction can be 0 is by having its numerator 0. Figure 2.11 shows the graph. In general, the graph of $s = k/t$, where k is a nonzero constant, is called a *rectangular hyperbola.*

Now Work Problem 11 ◁



t	5	−5	10	−10	20	−20	25	−25	50	−50
s	20	−20	10	−10	5	−5	4	−4	2	−2

FIGURE 2.11 Graph of $s = \dfrac{100}{t}$.



x	3	3	3
y	0	3	−2

FIGURE 2.12 Graph of $x = 3$.

EXAMPLE 3 Intercepts and Graph

Determine the intercepts of the graph of $x = 3$, and sketch the graph.

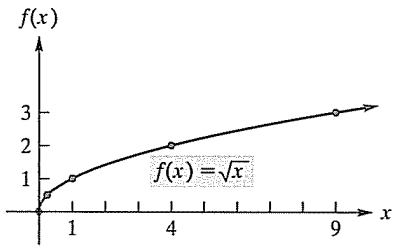
Solution: We can think of $x = 3$ as an equation in the variables x and y if we write it as $x = 3 + 0y$. Here y can be any value, but x must be 3. Because $x = 3$ when $y = 0$, the x-intercept is $(3, 0)$. There is no y-intercept, because x cannot be 0. (See Figure 2.12.) The graph is a vertical line.

Now Work Problem 13 ◁

Each function f gives rise to an equation, namely $y = f(x)$, which is a special case of the equations we have been graphing. Its **graph** consists of all points $(x, f(x))$, where x is in the domain of f. The vertical axis can be labeled either y or $f(x)$, where f is the name of the function, and is referred to as the **function-value axis**. *We always label the horizontal axis with the independent variable but note that economists label the vertical axis with the independent variable.* Observe that in graphing a function the "solutions" (x, y) that make the equation $y = f(x)$ true are handed to us. For each x in the domain of f, we have exactly one y obtained by evaluating $f(x)$. The resulting pair $(x, f(x))$ is a point on the graph and these are the only points on the graph of the equation $y = f(x)$.

The x-intercepts of the graph of a real-valued function f are all those real numbers x for which $f(x) = 0$. As such they are also known as **roots** of the equation $f(x) = 0$ and still further as **zeros** of the function f.

A useful geometric observation is that the graph of a function has at most one point of intersection with any vertical line in the plane. Recall that the equation of a vertical line is necessarily of the form $x = a$, where a is a constant. If a is not in the domain of the function f, then $x = a$ will not intersect the graph of $y = f(x)$. If a is in the domain of the function f, then $x = a$ will intersect the graph of $y = f(x)$ at the point $(a, f(a))$ and only there. Conversely, if a set of points in the plane has the property that any vertical line intersects the set at most once, then the set of points is actually the graph of a function. (The domain of the function is the set of all real numbers a with the property that the line $x = a$ does intersect the given set of points and for such an a the corresponding function value is the y-coordinate of the unique point of intersection of the line $x = a$ and the given set of points.) This is the basis of the **vertical-line test** that we will discuss after Example 7.



x	0	$\frac{1}{4}$	1	4	9
$f(x)$	0	$\frac{1}{2}$	1	2	3

FIGURE 2.13 Graph of $f(x) = \sqrt{x}$.

EXAMPLE 4 Graph of the Square-Root Function

Graph the function $f\colon (-\infty, \infty) \longrightarrow (-\infty, \infty)$ given by $f(x) = \sqrt{x}$.

Solution: The graph is shown in Figure 2.13. We label the vertical axis as $f(x)$. Recall that $\sqrt{x}$ denotes the *principal* square root of x. Thus, $f(9) = \sqrt{9} = 3$, not ± 3. Also, the domain of f is $[0, \infty)$ because its values are declared to be real numbers. Let us now

consider intercepts. If $f(x) = 0$, then $\sqrt{x} = 0$, so that $x = 0$. Also, if $x = 0$, then $f(x) = 0$. Thus, the x-intercept and the vertical-axis intercept are the same, namely, $(0, 0)$.

Now Work Problem 29 ◁

EXAMPLE 5 **Graph of the Absolute-Value Function**

APPLY IT ▷

13. Brett rented a bike from a rental shop, rode at a constant rate of 12 mi/h for 2.5 hours along a bike path, and then returned along the same path. Graph the absolute-value-like function that represents Brett's distance from the rental shop as a function of time over the appropriate domain.

Graph $p = G(q) = |q|$.

Solution: We use the independent variable q to label the horizontal axis. The function-value axis can be labeled either $G(q)$ or p. (See Figure 2.14.) Notice that the q- and p-intercepts are the same point, $(0, 0)$.

Now Work Problem 31 ◁

q	0	1	−1	3	−3	5	−5
p	0	1	1	3	3	5	5

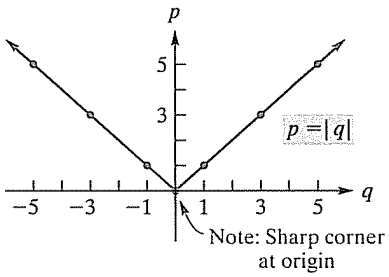


FIGURE 2.14 Graph of $p = |q|$.

TECHNOLOGY

To solve the equation $x^3 = 3x - 1$ with a graphing calculator, we first express the equation in the form $f(x) = 0$:

$$f(x) = x^3 - 3x + 1 = 0$$

Next we graph f and then estimate the x-intercepts, either by using zoom and trace or by using the root operation. (See Figure 2.15.) Note that we defined our window for $-4 \le x \le 4$ and $-5 \le y \le 5$.

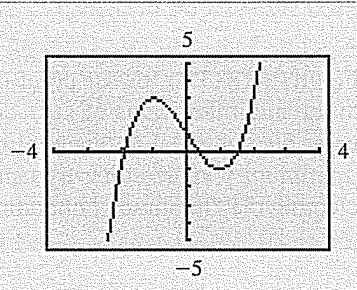


FIGURE 2.15 The roots of $x^3 - 3x + 1 = 0$ are approximately -1.88, 0.35, and 1.53.

Figure 2.16 shows the graph of a function $y = f(x)$. The point $(x, f(x))$ tells us that corresponding to the input number x on the horizontal axis is the output number $f(x)$ on the vertical axis, as indicated by the arrow. For example, corresponding to the input 4 is the output 3, so $f(4) = 3$.

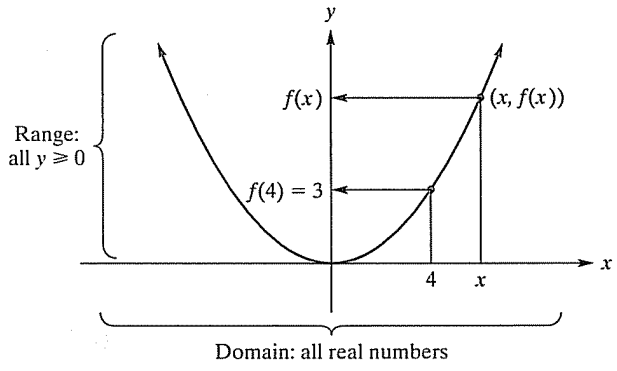


FIGURE 2.16 Domain, range, function values.

From the shape of the graph, it seems reasonable to assume that, for any value of x, there is an output number, so the domain of f is all real numbers. Notice that the set of all y-coordinates of points on the graph is the set of all nonnegative numbers. Thus, the range of f is all $y \geq 0$. This shows that we can make an "educated" guess about the domain and range of a function by looking at its graph. *In general, the domain consists of all x-values that are included in the graph, and the range is all y-values that are included.* For example, Figure 2.13 tells us that both the domain and range of $f(x) = \sqrt{x}$ are all nonnegative numbers. From Figure 2.14, it is clear that the domain of $p = G(q) = |q|$ is all real numbers and the range is all $p \geq 0$.

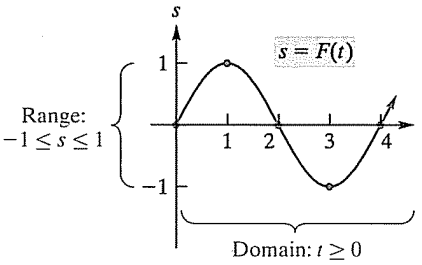


FIGURE 2.17 Domain, range, function values.

EXAMPLE 6 **Domain, Range, and Function Values**

Figure 2.17 shows the graph of a function F. To the right of 4, assume that the graph repeats itself indefinitely. Then the domain of F is all $t \geq 0$. The range is $-1 \leq s \leq 1$. Some function values are

$$F(0) = 0 \quad F(1) = 1 \quad F(2) = 0 \quad F(3) = -1$$

Now Work Problem 5 ◁

TECHNOLOGY

Using a graphing calculator, we can estimate the range of a function. The graph of

$$f(x) = 6x^4 - 8.1x^3 + 1$$

is shown in Figure 2.18. The lowest point on the graph corresponds to the minimum value of $f(x)$, and the range is all reals greater than or equal to this minimum. We can estimate this minimum y-value either by using trace and zoom or by selecting the "minimum" operation.

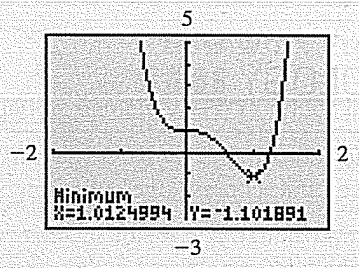


FIGURE 2.18 The range of $f(x) = 6x^4 - 8.1x^3 + 1$ is approximately $[-1.10, \infty)$.

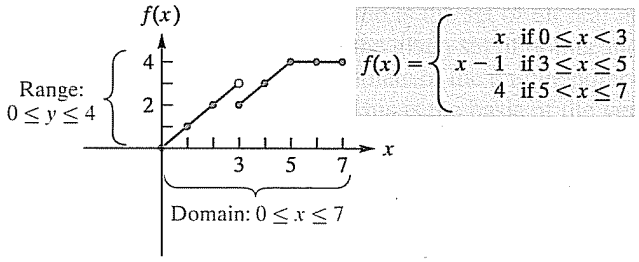
APPLY IT ▷

14. To encourage conservation, a gas company charges two rates. You pay \$0.53 per therm for 0–70 therms and \$0.74 for each therm over 70. Graph the case-defined function that represents the monthly cost of t therms of gas.

EXAMPLE 7 **Graph of a Case-Defined Function**

Graph the case-defined function

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < 3 \\ x - 1 & \text{if } 3 \leq x \leq 5 \\ 4 & \text{if } 5 < x \leq 7 \end{cases}$$



x	0	1	2	3	4	5	6	7
$f(x)$	0	1	2	2	3	4	4	4

FIGURE 2.19 Graph of case-defined function.

Solution: The domain of f is $0 \le x \le 7$. The graph is given in Figure 2.19, where the *hollow dot* means that the point is *not* included in the graph. Notice that the range of f is all real numbers y such that $0 \le y \le 4$.

Now Work Problem 35 ◁

There is an easy way to tell whether a curve is the graph of a function. In Figure 2.20(a), notice that with the given x there are associated *two* values of y: y_1 and y_2. Thus, the curve is *not* the graph of a function of x. Looking at it another way, we have the following general rule, called the **vertical-line test.** If a *vertical* line L can be drawn that intersects a curve in at least two points, then the curve is *not* the graph of a function of x. When no such vertical line can be drawn, the curve *is* the graph of a function of x. Consequently, the curves in Figure 2.20 do not represent functions of x, but those in Figure 2.21 do.

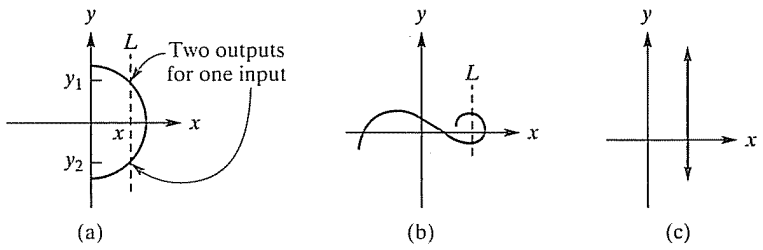


FIGURE 2.20 y is not a function of x.

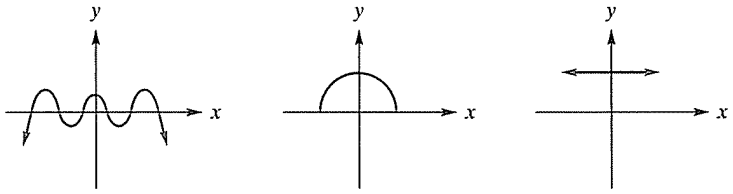


FIGURE 2.21 Functions of x.

EXAMPLE 8 **A Graph That Does Not Represent a Function of x**

Graph $x = 2y^2$.

Solution: Here it is easier to choose values of y and then find the corresponding values of x. Figure 2.22 shows the graph. By the vertical-line test, the equation $x = 2y^2$ does not define a function of x.

Now Work Problem 39 ◁

After we have determined whether a curve is the graph of a function, perhaps using the vertical-line test, there is an easy way to tell whether the function in question is one-to-one. In Figure 2.16 we see that $f(4) = 3$ and, apparently, also $f(-4) = 3$.

x	0	2	2	8	8	18	18
y	0	1	−1	2	−2	3	−3

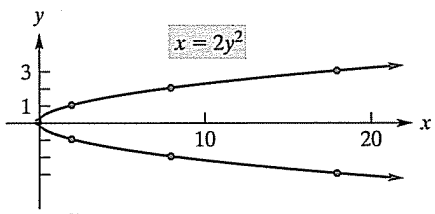


FIGURE 2.22 Graph of $x = 2y^2$.

Since the distinct input values -4 and 4 produce the same output, the function is not one-to-one. Looking at it another way, we have the following general rule, called the **horizontal-line test**. If a *horizontal* line L can be drawn that intersects the graph of a function in at least two points, then the function is *not* one-to-one. When no such horizontal line can be drawn, the function is one-to-one.

PROBLEMS 2.5

In Problems 1 and 2, locate and label each of the points, and give the quadrant, if possible, in which each point lies.

1. $(-2,-5), (3,-1), \left(-\frac{1}{3}, 4\right), (1,0)$
2. $(-4,5), (3,0), (1,1), (0,-6)$
3. Figure 2.23(a) shows the graph of $y = f(x)$.
   (a) Estimate $f(0), f(2), f(4)$, and $f(-2)$.
   (b) What is the domain of f?
   (c) What is the range of f?
   (d) What is an x-intercept of f?
4. Figure 2.23(b) shows the graph of $y = f(x)$.
   (a) Estimate $f(0)$ and $f(2)$.
   (b) What is the domain of f?
   (c) What is the range of f?
   (d) What is an x-intercept of f?

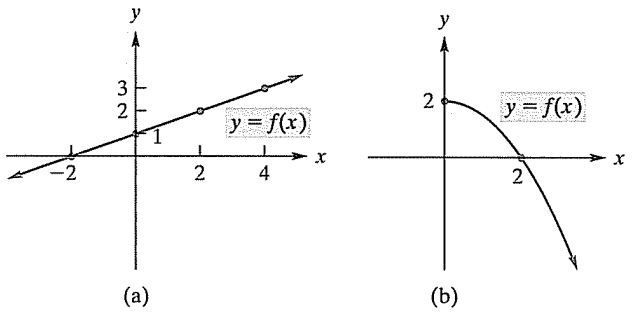


FIGURE 2.23 Diagram for Problems 3 and 4.

5. Figure 2.24(a) shows the graph of $y = f(x)$.
   (a) Estimate $f(0), f(1)$, and $f(-1)$.
   (b) What is the domain of f?
   (c) What is the range of f?
   (d) What is an x-intercept of f?
6. Figure 2.24(b) shows the graph of $y = f(x)$.
   (a) Estimate $f(0), f(2), f(3)$, and $f(4)$.
   (b) What is the domain of f?
   (c) What is the range of f?
   (d) What is an x-intercept f?

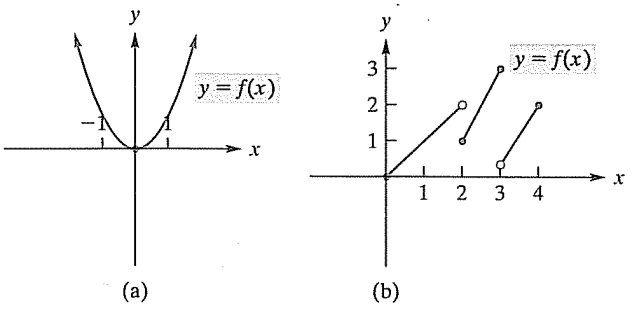


FIGURE 2.24 Diagram for Problems 5 and 6.

In Problems 7–20, determine the intercepts of the graph of each equation, and sketch the graph. Based on your graph, is y a function of x, and, if so, is it one-to-one and what are the domain and range?

7. $y = 2x$
8. $y = x + 1$
9. $y = 3x - 5$
10. $y = 3 - 2x$
11. $y = x^3 + x$
12. $y = \frac{2}{x^2}$
13. $x = 0$
14. $y = 4x^2 - 16$
15. $y = x^3$
16. $x = 3$
17. $x = -|y|$
18. $x^2 = y^2$
19. $2x + y - 2 = 0$
20. $x + y = 1$

In Problems 21–34, graph each function and give the domain and range. Also, determine the intercepts.

21. $u = f(v) = 2 + v^2$
22. $f(x) = 5 - 2x^2$
23. $y = h(x) = 3$
24. $g(s) = -17$
25. $y = h(x) = x^2 - 4x + 1$
26. $y = f(x) = -x^2 + x + 6$
27. $f(t) = -t^3$
28. $p = h(q) = 1 + 2q + q^2$
29. $s = f(t) = \sqrt{t^2 - 9}$
30. $F(r) = -\frac{1}{r}$
31. $f(x) = |3x + 2|$
32. $v = H(u) = |u - 3|$
33. $F(t) = \frac{16}{t^2}$
34. $y = f(x) = \frac{2}{x - 4}$

In Problems 35–38, graph each case-defined function and give the domain and range.

35. $c = g(p) = \begin{cases} p + 1 & \text{if } 0 \le p < 7 \\ 5 & \text{if } p \ge 7 \end{cases}$

36. $\gamma(x) = \begin{cases} 3x & \text{if } 0 \le x < 2 \\ 10 - x^2 & \text{if } x \ge 2 \end{cases}$

37. $g(x) = \begin{cases} x + 6 & \text{if } x \ge 3 \\ x^2 & \text{if } x < 3 \end{cases}$

38. $f(x) = \begin{cases} x + 1 & \text{if } 0 < x \le 3 \\ 4 & \text{if } 3 < x \le 5 \\ x - 1 & \text{if } x > 5 \end{cases}$

39. Which of the graphs in Figure 2.25 represent functions of x?

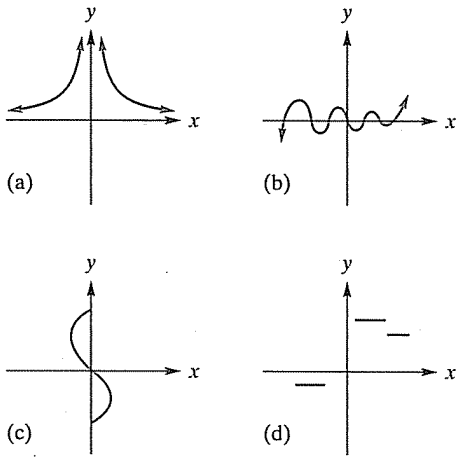


FIGURE 2.25 Diagram for Problem 39.

40. Which of the graphs in Figure 2.26 represent one-to-one functions of x?

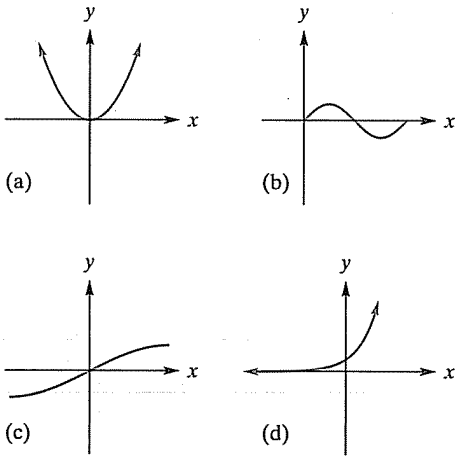


FIGURE 2.26 Diagram for Problem 40.

41. Debt Payments Allison has charged \$9200 on her credit cards. She plans to pay them off at the rate of \$325 per month. Write an equation to represent the amount she owes, excluding any finance charges, after she has made x payments and identify the intercepts, explaining their financial significance if any.

42. Pricing To encourage an even flow of customers, a restaurant varies the price of an item throughout the day. From 6:00 P.M. to 8:00 P.M., customers pay full price. At lunch, from 10:30 A.M. until 2:30 P.M., customers pay half price. From 2:30 P.M. until 4:30 P.M., customers get a dollar off the lunch price. From 4:30 P.M. until 6:00 P.M., customers get \$5.00 off the dinner price. From 8:00 P.M. until closing time at 10:00 P.M., customers get \$5.00 off the dinner price. Graph the case-defined function that represents the cost of an item throughout the day for a dinner price of \$18.

43. Supply Schedule Given the following supply schedule (see Example 6 of Section 2.1), plot each quantity–price pair by choosing the horizontal axis for the possible quantities. Approximate the points in between the data by connecting the data points with a smooth curve. The result is a *supply curve*. From the graph, determine the relationship between price and supply. (That is, as price increases, what happens to the quantity supplied?) Is price per unit a function of quantity supplied?

Quantity Supplied per Week, q	Price per Unit, p
30	\$10
100	20
150	30
190	40
210	50

44. Demand Schedule The following table is called a *demand schedule*. It indicates the quantities of brand X that consumers will demand (that is, purchase) each week at certain prices per unit (in dollars). Plot each quantity–price pair by choosing the vertical axis for the possible prices. Connect the points with a smooth curve. In this way, we approximate points in between the given data. The result is called a *demand curve*. From the graph, determine the relationship between the price of brand X and the amount that will be demanded. (That is, as price decreases, what happens to the quantity demanded?) Is price per unit a function of quantity demanded?

Quantity Demanded, q	Price per Unit, p
5	\$20
10	10
20	5
25	4

45. Inventory Sketch the graph of

$$y = f(x) = \begin{cases} -100x + 1000 & \text{if } 0 \le x < 7 \\ -100x + 1700 & \text{if } 7 \le x < 14 \\ -100x + 2400 & \text{if } 14 \le x < 21 \end{cases}$$

A function such as this might describe the inventory y of a company at time x.

46. Psychology In a psychological experiment on visual information, a subject briefly viewed an array of letters and was then asked to recall as many letters as possible from the array. The procedure was repeated several times. Suppose that y is the average number of letters recalled from arrays with x letters. The graph of the results approximately fits the graph of

$$y = f(x) = \begin{cases} x & \text{if } 0 \le x \le 4 \\ \frac{1}{2}x + 2 & \text{if } 4 < x \le 5 \\ 4.5 & \text{if } 5 < x \le 12 \end{cases}$$

Plot this function.[5]

In Problems 47–50, use a graphing calculator to find all real roots, if any, of the given equation. Round answers to two decimal places.

47. $5x^3 + 7x = 3$

48. $x^2(x - 3) = 2x^4 - 1$

49. $(9x + 3.1)^2 = 7.4 - 4x^2$

50. $(x - 2)^3 = x^2 - 3$

In Problems 51–54, use a graphing calculator to find all x-intercepts of the graph of the given function. Round answers to two decimal places.

51. $f(x) = x^3 + 5x + 7$

52. $f(x) = 2x^4 - 1.5x^3 + 2$

53. $g(x) = x^4 - 1.7x^2 + 2x$

54. $g(x) = \sqrt{3}x^5 - 4x^2 + 1$

[5] Adapted from G. R. Loftus and E. F. Loftus, *Human Memory: The Processing of Information* (New York: Lawrence Erlbaum Associates, Inc., distributed by the Halsted Press, Division of John Wiley & Sons, Inc., 1976).

In Problems 55–57, use a graphing calculator to find (a) the maximum value of $f(x)$ and (b) the minimum value of $f(x)$ for the indicated values of x. Round answers to two decimal places.

55. $f(x) = x^4 - 4.1x^3 + x^2 + 10 \quad 1 \le x \le 4$

56. $f(x) = x(2.1x^2 - 3)^2 - x^3 + 1 \quad -1 \le x \le 1$

57. $f(x) = \dfrac{x^2 - 4}{2x - 5} \quad 3 \le x \le 5$

58. From the graph of $f(x) = \sqrt{2}x^3 + 1.1x^2 + 4$, find (a) the range and (b) the intercepts. Round values to two decimal places.

59. From the graph of $f(x) = 1 - 4x^3 - x^4$, find (a) the maximum value of $f(x)$, (b) the range of f, and (c) the real zeros of f. Round values to two decimal places.

60. From the graph of $f(x) = \dfrac{x^3 + 1.1}{3.8 + x^{2/3}}$, find **(a)** the range of f and **(b)** the intercepts. **(c)** Does f have any real zeros? Round values to two decimal places.

61. Graph $f(x) = \dfrac{4.1x^3 + \sqrt{2}}{x^2 - 3}$ for $2 \le x \le 5$. Determine (a) the maximum value of $f(x)$, (b) the minimum value of $f(x)$, (c) the range of f, and (d) all intercepts. Round values to two decimal places.

2.6 Symmetry

Objective

To study symmetry about the x-axis, the y-axis, and the origin, and to apply symmetry to curve sketching.

Examining the graphical behavior of equations is a basic part of mathematics. In this section, we examine equations to determine whether their graphs have *symmetry*. In a later chapter, you will see that calculus is a *great* aid in graphing because it helps determine the shape of a graph. It provides powerful techniques for determining whether or not a curve "wiggles" between points.

Consider the graph of $y = x^2$ in Figure 2.27. The portion to the left of the y-axis is the reflection (or mirror image) through the y-axis of that portion to the right of the y-axis, and vice versa. More precisely, if (a, b) is any point on this graph, then the point $(-a, b)$ must also lie on the graph. We say that this graph is *symmetric about the y-axis*.

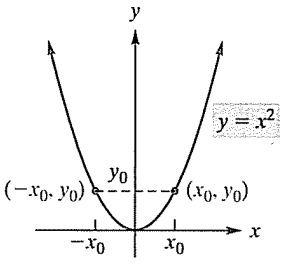


FIGURE 2.27 Symmetry about the y-axis.

Definition

A graph is ***symmetric about the y-axis*** if and only if $(-a, b)$ lies on the graph when (a, b) does.

EXAMPLE 1 *y*-Axis Symmetry

Use the preceding definition to show that the graph of $y = x^2$ is symmetric about the y-axis.

Solution: Suppose (a, b) is *any* point on the graph of $y = x^2$. Then

$$b = a^2$$

We must show that the coordinates of $(-a, b)$ satisfy $y = x^2$. But

$$(-a)^2 = a^2 = b$$

shows this to be true. Thus we have *proved* with simple algebra what the picture of the graph led us to believe: The graph of $y = x^2$ is symmetric about the y-axis.

Now Work Problem 7 ◁

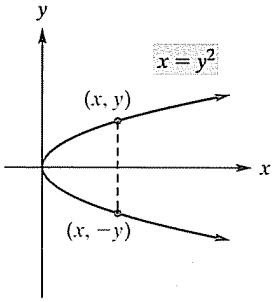


FIGURE 2.28 Symmetry about the x-axis.

When one is testing for symmetry in Example 1, (a, b) can be any point on the graph. In the future, for convenience, we write (x, y) for a typical point on the graph. This means that a graph is symmetric about the y-axis if replacing x by $-x$ in its equation results in an equivalent equation.

Another type of symmetry is shown by the graph of $x = y^2$ in Figure 2.28. Here the portion below the x-axis is the reflection through the x-axis of that portion above

the x-axis, and vice versa. If the point (x, y) lies on the graph, then $(x, -y)$ also lies on it. This graph is said to be *symmetric about the x-axis*.

Definition

A graph is ***symmetric about the x-axis*** if and only if $(x, -y)$ lies on the graph when (x, y) does.

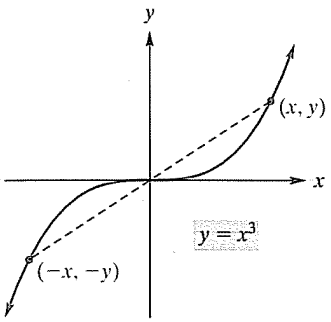


FIGURE 2.29 Symmetry about the origin.

Thus, the graph of an equation in x and y has x-axis symmetry if replacing y by $-y$ results in an equivalent equation. For example, applying this test to the graph of $x = y^2$, we see that $(-y)^2 = x$ if and only if $y^2 = x$, simply because $(-y)^2 = y^2$. Hence the graph of $x = y^2$ is symmetric about the x-axis.

A third type of symmetry, *symmetry about the origin*, is illustrated by the graph of $y = x^3$ (Figure 2.29). Whenever the point (x, y) lies on the graph, $(-x, -y)$ also lies on it.

Definition

A graph is ***symmetric about the origin*** if and only if $(-x, -y)$ lies on the graph when (x, y) does.

Thus, the graph of an equation in x and y has symmetry about the origin if replacing x by $-x$ and y by $-y$ results in an equivalent equation. For example, applying this test to the graph of $y = x^3$ shown in Figure 2.29 gives

$$-y = (-x)^3$$
$$-y = -x^3$$
$$y = x^3$$

where all three equations are equivalent, in particular the first and last. Accordingly, the graph is symmetric about the origin.

Table 2.1 summarizes the tests for symmetry. When we know that a graph has symmetry, we can sketch it by plotting fewer points than would otherwise be needed.

Table 2.1 Tests for Symmetry

Symmetry about x-axis	Replace y by $-y$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about y-axis	Replace x by $-x$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about origin	Replace x by $-x$ and y by $-y$ in given equation. Symmetric if equivalent equation is obtained.

EXAMPLE 2 Graphing with Intercepts and Symmetry

Test $y = \dfrac{1}{x}$ for symmetry about the x-axis, the y-axis, and the origin. Then find the intercepts and sketch the graph.

Solution:

Symmetry *x-axis:* Replacing y by $-y$ in $y = 1/x$ gives

$$-y = \frac{1}{x} \quad \text{equivalently} \quad y = -\frac{1}{x}$$

which is not equivalent to the given equation. Thus, the graph is *not* symmetric about the x-axis.

y-axis: Replacing x by $-x$ in $y = 1/x$ gives

$$y = \frac{1}{-x} \quad \text{equivalently} \quad y = -\frac{1}{x}$$

which is not equivalent to the given equation. Hence, the graph is *not* symmetric about the y-axis.

Origin: Replacing x by $-x$ and y by $-y$ in $y = 1/x$ gives

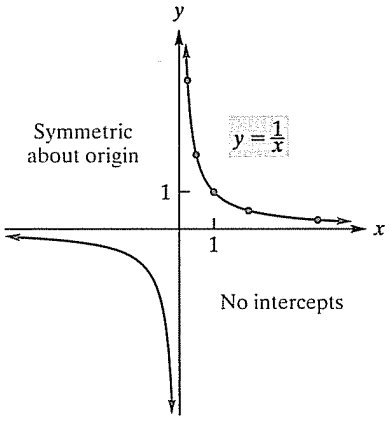
$$-y = \frac{1}{-x} \quad \text{equivalently} \quad y = \frac{1}{x}$$

which is the given equation. Consequently, the graph *is* symmetric about the origin.

Intercepts Since x cannot be 0, the graph has no y-intercept. If y is 0, then $0 = 1/x$, and this equation has no solution. Thus, no x-intercept exists.

Discussion Because no intercepts exist, the graph cannot intersect either axis. If $x > 0$, we obtain points only in quadrant I. Figure 2.30 shows the portion of the graph in quadrant I. By symmetry, we reflect that portion through the origin to obtain the entire graph.

Now Work Problem 9 ◁



x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

FIGURE 2.30 Graph of $y = \frac{1}{x}$.

EXAMPLE 3 Graphing with Intercepts and Symmetry

Test $y = f(x) = 1 - x^4$ for symmetry about the x-axis, the y-axis, and the origin. Then find the intercepts and sketch the graph.

Solution:

Symmetry *x-axis:* Replacing y by $-y$ in $y = 1 - x^4$ gives

$$-y = 1 - x^4 \quad \text{equivalently} \quad y = -1 + x^4$$

which is not equivalent to the given equation. Thus, the graph is *not* symmetric about the x-axis.

y-axis: Replacing x by $-x$ in $y = 1 - x^4$ gives

$$y = 1 - (-x)^4 \quad \text{equivalently} \quad y = 1 - x^4$$

which is the given equation. Hence, the graph *is* symmetric about the y-axis.

Origin: Replacing x by $-x$ and y by $-y$ in $y = 1 - x^4$ gives

$$-y = 1 - (-x)^4 \quad \text{equivalently} \quad -y = 1 - x^4 \quad \text{equivalently} \quad y = -1 + x^4$$

which is not equivalent to the given equation. Thus, the graph is *not* symmetric about the origin.

Intercepts Testing for x-intercepts, we set $y = 0$ in $y = 1 - x^4$. Then

$$1 - x^4 = 0$$
$$(1 - x^2)(1 + x^2) = 0$$
$$(1 - x)(1 + x)(1 + x^2) = 0$$
$$x = 1 \quad \text{or} \quad x = -1$$

x	y
0	1
$\frac{1}{2}$	$\frac{15}{16}$
$\frac{3}{4}$	$\frac{175}{256}$
1	0
$\frac{3}{2}$	$-\frac{65}{16}$

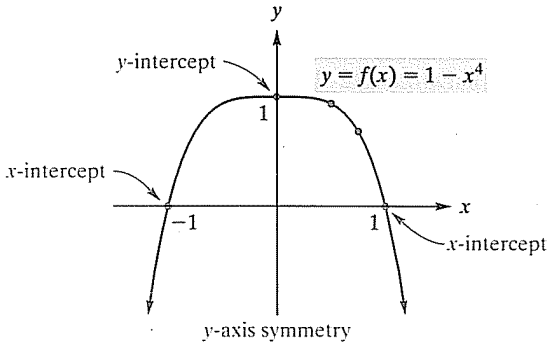


FIGURE 2.31 Graph of $y = 1 - x^4$.

The x-intercepts are therefore $(1, 0)$ and $(-1, 0)$. Testing for y-intercepts, we set $x = 0$. Then $y = 1$, so $(0, 1)$ is the only y-intercept.

Discussion If the intercepts and some points (x, y) to the right of the y-axis are plotted, we can sketch the *entire* graph by using symmetry about the y-axis (Figure 2.31).

Now Work Problem 19 ◁

The only *function* whose graph is symmetric about the x-axis is the function constantly 0.

The constant function $f(x) = 0$, for all x, is easily seen to be symmetric about the x-axis. In Example 3, we showed that the graph of $y = f(x) = 1 - x^4$ does not have x-axis symmetry. For any *function* f, suppose that the graph of $y = f(x)$ has x-axis symmetry. According to the definition, this means that we also have $-y = f(x)$. This tells us that for an arbitrary x in the domain of f we have $f(x) = y$ and $f(x) = -y$. Since for a function each x-value determines a unique y-value, we must have $y = -y$, and this implies $y = 0$. Since x was arbitrary, it follows that if the graph of a *function* is symmetric about the x-axis, then the function must be the constant 0.

EXAMPLE 4 **Graphing with Intercepts and Symmetry**

Test the graph of $4x^2 + 9y^2 = 36$ for intercepts and symmetry. Sketch the graph.

Solution:

Intercepts If $y = 0$, then $4x^2 = 36$, so $x = \pm 3$. Thus, the x-intercepts are $(3, 0)$ and $(-3, 0)$. If $x = 0$, then $9y^2 = 36$, so $y = \pm 2$. Hence, the y-intercepts are $(0, 2)$ and $(0, -2)$.

Symmetry *x-axis:* Replacing y by $-y$ in $4x^2 + 9y^2 = 36$ gives

$$4x^2 + 9(-y)^2 = 36 \quad \text{equivalently} \quad 4x^2 + 9y^2 = 36$$

which is the original equation, so there is symmetry about the x-axis.

y-axis: Replacing x by $-x$ in $4x^2 + 9y^2 = 36$ gives

$$4(-x)^2 + 9y^2 = 36 \quad \text{equivalently} \quad 4x^2 + 9y^2 = 36$$

which is the original equation, so there is also symmetry about the y-axis.

Origin: Replacing x by $-x$ and y by $-y$ in $4x^2 + 9y^2 = 36$ gives

$$4(-x)^2 + 9(-y)^2 = 36 \quad \text{equivalently} \quad 4x^2 + 9y^2 = 36$$

which is the original equation, so the graph is also symmetric about the origin.

Discussion In Figure 2.32, the intercepts and some points in the first quadrant are plotted. The points in that quadrant are then connected by a smooth curve. By symmetry about the x-axis, the points in the fourth quadrant are obtained. Then, by symmetry about the y-axis, the complete graph is found. There are other ways of graphing the equation by using symmetry. For example, after plotting the intercepts and some points in the first quadrant, we can obtain the points in the third quadrant by symmetry about the origin. By symmetry about the x-axis (or y-axis), we can then obtain the entire graph.

Now Work Problem 23 ◁

x	y
± 3	0
0	± 2
1	$\frac{4\sqrt{2}}{3}$
2	$\frac{2\sqrt{5}}{3}$
$\frac{5}{2}$	$\frac{\sqrt{11}}{3}$

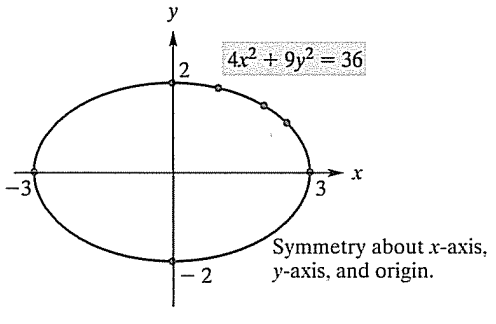


FIGURE 2.32 Graph of $4x^2 + 9y^2 = 36$.

This fact can be a time-saving device in checking for symmetry.

In Example 4, the graph is symmetric about the x-axis, the y-axis, and the origin. It can be shown that **for any graph, if any two of the three types of symmetry exist, then the remaining type must also exist.**

EXAMPLE 5 Symmetry about the Line $y = x$

Definition

A graph is ***symmetric about the line*** $y = x$ if and only if (b, a) lies on the graph when (a, b) does.

Another way of stating the definition is to say that interchanging the roles of x and y in the given equation results in an equivalent equation.

Use the preceding definition to show that $x^2 + y^2 = 1$ is symmetric about the line $y = x$.

Solution: Interchanging the roles of x and y produces $y^2 + x^2 = 1$, which is equivalent to $x^2 + y^2 = 1$. Thus $x^2 + y^2 = 1$ is symmetric about $y = x$.

◁

The point with coordinates (b, a) is the mirror image in the line $y = x$ of the point (a, b). If f is a one-to-one function, $b = f(a)$ if and only if $a = f^{-1}(b)$. Thus the graph of f^{-1} is the mirror image in the line $y = x$ of the graph of f. It is interesting to note that for *any* function f we can form the mirror image of the graph of f. However, the resulting graph need not be the graph of a function. For this mirror image to be itself the graph of a function, it must pass the vertical-line test. However vertical lines and horizontal lines are mirror images in the line $y = x$, and we see that for the mirror image of the graph of f to pass the vertical-line test is for the graph of f to pass the horizontal-line test. This last happens precisely if f is one-to-one, which is the case if and only if f has an inverse.

EXAMPLE 6 Symmetry and Inverse Functions

Sketch the graph of $g(x) = 2x + 1$ and its inverse in the same plane.

Solution: As we shall study in greater detail in Chapter 3, the graph of g is the straight line with slope (see Section 3.1) 2 and y-intercept 1. This line, the line $y = x$, and the reflection of $y = 2x + 1$ in $y = x$ are shown in Figure 2.33.

Now Work Problem 27 ◁

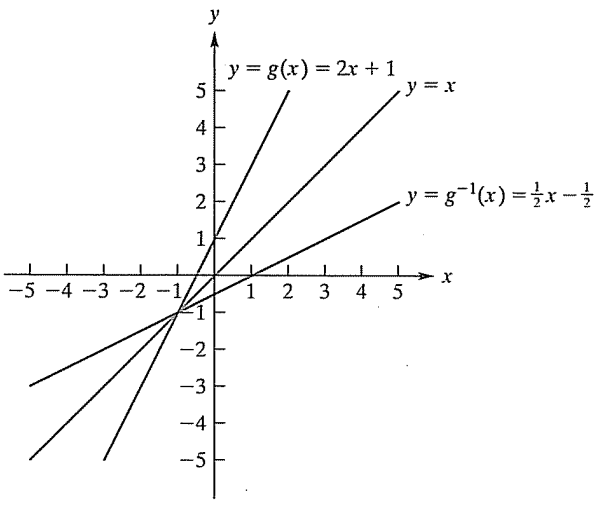


FIGURE 2.33 Graph of $y = g(x)$ and $y = g^{-1}(x)$.

PROBLEMS 2.6

In Problems 1–16, find the x- and y-intercepts of the graphs of the equations. Also, test for symmetry about the x-axis, the y-axis, the origin, and the line $y = x$. Do not sketch the graphs.

1. $y = 5x$

2. $y = f(x) = x^2 - 4$

3. $2x^2 + y^2x^4 = 8 - y$

4. $x = y^3$

5. $25x^2 + 144y^2 = 169$

6. $y = 57$

7. $x = -2$

8. $y = |2x| - 2$

9. $x = -y^{-4}$

10. $y = \sqrt{x^2 - 36}$

11. $x - 4y - y^2 + 21 = 0$

12. $x^2 + xy + y^3 = 0$

13. $y = f(x) = \dfrac{x^3 - 2x^2 + x}{x^2 + 1}$

14. $x^2 + xy + y^2 = 0$

15. $y = \dfrac{2}{x^3 + 27}$

16. $y = \dfrac{x^4}{x + y}$

In Problems 17–24, find the x- and y-intercepts of the graphs of the equations. Also, test for symmetry about the x-axis, the y-axis, the origin, and the line $y = x$. Then sketch the graphs.

17. $3x + y^2 = 9$

18. $x - 1 = y^4 + y^2$

19. $y = f(x) = x^3 - 4x$

20. $2y = 5 - x^2$

21. $|x| - |y| = 0$

22. $x^2 + y^2 = 16$

23. $9x^2 + 4y^2 = 25$

24. $x^2 - y^2 = 4$

25. Prove that the graph of $y = f(x) = 5 - 1.96x^2 - \pi x^4$ is symmetric about the y-axis, and then graph the function. (a) Make use of symmetry, where possible, to find all intercepts. Determine (b) the maximum value of $f(x)$ and (c) the range of f. Round all values to two decimal places.

26. Prove that the graph of $y = f(x) = 2x^4 - 7x^2 + 5$ is symmetric about the y-axis, and then graph the function. Find all real zeros of f. Round your answers to two decimal places.

27. Sketch the graph of $f(x) = -3x + 2$ and its inverse in the same plane.

Objective

To become familiar with the shapes of the graphs of six basic functions and to consider translation, reflection, and vertical stretching or shrinking of the graph of a function.

2.7 Translations and Reflections

Up to now, our approach to graphing has been based on plotting points and making use of any symmetry that exists. But this technique is not necessarily the preferred way. Later in this book, we will analyze graphs by using other techniques. However, some functions and their associated graphs occur so frequently that we find it worthwhile to memorize them. Figure 2.34 shows six such functions.

At times, by altering a function through an *algebraic* manipulation, the graph of the new function can be obtained from the graph of the original function by performing a *geometric* manipulation. For example, we can use the graph of $f(x) = x^2$ to graph $y = x^2 + 2$. Note that $y = f(x) + 2$. Thus, for each x, the corresponding ordinate for the

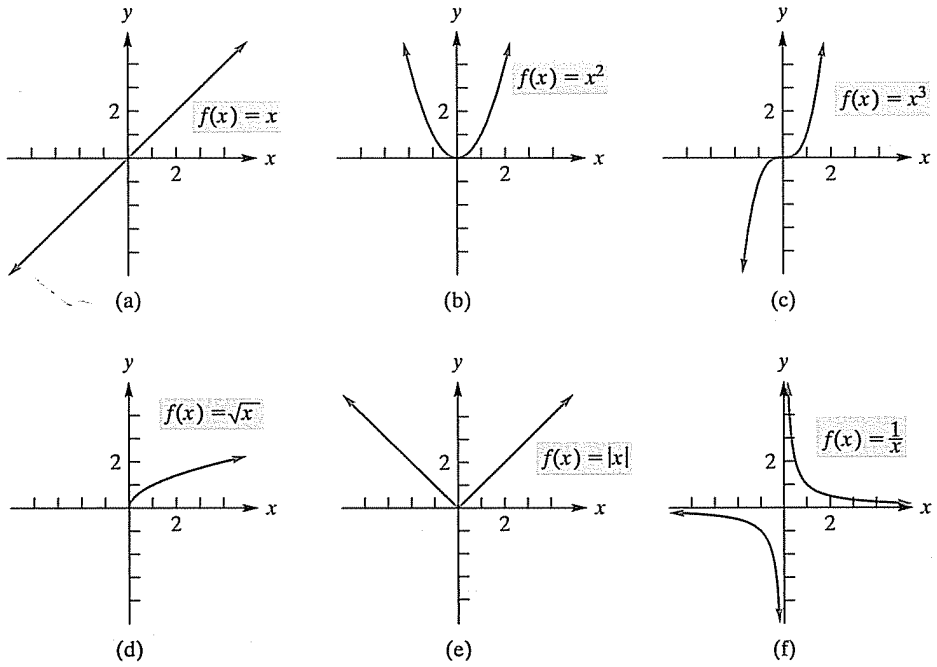


FIGURE 2.34 Functions frequently used.

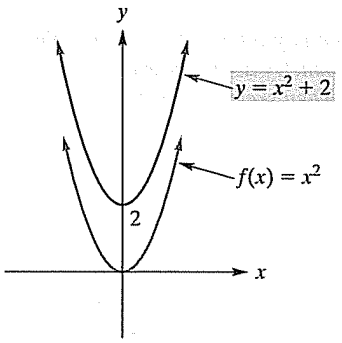


FIGURE 2.35 Graph of $y = x^2 + 2$.

graph of $y = x^2 + 2$ is 2 more than the ordinate for the graph of $f(x) = x^2$. This means that the graph of $y = x^2 + 2$ is simply the graph of $f(x) = x^2$, shifted, or *translated*, 2 units upward. (See Figure 2.35.) We say that the graph of $y = x^2 + 2$ is a *transformation* of the graph of $f(x) = x^2$. Table 2.2 gives a list of basic types of transformations.

Table 2.2 Transformations, $c > 0$

Equation	How to Transform Graph of $y = f(x)$ to Obtain Graph of Equation
$y = f(x) + c$	shift c units upward
$y = f(x) - c$	shift c units downward
$y = f(x - c)$	shift c units to right
$y = f(x + c)$	shift c units to left
$y = -f(x)$	reflect about x-axis
$y = f(-x)$	reflect about y-axis
$y = cf(x)$ $c > 1$	vertically stretch away from x-axis by a factor of c
$y = cf(x)$ $c < 1$	vertically shrink toward x-axis by a factor of c

EXAMPLE 1 Horizontal Translation

Sketch the graph of $y = (x - 1)^3$.

Solution: We observe that $(x - 1)^3$ is x^3 with x replaced by $x - 1$. Thus, if $f(x) = x^3$, then $y = (x-1)^3 = f(x-1)$, which has the form $f(x-c)$, where $c = 1$. From Table 2.2, the graph of $y = (x - 1)^3$ is the graph of $f(x) = x^3$, shifted 1 unit to the right. (See Figure 2.36.)

Now Work Problem 3 ◁

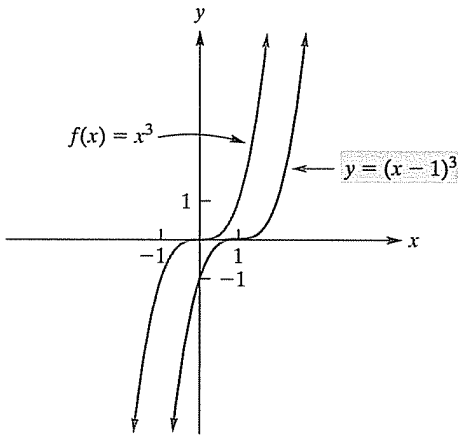


FIGURE 2.36 Graph of $y = (x - 1)^3$.

EXAMPLE 2 Shrinking and Reflection

Sketch the graph of $y = -\frac{1}{2}\sqrt{x}$.

Solution: We can do this problem in two steps. First, observe that $\frac{1}{2}\sqrt{x}$ is $\sqrt{x}$ multiplied by $\frac{1}{2}$. Thus, if $f(x) = \sqrt{x}$, then $\frac{1}{2}\sqrt{x} = \frac{1}{2}f(x)$, which has the form $cf(x)$, where $c = \frac{1}{2}$. So the graph of $y = \frac{1}{2}\sqrt{x}$ is the graph of f shrunk vertically toward

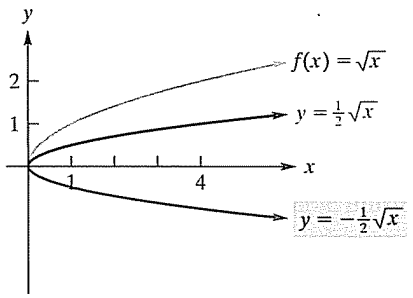


FIGURE 2.37 To graph $y = -\frac{1}{2}\sqrt{x}$, shrink $y = \sqrt{x}$ and reflect result about x-axis.

the x-axis by a factor of $\frac{1}{2}$ (transformation 8, Table 2.2; see Figure 2.37). Second, the minus sign in $y = -\frac{1}{2}\sqrt{x}$ causes a reflection in the graph of $y = \frac{1}{2}\sqrt{x}$ about the x-axis (transformation 5, Table 2.2; see Figure 2.37).

Now Work Problem 5 ◁

PROBLEMS 2.7

In Problems 1–12, use the graphs of the functions in Figure 2.34 and transformation techniques to plot the given functions.

1. $y = x^3 - 1$ **2.** $y = -x^2$ **3.** $y = \dfrac{1}{x-2}$

4. $y = -\sqrt{x-2}$ **5.** $y = \dfrac{2}{3x}$ **6.** $y = |x| - 2$

7. $y = |x+1| - 2$ **8.** $y = -\dfrac{1}{3}\sqrt{x}$ **9.** $y = 2 + (x+3)^3$

10. $y = (x-1)^2 + 1$ **11.** $y = \sqrt{-x}$ **12.** $y = \dfrac{5}{2-x}$

In Problems 13–16, describe what must be done to the graph of $y = f(x)$ to obtain the graph of the given equation.

13. $y = -2f(x+3) + 2$ **14.** $y = 2(f(x-1) - 4)$

15. $y = f(-x) - 5$ **16.** $y = f(3x)$

17. Graph the function $y = \sqrt[3]{x} + k$ for $k = 0, 1, 2, 3, -1, -2,$ and -3. Observe the vertical translations compared to the first graph.

18. Graph the function $y = \sqrt[5]{x+k}$ for $k = 0, 1, 2, 3, -1, -2,$ and -3. Observe the horizontal translations compared to the first graph.

19. Graph the function $y = kx^3$ for $k = 1, 2, \frac{1}{2},$ and 3. Observe the vertical stretching and shrinking compared to the first graph. Graph the function for $k = -2$. Observe that the graph is the same as that obtained by stretching the reflection of $y = x^3$ about the x-axis by a factor of 2.

Objective

To discuss functions of several variables and to compute function values. To discuss three-dimensional coordinates and sketch simple surfaces.

2.8 Functions of Several Variables

When we defined a *function* $f : X \longrightarrow Y$ from X to Y in Section 2.1, we did so for *sets* X and Y without requiring that they be sets of numbers. We have not often used that generality yet. Most of our examples have been functions from $(-\infty, \infty)$ to $(-\infty, \infty)$. We also saw in Section 2.1 that, for sets X and Y, we can construct the new set $X \times Y$ whose elements are ordered pairs (x, y) with x in X and y in Y. It follows that, for any three sets X, Y, and Z, the notion of a function $f : X \times Y \longrightarrow Z$ is already covered by the basic definition. Such an f is simply a rule which assigns to each element (x, y) in $X \times Y$ at most one element of Z, denoted by $f((x, y))$. There is general agreement that in this situation one should drop a layer of parentheses and write simply $f(x, y)$ for $f((x, y))$. Do note here that even if each of X and Y are sets of numbers, say $X = (-\infty, \infty) = Y$, then $X \times Y$ is definitely *not* a set of numbers. In other words, *an ordered pair of numbers is not a number.*

The *graph* of a function $f : X \longrightarrow Y$ is the subset of $X \times Y$ consisting of all ordered pairs of the form $(x, f(x))$, where x is in the domain of f. It follows that the graph of a function $f : X \times Y \longrightarrow Z$ is the subset of $(X \times Y) \times Z$ consisting of all ordered pairs of the form $((x, y), f(x, y))$, where (x, y) is in the domain of f. The ordered pair $((x, y), f(x, y))$ has its first coordinate given by (x, y), itself an ordered pair, while its

second coordinate is the element $f(x, y)$ in Z. Most people prefer to replace $(X \times Y) \times Z$ with $X \times Y \times Z$, an element of which is an **ordered triple** (x, y, z), with x in X, y in Y, and z in Z. These elements are easier to read than the $((x, y), z)$, which are the "official" elements of $(X \times Y) \times Z$. In fact, we can *define* an ordered triple (x, y, z) to be a shorthand for $((x, y), z)$ if we wish.

Before going further, it is important to realize that these very general considerations have been motivated by a desire to make mathematics applicable. Many people when confronted with mathematical models built around functions, and equations relating them, express both an appreciation for the elegance of the ideas and a skepticism about their practical value. A common complaint is that in practice there are "factors" unaccounted for in a particular mathematical model. Translated into the context we are developing, this complaint frequently means that the functions in a mathematical model should involve more variables than the modeler originally contemplated. Being able to add new variables, to account for phenomena that were earlier thought to be insignificant, is an important aspect of robustness that a mathematical model should possess. If we know how to go from one variable to two variables, where the "two variables" can be construed as an ordered pair and hence a single variable of a new kind, then we can iterate the procedure and deal with functions of as many variables as we like.

For sets $X_1, X_2, \ldots, X_n$ and Y, a function $f : X_1 \times X_2 \times \cdots \times X_n \longrightarrow Y$ in our general sense provides the notion of a Y-valued function of n-variables. In this case, an element of the domain of f is an **ordered n-tuple** $(x_1, x_2, \cdots, x_n)$, with x_i in X_i for $i = 1, 2, \cdots, n$, for which $f(x_1, x_2, \cdots, x_n)$ is defined. The **graph** of f is the set of all ordered $n+1$-tuples of the form $(x_1, x_2, \cdots, x_n, f(x_1, x_2, \cdots, x_n))$, where $(x_1, x_2, \cdots, x_n)$ is in the domain of f.

Suppose a manufacturer produces two products, X and Y. Then the total cost depends on the levels of production of *both* X and Y. Table 2.3 is a schedule that indicates total cost at various levels. For example, when 5 units of X and 6 units of Y are produced, the total cost c is 17. In this situation, it seems natural to associate the number 17 with the *ordered pair* (5, 6):

$$(5, 6) \mapsto 17$$

The first element of the ordered pair, 5, represents the number of units of X produced, while the second element, 6, represents the number of units of Y produced. Corresponding to the other production situations shown, we have

$$(5, 7) \mapsto 19$$

$$(6, 6) \mapsto 18$$

and

$$(6, 7) \mapsto 20$$

Table 2.3

No. of Units of X Produced, x	No. of Units of Y Produced, y	Total Cost of Production, c
5	6	17
5	7	19
6	6	18
6	7	20

This listing can be considered to be the defintion of a function c: $X \times Y \longrightarrow (-\infty, \infty)$, where $X = \{5, 6\}$ and and $Y = \{6, 7\}$.

$$c(5, 7) = 19 \quad c(6, 7) = 20$$

$$c(5, 6) = 17 \quad c(6, 6) = 18$$

We say that the total-cost schedule can be described by $c = c(x, y)$, a function of the two independent variables x and y. The letter c is used here for both the dependent variable and the name of the rule that defines the function. Of course the range of c is the subset $\{17, 18, 19, 20\}$ of $(-\infty, \infty)$. Because negative costs are unlikely to ever make sense, we might want to refine c and construe it to be a function $c : X \times Y \longrightarrow [0, \infty)$.

Most people were acquainted with certain functions of two variables long before they ever heard of functions, as the following example illustrates.

EXAMPLE 1 **Functions of Two Variables**

a. $a(x, y) = x + y$ is a function of two variables. Some function values are

$$a(1, 1) = 1 + 1 = 2$$

$$a(2, 3) = 2 + 3 = 5$$

We have a: $(-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$.

b. $m(x, y) = xy$ is a function of two variables. Some function values are

$$m(2, 2) = 2 \cdot 2 = 4$$

$$m(3, 2) = 3 \cdot 2 = 6$$

The domain of both a and m is all of $(-\infty, \infty) \times (-\infty, \infty)$. Observe that if you were to define division as a function d: $(-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$ with $d(x, y) = x \div y$ then the domain of d is $(-\infty, \infty) \times ((-\infty, \infty) - \{0\})$, where $(-\infty, \infty) - \{0\}$ is the set all real numbers except 0.

◁

Turning to another function of two variables, we see that the equation

$$z = \frac{2}{x^2 + y^2}$$

defines z as a function of x and y:

$$z = f(x, y) = \frac{2}{x^2 + y^2}$$

The domain of f is all ordered pairs of real numbers (x, y) for which the equation has meaning when the first and second elements of (x, y) are substituted for x and y, respectively, in the equation. Thus, the domain of f is $(-\infty, \infty) \times (-\infty, \infty) - \{(0, 0)\}$, the set of all ordered pairs of real numbers except $(0, 0)$. To find $f(2, 3)$, for example, we substitute $x = 2$ and $y = 3$ into the expression $2/(x^2 + y^2)$. We obtain $f(2, 3) = 2/(2^2 + 3^2) = 2/13$.

EXAMPLE 2 **Functions of Two Variables**

a. $f(x, y) = \dfrac{x + 3}{y - 2}$ is a function of two variables. Because the denominator is zero when $y = 2$, the domain of f is all (x, y) such that $y \neq 2$. Some function values are

$$f(0, 3) = \frac{0 + 3}{3 - 2} = 3$$

$$f(3, 0) = \frac{3 + 3}{0 - 2} = -3$$

Note that $f(0, 3) \neq f(3, 0)$.

b. $h(x, y) = 4x$ defines h as a function of x and y. The domain is all ordered pairs of real numbers. Some function values are

$$h(2, 5) = 4(2) = 8$$

$$h(2, 6) = 4(2) = 8$$

Note that the function values are independent of the choice of y.

APPLY IT ▷

15. The cost per day for manufacturing both 12-ounce and 20-ounce coffee mugs is given by $c = 160 + 2x + 3y$, where x is the number of 12-ounce mugs and y is the number of 20-ounce mugs. What is the cost per day of manufacturing

a. 500 12-ounce and 700 20-ounce mugs?

b. 1000 12-ounce and 750 20-ounce mugs?

c. If $z^2 = x^2 + y^2$ and $x = 3$ and $y = 4$, then $z^2 = 3^2 + 4^2 = 25$. Consequently, $z = \pm 5$. Thus, with the ordered pair (3, 4), we *cannot* associate exactly one output number. Hence $z^2 = x^2 + y^2$ does not define z as a function of x and y.

Now Work Problem 1 ◁

EXAMPLE 3 **Temperature–Humidity Index**

On hot and humid days, many people tend to feel uncomfortable. The degree of discomfort is numerically given by the temperature–humidity index, THI, which is a function of two variables, t_d and t_w:

$$\text{THI} = f(t_d, t_w) = 15 + 0.4(t_d + t_w)$$

where t_d is the dry-bulb temperature (in degrees Fahrenheit) and t_w is the wet-bulb temperature (in degrees Fahrenheit) of the air. Evaluate the THI when $t_d = 90$ and $t_w = 80$.

Solution: We want to find $f(90, 80)$:

$$f(90, 80) = 15 + 0.4(90 + 80) = 15 + 68 = 83$$

When the THI is greater than 75, most people are uncomfortable. In fact, the THI was once called the "discomfort index." Many electric utilities closely follow this index so that they can anticipate the demand that air-conditioning places on their systems.

Now Work Problem 3 ◁

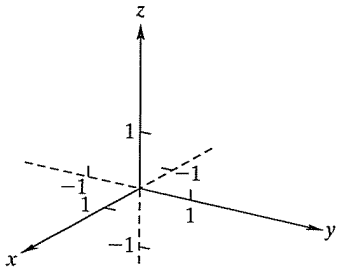


FIGURE 2.38 Three-dimensional rectangular coordinate system.

From the second paragraph in this section it follows that a function $f : (-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$, where we write $z = f(x, y)$, will have a graph consisting of ordered triples of real numbers. The set of all ordered triples of real numbers can be pictured as providing a ***three*-dimensional rectangular coordinate system**. Such a system is formed when three mutually perpendicular real-number lines in space intersect at the origin of each line, as in Figure 2.38. The three number lines are called the x-, y-, and z-axes, and their point of intersection is called the origin of the system. The arrows indicate the positive directions of the axes, and the negative portions of the axes are shown as dashed lines.

To each point P in space, we can assign a unique ordered triple of numbers, called the *coordinates* of P. To do this [see Figure 14.24(a)], from P, we construct a line perpendicular to the x, y-plane—that is, the plane determined by the x- and y-axes. Let Q be the point where the line intersects this plane. From Q, we construct perpendiculars to the x- and y-axes. These lines intersect the x- and y-axes at x_0 and y_0, respectively. From P, a perpendicular to the z-axis is constructed that intersects the axis at z_0. Thus, we assign to P the ordered triple (x_0, y_0, z_0). It should also be evident that with each ordered triple of numbers we can assign a unique point in space. Due to this one-to-one correspondence between points in space and ordered triples, an ordered triple can be

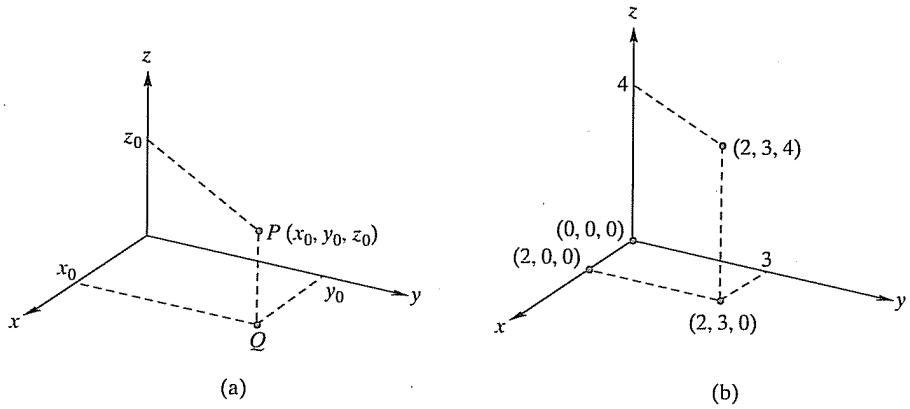


FIGURE 2.39 Points in space.

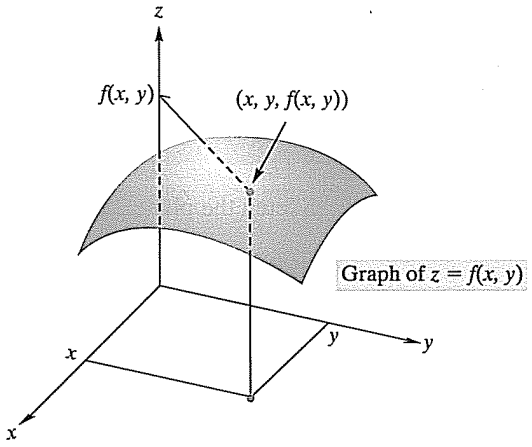


FIGURE 2.40 Graph of a function of two variables.

called a point. In Figure 14.24(b), points (2, 0, 0), (2, 3, 0), and (2, 3, 4) are shown. Note that the origin corresponds to (0, 0, 0). Typically, the negative portions of the axes are not shown unless required.

We represent a function of two variables, $z = f(x, y)$, geometrically as follows: To each ordered pair (x, y) in the domain of f, we assign the point $(x, y, f(x, y))$. The set of all such points is the *graph* of f. Such a graph appears in Figure 14.25. We can consider $z = f(x, y)$ as representing a *surface in space* in the same way that we have considered $y = f(x)$ as representing a *curve in the plane*. [Not all functions $y = f(x)$ describe aesthetically pleasing curves—in fact most do not—and in the same way we stretch the meaning of the word *surface*.]

We now give a brief discussion of sketching surfaces in space. We begin with planes that are parallel to a coordinate plane. By a "coordinate plane" we mean a plane containing two coordinate axes. For example, the plane determined by the x- and y-axes is the x, y-**plane.** Similarly, we speak of the x, z-**plane** and the y, z-**plane**. The coordinate planes divide space into eight parts, called *octants*. In particular, the part containing all points (x, y, z) such that x, y, and z are positive is called the **first octant.**

Names are not usually assigned to the remaining seven octants.

Suppose S is a plane that is parallel to the x, y-plane and passes through the point (0, 0, 5). [See Figure 14.26(a).] Then the point (x, y, z) will lie on S if and only if $z = 5$; that is, x and y can be any real numbers, but z must equal 5. For this reason, we say that $z = 5$ is an equation of S. Similarly, an equation of the plane parallel to the x, z-plane and passing through the point (0, 2, 0) is $y = 2$ [Figure 14.26(b)]. The equation $x = 3$ is an equation of the plane passing through (3, 0, 0) and parallel to the y, z-plane [Figure 14.26(c)]. Next, we look at planes in general.

In space, the graph of an equation of the form

$$Ax + By + Cz + D = 0$$

where D is a constant and A, B, and C are constants that are not all zero, is a plane. Since three distinct points (not lying on the same line) determine a plane, a convenient

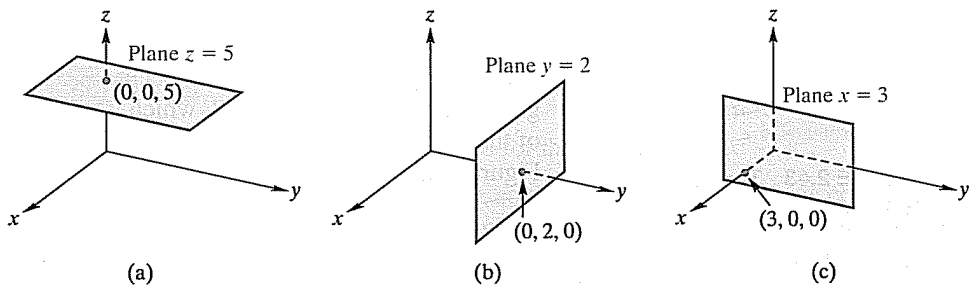


FIGURE 2.41 Planes parallel to coordinate planes.

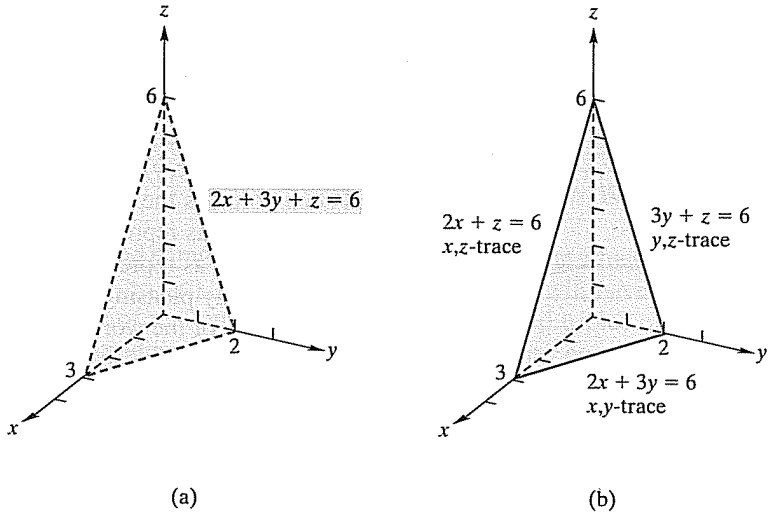
way to sketch a plane is to first determine the points, if any, where the plane intersects the x-, y-, and z-axes. These points are called *intercepts*.

EXAMPLE 4 **Graphing a Plane**

Sketch the plane $2x + 3y + z = 6$.

Solution: The plane intersects the x-axis when $y = 0$ and $z = 0$. Thus $2x = 6$ which gives $x = 3$. Similarly, if $x = z = 0$, then $y = 2$; if $x = y = 0$, then $z = 6$. Therefore, the intercepts are $(3, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 6)$. After these points are plotted, a plane is passed through them. The portion of the plane in the first octant is shown in Figure 2.42(a); note, however, that the plane extends indefinitely into space.

Now Work Problem 19 ◁



(a) (b)

FIGURE 2.42 The plane $2x + 3y + z = 6$ and its traces.

A surface can be sketched with the aid of its **traces.** These are the intersections of the surface with the coordinate planes. To illustrate, for the plane $2x + 3y + z = 6$ in Example 4, the trace in the x,y-plane is obtained by setting $z = 0$. This gives $2x + 3y = 6$, which is an equation of a *line* in the x,y-plane. Similarly, setting $x = 0$ gives the trace in the y,z-plane: the line $3y + z = 6$. The x,z-trace is the line $2x + z = 6$. [See Figure 2.42(b).]

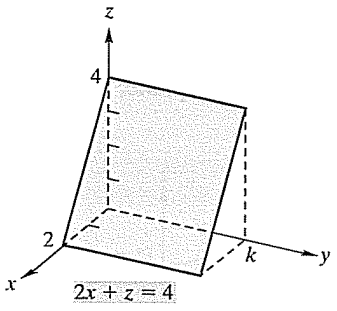


FIGURE 2.43 The plane $2x + z = 4$.

EXAMPLE 5 **Sketching a Surface**

Sketch the surface $2x + z = 4$.

Solution: This equation has the form of a plane. The x- and z-intercepts are $(2, 0, 0)$ and $(0, 0, 4)$, and there is no y-intercept, since x and z cannot both be zero. Setting $y = 0$ gives the x,z-trace $2x + z = 4$, which is a line in the x,z-plane. In fact, the intersection of the surface with *any* plane $y = k$ is also $2x + z = 4$. Hence, the plane appears as in Figure 2.43.

Now Work Problem 21 ◁

Our final examples deal with surfaces that are not planes but whose graphs can be easily obtained.

EXAMPLE 6 **Sketching a Surface**

Sketch the surface $z = x^2$.

Note that this equation places no restriction on y.

Solution: The x,z-trace is the curve $z = x^2$, which is parabola. In fact, for *any* fixed value of y, we get $z = x^2$. Thus, the graph appears as in Figure 2.44.

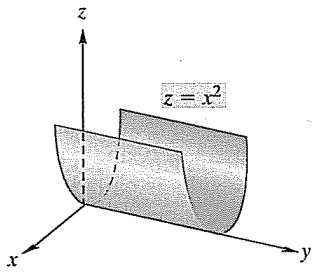


FIGURE 2.44 The surface $z = x^2$.

Now Work Problem 25 ◁

EXAMPLE 7 **Sketching a Surface**

Sketch the surface $x^2 + y^2 + z^2 = 25$.

Solution: Setting $z = 0$ gives the x,y-trace $x^2 + y^2 = 25$, which is a circle of radius 5. Similarly, the y,z-, and x,z-traces are the circles $y^2 + z^2 = 25$ and $x^2 + z^2 = 25$, respectively. Note also that since $x^2 + y^2 = 25 - z^2$, the intersection of the surface with the plane $z = k$, where $-5 \leq k \leq 5$, is a circle. For example, if $z = 3$, the intersection is the circle $x^2 + y^2 = 16$. If $z = 4$, the intersection is $x^2 + y^2 = 9$. That is, cross sections of the surface that are parallel to the x,y-plane are circles. The surface appears in Figure 2.45; it is a sphere.

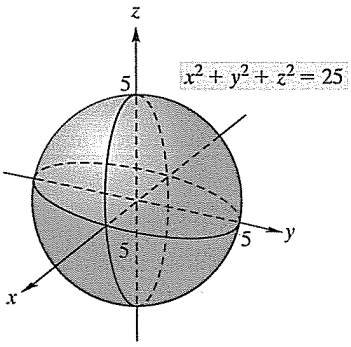


FIGURE 2.45 The surface $x^2 + y^2 + z^2 = 25$.

Now Work Problem 27 ◁

For a function $f : X \times Y \longrightarrow Z$, we have seen that the graph of f, being a subset of $X \times Y \times Z$, is three dimensional for numerical examples. Admittedly, constructing such a graph on paper can be challenging. There is another pictorial presentation of a function $z = f(x, y)$, for $f : (-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$ which is entirely two dimensional. Let l be a number in the range of f. The *equation* $f(x, y) = l$ has a graph in the x, y-plane that, in principle, can be constructed and labeled. If we repeat this contruction in the same plane, for several other values, l_i say, in the range of f then we have a set of curves, called **level curves**, which may provide us with a useful visualization of f.

There are are at least two examples of this technique that are within everyday experience for many people. For the first, consider a geographic region that is small enough to be considered planar and coordinatize it. (A city with a rectangular grid of numbered avenues and streets can be considered to be coordinatized by these.) At any given time, temperature T in degrees Farenheit is a function of place (x, y). We might write $T = T(x, y)$. On a map of the region we might connect all places that currently have a temperature of 70°F with a curve. This is the curve $T(x, y) = 70$. If we put several other curves, such as $T(x, y) = 68$ and $T(x, y) = 72$, on the same map, then we have the kind of map that appears on televised weather reports. The curves in this case are called *isotherms*; the prefix *iso* comes from the Greek *isos* meaning "equal". For the next, again referring to geography, observe that each place (x, y) has a definite altitude $A = A(x, y)$. A map of a mountainous region with points of equal altitude connected by what are called *contour lines* is called a *togographic map*, and the generic term *level curves* is particularly apt in this case.

In Chapter 7 we will encounter a number of *linear* functions of several variables. If we have $P = ax + by$, expressing profit P as a function of production x of a product X and production y of a product Y, then the level curves $ax + by = l$ are called *isoprofit lines*.

EXAMPLE 8 **Level Curves**

Sketch a family of at least four level curves for the function $z = x^2 + y^2$.

Solution: For any pair (x, y), $x^2 + y^2 \geq 0$, so the range of $z = x^2 + y^2$ is contained in $[0, \infty)$. On the other hand, for any $l \geq 0$ we can write $l = (\sqrt{l})^2 + 0^2$, which shows that the range of $z = x^2 + y^2$ is all of $[0, \infty)$. For $l \geq 0$ we recognize the graph of $x^2 + y^2 = l$ as a circle of radius $\sqrt{l}$ centered at the origin $(0, 0)$. If we take l to be 4, 9, 16, and 25, then our level curves are concentric circles of radii 2, 3, 4, and 5, respectively.

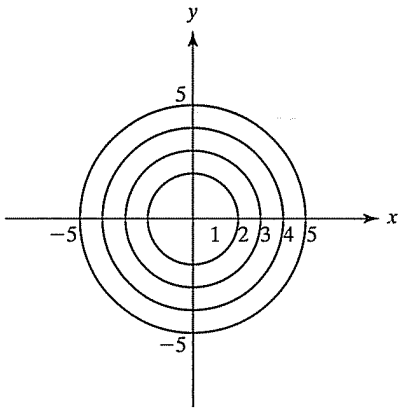


FIGURE 2.46 Level curves for $z = x^2 + y^2$.

See Figure 2.46. Note that the level "curve" $x^2 + y^2 = 0$ consists of the point $(0, 0)$ and no others.

Now Work Problem 29 ◁

An example of a function of three variables is $v = v(x, y, z) = xyz$. It provides the volume of a "brick" with side lengths x, y, and z if x, y, and z are all positive.

An *ellipsoid* is a surface which in "standard position" is given by an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, for a, b, and c positive numbers called the radii. No one of the variables is a function of the other two. If two of the numbers a, b, and c are equal and the third is larger, then the special kind of ellipsoid that results is called a *prolate spheroid*, examples of which are provided by both a football and a rugby ball. In any event, the volume of space enclosed by an ellipsoid with radii a, b, and c is given by $V = V(a, b, c) = \frac{4}{3}\pi abc$, and this is another example of a function of three (positive) variables.

In the context of functions of several variables, it is also interesting to consider functions whose *values* are ordered pairs. For any set X, one of the simplest is the *diagonal* function $\Delta : X \longrightarrow X \times X$ given by $\Delta(x) = (x, x)$. We mentioned in Example 1(b) that ordinary multiplication is a function $m : (-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$. If we let Δ denote the diagonal function for $(-\infty, \infty)$, then we see that the composite $m \circ \Delta$ of the rather simple-minded functions Δ and m is the more interesting function $y = x^2$.

PROBLEMS 2.8

In Problems 1–12, determine the indicated function values for the given functions.

1. $f(x, y) = 4x - y^2 + 3$; $f(1, 2)$
2. $f(x, y) = 3x^2y - 4y$; $f(2, -1)$
3. $g(x, y, z) = 2x(3y + z)$; $g(3, 0, -1)$
4. $g(x, y, z) = x^2yz + xy^2z + xyz^2$; $g(3, 1, -2)$
5. $h(r, s, t, u) = \frac{rs}{t^2 - u^2}$; $h(-3, 3, 5, 4)$
6. $h(r, s, t, u) = ru$; $h(1, 5, 3, 1)$
7. $g(p_A, p_B) = 2p_A(p_A^2 - 5)$; $g(4, 8)$
8. $g(p_A, p_B) = p_A^2\sqrt{p_B} + 9$; $g(4, 9)$
9. $F(x, y, z) = 3$; $F(2, 0, -1)$
10. $F(x, y, z) = \frac{2x}{(y + 1)z}$; $F(1, 0, 3)$
11. $f(x, y) = (x + y)^2$; $f(a + h, b)$
12. $f(x, y) = x^2y - 3y^3$; $f(r + t, r)$

13. Ecology A method of ecological sampling to determine animal populations in a given area involves first marking all the animals obtained in a sample of R animals from the area and then releasing them so that they can mix with unmarked animals. At a later date a second sample is taken of M animals, and the number of these that are marked, S, is noted. Based on R, M, and S, an estimate of the total population of animals in the sample area is given by

$$N = f(R, M, S) = \frac{RM}{S}$$

Find $f(200, 200, 50)$. This method is called the *mark-and-recapture procedure*.[6]

14. Genetics Under certain conditions, if two brown-eyed parents have exactly k children, the probability that there will be exactly r blue-eyed children is given by

$$P(r, k) = \frac{k!\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{k-r}}{r!(k - r)!} \qquad r = 0, 1, 2, \ldots, k$$

Find the probability that, out of a total of four children, exactly three will be blue-eyed.

In Problems 15–18 find equations of the planes that satisfy the given conditions.

15. Parallel to the x,z-plane and also passes through the point $(0, 2, 0)$

16. Parallel to the y,z-plane and also passes through the point $(-2, 0, 0)$

17. Parallel to the x,y-plane and also passes through the point $(2, 7, 6)$

18. Parallel to the y, z-plane and also passes through the point $(96, -2, 2)$

In Problems 19–28, sketch the given surfaces.

19. $x + y + z = 1$
20. $2x + y + 2z = 6$
21. $3x + 6y + 2z = 12$
22. $2x + 3y + 5z = 1$
23. $3x + y = 6$
24. $y = 3z + 2$
25. $z = 4 - x^2$
26. $y = z^2$
27. $x^2 + y^2 + z^2 = 9$
28. $x^2 + 4y^2 = 1$

In Problems 29–30, sketch at least three level curves for the given function.

29. $z = 5x + 8y$
30. $z = x^2 - y^2$

[6] E. P. Odum, *Ecology* (New York: Holt, Rinehart and Winston, 1966).

Chapter 2 Review

Important Terms and Symbols

Examples

Section 2.1 Functions

function domain range independent variable — Ex. 2, p. 84
dependent variable function value, $f(x)$ — Ex. 3, p. 84
difference quotient, $\frac{f(x+h)-f(x)}{h}$ — Ex. 4, p. 85
demand function supply function — Ex. 5, Ex. 6, p. 85, 86

Section 2.2 Special Functions

constant function polynomial function (linear and quadratic) — Ex. 1, Ex. 2, p. 87, 88
rational function case-defined function — Ex. 3, Ex. 4, p. 88, 89
absolute value, $|x|$ factorial, $r!$ — Ex. 5, Ex. 6, p. 89

Section 2.3 Combinations of Functions

$f+g$ $f-g$ fg f/g composite function, $f \circ g$ — Ex. 1, Ex. 2, p. 92, 94

Section 2.4 Inverse Functions

inverse function, f^{-1} one-to-one function — Ex. 1, p. 97

Section 2.5 Graphs in Rectangular Coordinates

rectangular coordinate system coordinate axes origin x, y-plane
ordered pair, (x, y) coordinates of a point quadrant graph of equation
x-intercept y-intercept — Ex. 1, p. 101
graph of function function-value axis zeros of function — Ex. 4, p. 102
vertical-line test horizontal-line test — Ex. 8, p. 105

Section 2.6 Symmetry

x-axis symmetry y-axis symmetry — Ex. 1, p. 108
symmetry about origin symmetry about $y = x$ — Ex. 6, p. 112

Section 2.7 Translations and Reflections

horizontal and vertical translations — Ex. 1, p. 114
stretching and reflection — Ex. 2, p. 114

Section 15. Functions of Several Variables

$z = f(x, y)$ — Ex. 2, p. 117
graph of $z = f(x, y)$ — Ex. 4, p. 120
level curves — Ex. 8, p. 121

Summary

A function f is a rule that assigns at most one output $f(x)$ to each possible input x. A function is often specified by a formula that prescribes what must be done to an input x to obtain $f(x)$. To obtain a particular function value $f(a)$, we replace each x in the formula by a.

The domain of a function $f : X \longrightarrow Y$ consists of all inputs x for which the rule defines $f(x)$ as an element of Y; the range consists of all elements of Y of the form $f(x)$.

Some special types of functions are constant functions, polynomial functions, and rational functions. A function that is defined by more than one expression depending on the kind of input is called a case-defined function.

A function has an inverse if and only if it is one-to-one.

In economics, supply (demand) functions give a correspondence between the price p of a product and the number of units q of the product that producers (consumers) will supply (buy) at that price.

Two functions f and g can be combined to form a sum, difference, product, quotient, or composite as follows:

$$(f+g)(x) = f(x) + g(x)$$
$$(f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$(f \circ g)(x) = f(g(x))$$

A rectangular coordinate system allows us to represent equations in two variables (in particular, those arising from functions) geometrically. The graph of an equation in x and y consists of all points (x, y) that correspond to the solutions of the equation. We plot a sufficient number of points and connect them (where appropriate) so that the basic shape of the graph is apparent. Points where the graph intersects the

x- and y-axes are called x-intercepts and y-intercepts, respectively. An x-intercept is found by letting y be 0 and solving for x; a y-intercept is found by letting x be 0 and solving for y.

The graph of a function f is the graph of the equation $y = f(x)$ and consists of all points $(x, f(x))$ such that x is in the domain of f. From the graph of a function, it is easy to determine the domain and range.

The fact that a graph represents a function can be determined by using the vertical-line test. A vertical line cannot cut the graph of a function at more than one point.

The fact that a function is one-to-one can be determined by using the horizontal-line test on its graph. A horizontal line cannot cut the graph of a one-to-one function at more than one point. When a function passes the horizontal-line test, the graph of the inverse can be obtained by reflecting the original graph in the line $y = x$.

When the graph of an equation has symmetry, the mirror-image effect allows us to sketch the graph by plotting fewer points than would otherwise be needed. The tests for symmetry are as follows:

Symmetry about x-axis	Replace y by $-y$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about y-axis	Replace x by $-x$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about origin	Replace x by $-x$ and y by $-y$ in given equation. Symmetric if equivalent equation is obtained.
Symmetry about $y = x$	Interchange x and y in given equation. Symmetric if equivalent equation is obtained.

Sometimes the graph of a function can be obtained from that of a familiar function by means of a vertical shift upward or downward, a horizontal shift to the right or left, a reflection about the x-axis or y-axis, or a vertical stretching or shrinking away from or toward the x-axis. Such transformations are indicated in Table 2.2 in Section 2.7.

A function of two variables is a function whose domain consists of ordered pairs. A function of n variables is a function whose domain consists of ordered n-tuples. The graph of a real-valued function of two variables requires a three-dimensional coordinate system. Level curves provide another technique to visualize functions of two variables.

Review Problems

In Problems 1–6, give the domain of each function.

1. $f(x) = \dfrac{x}{x^2 - 6x + 5}$

2. $g(x) = x^2 + 3|x + 2|$

3. $F(t) = 7t + 4t^2$

4. $G(x) = 18$

5. $h(x) = \dfrac{\sqrt{x}}{x - 1}$

6. $H(s) = \dfrac{\sqrt{s - 5}}{4}$

In Problems 7–14, find the function values for the given function.

7. $f(x) = 2x^2 - 3x + 5$; $f(0)$, $f(-2)$, $f(5)$, $f(\pi)$

8. $h(x) = 7$; $h(4)$, $h\left(\dfrac{1}{100}\right)$, $h(-156)$, $h(x + 4)$

9. $G(x) = \sqrt[3]{x - 3}$; $G(3)$, $G(19)$, $G(t + 1)$, $G(x^3)$

10. $F(x) = \dfrac{x - 3}{x + 4}$; $F(-1)$, $F(0)$, $F(5)$, $F(x + 3)$

11. $h(u) = \dfrac{\sqrt{u + 4}}{u}$; $h(5)$, $h(-4)$, $h(x)$, $h(u - 4)$

12. $H(t) = \dfrac{(t - 2)^3}{5}$; $H(-1)$, $H(0)$, $H\left(\dfrac{1}{3}\right)$, $H(x^2)$

13. $f(x) = \begin{cases} -3 & \text{if } x < 1 \\ 4 + x^2 & \text{if } x > 1 \end{cases}$;

$f(4), f(-2), f(0), f(1)$

14. $f(q) = \begin{cases} -q + 1 & \text{if } -1 \le q < 0 \\ q^2 + 1 & \text{if } 0 \le q < 5 \\ q^3 - 99 & \text{if } 5 \le q \le 7 \end{cases}$;

$f\left(-\dfrac{1}{2}\right), f(0), f\left(\dfrac{1}{2}\right), f(5), f(6)$

In Problems 15–18, find (a) $f(x + h)$ and (b) $\dfrac{f(x + h) - f(x)}{h}$, and simplify your answers.

15. $f(x) = 3 - 7x$

16. $f(x) = 11x^2 + 4$

17. $f(x) = 3x^2 + x - 2$

18. $f(x) = \dfrac{7}{x + 1}$

19. If $f(x) = 3x - 1$ and $g(x) = 2x + 3$, find the following.

(a) $(f + g)(x)$ **(b)** $(f + g)(4)$ **(c)** $(f - g)(x)$

(d) $(fg)(x)$ **(e)** $(fg)(1)$ **(f)** $\dfrac{f}{g}(x)$

(g) $(f \circ g)(x)$ **(h)** $(f \circ g)(5)$ **(i)** $(g \circ f)(x)$

20. If $f(x) = -x^2$ and $g(x) = 3x - 2$, find the following.

(a) $(f + g)(x)$ **(b)** $(f - g)(x)$ **(c)** $(f - g)(-3)$

(d) $(fg)(x)$ **(e)** $\dfrac{f}{g}(x)$ **(f)** $\dfrac{f}{g}(2)$

(g) $(f \circ g)(x)$ **(h)** $(g \circ f)(x)$ **(i)** $(g \circ f)(-4)$

In Problems 21–24, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

21. $f(x) = \dfrac{1}{x^2}$, $g(x) = x + 1$

22. $f(x) = \dfrac{x - 2}{3}$, $g(x) = \dfrac{1}{\sqrt{x}}$

23. $f(x) = \sqrt{x + 2}$, $g(x) = x^3$

24. $f(x) = 2$, $g(x) = 3$

In Problems 25 and 26, find the intercepts of the graph of each equation, and test for symmetry about the x-axis, the y-axis, the origin, and $y = x$. Do not sketch the graph.

25. $y = 3x - x^3$

26. $\dfrac{x^2y^2}{x^2 + y^2 + 1} = 4$

In Problems 27 and 28, find the x- and y-intercepts of the graphs of the equations. Also, test for symmetry about the x-axis, the y-axis, and the origin. Then sketch the graphs.

27. $y = 4 + x^2$ **28.** $y = 3x - 7$

In Problems 29–32, graph each function and give its domain and range. Also, determine the intercepts.

29. $G(u) = \sqrt{u+4}$ **30.** $f(x) = |x| + 1$

31. $y = g(t) = \dfrac{2}{|t-4|}$ **32.** $v = \phi(u) = \sqrt{-u}$

33. Graph the following case-defined function, and give its domain and range:

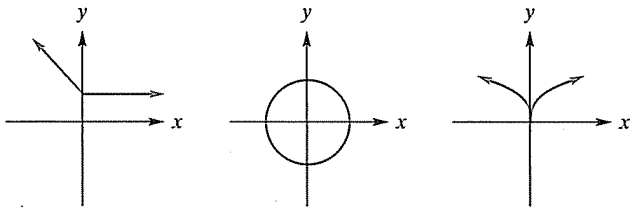
$$y = f(x) = \begin{cases} 2 & \text{if } x \le 0 \\ 2 - x & \text{if } x > 0 \end{cases}$$

34. Use the graph of $f(x) = \sqrt{x}$ to sketch the graph of $y = \sqrt{x-2} - 1$.

35. Use the graph of $f(x) = x^2$ to sketch the graph of $y = -\frac{1}{2}x^2 + 2$.

36. Trend Equation The projected annual sales (in dollars) of a new product are given by the equation $S = 150{,}000 + 3000t$, where t is the time in years from 2005. Such an equation is called a *trend equation*. Find the projected annual sales for 2010. Is S a function of t?

37. In Figure 2.47, which graphs represent functions of x?



(a) (b) (c)

FIGURE 2.47 Diagram for Problem 37.

38. If $f(x) = (x^2 - x + 7)^3$, find **(a)** $f(2)$ and **(b)** $f(1.1)$. Round your answers to two decimal places.

39. Find all real roots of the equation

$$5x^3 - 7x^2 = 4x - 2$$

Round your answers to two decimal places.

40. Find all real roots of the equation

$$x^4 - 4x^3 = (2x - 1)^2$$

Round your answers to two decimal places.

41. Find all real zeros of

$$f(x) = x(2.1x^2 - 3)^2 - x^3 + 1$$

Round your answers to two decimal places.

42. Determine the range of

$$f(x) = \begin{cases} -2.5x - 4 & \text{if } x < 0 \\ 6 + 4.1x - x^2 & \text{if } x \ge 0 \end{cases}$$

43. From the graph of $f(x) = -x^3 + 0.04x + 7$, find **(a)** the range and **(b)** the intercepts. Round values to two decimal places.

44. From the graph of $f(x) = \sqrt{x+5}(x^2 - 4)$, find **(a)** the minimum value of $f(x)$ **(b)** the range of f, and **(c)** all real zeros of f. Round values to two decimal places.

45. Graph $y = f(x) = x^2 + x^k$ for $k = 0, 1, 2, 3$, and 4. For which values of k does the graph have **(a)** symmetry about the y-axis? **(b)** symmetry about the origin?

46. Sketch the graph of $x + 2y + 3z = 6$.

47. Sketch the graph of $3x + y + 5z = 10$.

48. Construct three level curves for $P = 5x + 7y$.

49. Construct three level curves for $C = 2x + 10y$.

EXPLORE & EXTEND A Taxing Experience!

Occasionally, you will hear someone complain that an unexpected source of income is going to *bump* him or her *into the next tax bracket* with the further speculation that this will result in a *reduction in take-home earnings*. It is true that U.S. federal income tax is prescribed by case-defined functions (the cases defining what are often called *brackets*), but we will see that there are no *jumps* in tax paid as a function of income. It is an urban myth that an increase in pretax income can result in a decrease in take-home income.

We will examine the 2008 federal tax rates for a married couple filing a joint return. The relevant document is Schedule Y-1, available at `http://www.irs.gov/` and partially shown in Figure 2.48.

Schedule Y-1—Use if your **2008** filing status is **Married filing jointly** or **Qualifying widow(er)**

If line 5 is:		**The tax is:**	*of the amount over—*
Over—	*But not over—*		
$0	$16,050	..... 10%	$0
16,050	65,100	$1,605.00 + 15%	16,050
65,100	131,450	8,962.50 + 25%	65,100
131,450	200,300	25,550.00 + 28%	131,450
200,300	357,700	44,828.00 + 33%	200,300
357,700	.....	96,770.00 + 35%	357,700

FIGURE 2.48 Internal Revenue Service 2008 Schedule Y-1.

3 Lines, Parabolas, and Systems

3.1 Lines

3.2 Applications and Linear Functions

3.3 Quadratic Functions

3.4 Systems of Linear Equations

3.5 Nonlinear Systems

3.6 Applications of Systems of Equations

Chapter 3 Review

EXPLORE & EXTEND
Mobile Phone Billing Plans

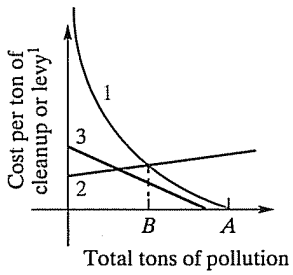
For the problem of industrial pollution, some people advocate a market-based solution: Let manufacturers pollute, but make them pay for the privilege. The more pollution, the greater the fee, or levy. The idea is to give manufacturers an incentive not to pollute more than necessary.

Does this approach work? In the figure below, curve 1 represents the cost per ton[1] of cutting pollution. A company polluting indiscriminately can normally do some pollution reduction at a small cost. As the amount of pollution is reduced, however, the cost per ton of further reduction rises and ultimately increases without bound. This is illustrated by curve 1 rising indefinitely as the total tons of pollution produced approaches 0. (You should try to understand why this *model* is reasonably accurate.)

Line 2 is a levy scheme that goes easy on clean-running operations but charges an increasing per-ton fee as the total pollution amount goes up. Line 3, by contrast, is a scheme in which low-pollution manufacturers pay a high per-ton levy while gross polluters pay less per ton (but more overall). Questions of fairness aside, how well will each scheme work as a pollution control measure?

Faced with a pollution levy, a company tends to cut pollution *so long as it saves more in levy costs than it incurs in reduction costs*. The reduction efforts continue until the reduction costs exceed the levy savings.

The latter half of this chapter deals with systems of equations. Here, curve 1 and line 2 represent one system of equations, and curve 1 and line 3 represent another. Once you have learned how to solve systems of equations, you can return to this page and verify that the line 2 scheme leads to a pollution reduction from amount A to amount B, while the line 3 scheme fails as a pollution control measure, leaving the pollution level at A.



[1]Technically, this is the *marginal* cost per ton (see Section 11.3).

Objective

To develop the notion of slope and different forms of equations of lines.

3.1 Lines

Slope of a Line

Many relationships between quantities can be represented conveniently by straight lines. One feature of a straight line is its "steepness." For example, in Figure 3.1, line L_1 rises faster as it goes from left to right than does line L_2. In this sense, L_1 is steeper.

To measure the steepness of a line, we use the notion of *slope*. In Figure 3.2, as we move along line L from (1, 3) to (3, 7), the x-coordinate increases from 1 to 3, and the y-coordinate increases from 3 to 7. The average rate of change of y with respect to x is the ratio

$$\frac{\text{change in } y}{\text{change in } x} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{7-3}{3-1} = \frac{4}{2} = 2$$

The ratio of 2 means that for each 1-unit increase in x, there is a 2-unit *increase* in y. Due to the increase, the line *rises* from left to right. It can be shown that, regardless of which two points on L are chosen to compute the ratio of the change in y to the change in x, the result is always 2, which we call the *slope* of the line.

Definition

Let (x_1, y_1) and (x_2, y_2) be two different points on a nonvertical line. The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \left(= \frac{\text{vertical change}}{\text{horizontal change}} \right) \tag{1}$$

CAUTION!

Having no slope does not mean having a slope of zero.

A vertical line does not have a slope, because any two points on it must have $x_1 = x_2$ [see Figure 3.3(a)], which gives a denominator of zero in Equation (1). For a horizontal line, any two points must have $y_1 = y_2$. [See Figure 3.3(b).] This gives a numerator of zero in Equation (1), and hence the slope of the line is zero.

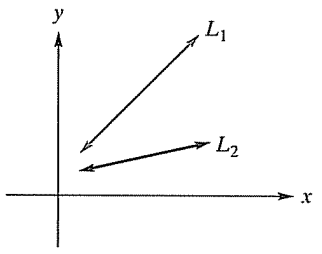


FIGURE 3.1 Line L_1 is "steeper" than L_2.

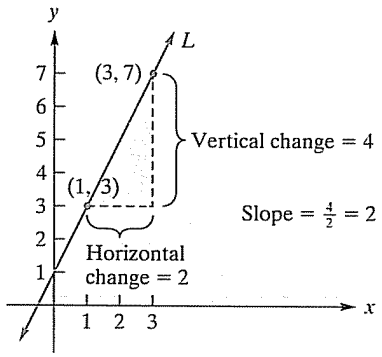
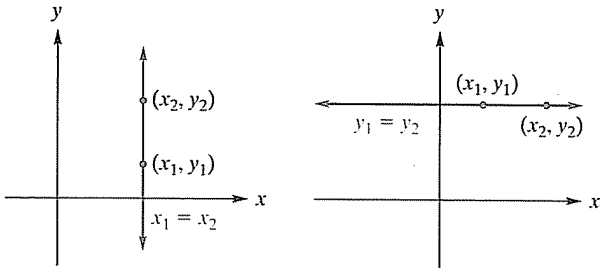


FIGURE 3.2 Slope of a line.



(a) No slope (b) Slope of zero

FIGURE 3.3 Vertical and horizontal lines.

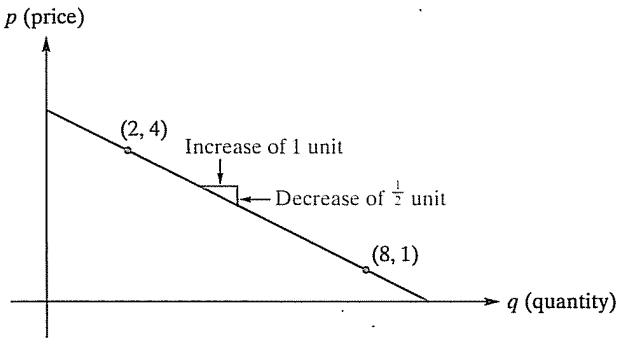


FIGURE 3.4 Price–quantity line.

This example shows how the slope can be interpreted.

APPLY IT ▶

1. A doctor purchased a new car in 2001 for \$32,000. In 2004, he sold it to a friend for \$26,000. Draw a line showing the relationship between the selling price of the car and the year in which it was sold. Find and interpret the slope.

EXAMPLE 1 **Price–Quantity Relationship**

The line in Figure 3.4 shows the relationship between the price p of a widget (in dollars) and the quantity q of widgets (in thousands) that consumers will buy at that price. Find and interpret the slope.

Solution: In the slope formula (1), we replace the x's by q's and the y's by p's. Either point in Figure 3.4 may be chosen as (q_1, p_1). Letting $(2, 4) = (q_1, p_1)$ and $(8, 1) = (q_2, p_2)$, we have

$$m = \frac{p_2 - p_1}{q_2 - q_1} = \frac{1 - 4}{8 - 2} = \frac{-3}{6} = -\frac{1}{2}$$

The slope is negative, $-\frac{1}{2}$. This means that, for each 1-unit increase in quantity (one thousand widgets), there corresponds a **decrease** in price of $\frac{1}{2}$ (dollar per widget). Due to this decrease, the line **falls** from left to right.

Now Work Problem 3 ◁

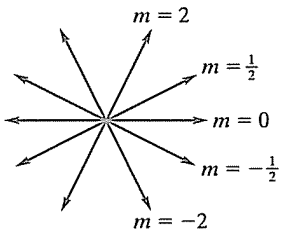


FIGURE 3.5 Slopes of lines.

In summary, we can characterize the orientation of a line by its slope:

Zero slope:	horizontal line
Undefined slope:	vertical line
Positive slope:	line rises from left to right
Negative slope:	line falls from left to right

Lines with different slopes are shown in Figure 3.5. Notice that *the closer the slope is to 0, the more nearly horizontal is the line. The greater the absolute value of the slope, the more nearly vertical is the line.* We remark that two lines are parallel if and only if they have the same slope or are both vertical.

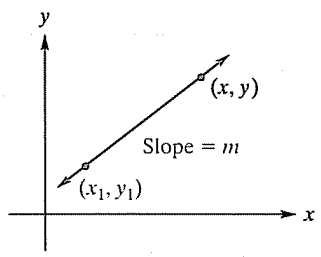


FIGURE 3.6 Line through (x_1, y_1) with slope m.

Equations of Lines

If we know a point on a line and the slope of the line, we can find an equation whose graph is that line. Suppose that line L has slope m and passes through the point (x_1, y_1). If (x, y) is *any* other point on L (see Figure 3.6), we can find an algebraic relationship between x and y. Using the slope formula on the points (x_1, y_1) and (x, y) gives

$$\frac{y - y_1}{x - x_1} = m$$

$$y - y_1 = m(x - x_1) \tag{2}$$

Every point on L satisfies Equation (2). It is also true that *every* point satisfying Equation (2) must lie on L. Thus, Equation (2) is an equation for L and is given a special name:

$$y - y_1 = m(x - x_1)$$

is a ***point-slope form*** *of an equation of the line through* (x_1, y_1) *with slope* m.

EXAMPLE 2 **Point-Slope Form**

APPLY IT ▷

2. A new applied mathematics program at a university has grown in enrollment by 14 students per year for the last five years. If the program enrolled 50 students in its third year, what is an equation for the number of students S in the program as a function of the number of years T since its inception?

Find an equation of the line that has slope 2 and passes through $(1, -3)$.

Solution: Using a point-slope form with $m = 2$ and $(x_1, y_1) = (1, -3)$ gives

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = 2(x - 1)$$
$$y + 3 = 2x - 2$$

which can be rewritten as

$$2x - y - 5 = 0$$

Now Work Problem 9 ◁

An equation of the line passing through two given points can be found easily, as Example 3 shows.

EXAMPLE 3 **Determining a Line from Two Points**

APPLY IT ▷

3. Find an equation of the line passing through the given points. A temperature of 41°F is equivalent to 5°C, and a temperature of 77°F is equivalent to 25°C.

Find an equation of the line passing through $(-3, 8)$ and $(4, -2)$.

Solution:

Strategy First we find the slope of the line from the given points. Then we substitute the slope and one of the points into a point-slope form.

The line has slope

$$m = \frac{-2 - 8}{4 - (-3)} = -\frac{10}{7}$$

Choosing $(4, -2)$ as (x_1, y_1) would give the same result.

Using a point-slope form with $(-3, 8)$ as (x_1, y_1) gives

$$y - 8 = -\frac{10}{7}[x - (-3)]$$
$$y - 8 = -\frac{10}{7}(x + 3)$$
$$7y - 56 = -10x - 30$$
$$10x + 7y - 26 = 0$$

Now Work Problem 13 ◁

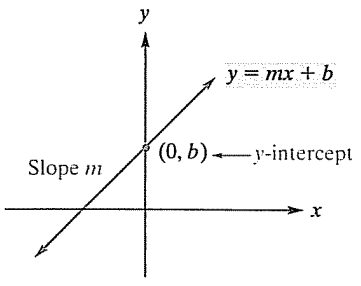


FIGURE 3.7 Line with slope m and y-intercept b.

Recall that a point $(0, b)$ where a graph intersects the y-axis is called a y-intercept (Figure 3.7). If the slope m and y-intercept b of a line are known, an equation for the line is [by using a point-slope form with $(x_1, y_1) = (0, b)$]

$$y - b = m(x - 0)$$

Solving for y gives $y = mx + b$, called the *slope-intercept form* of an equation of the line:

$$y = mx + b$$

is the ***slope-intercept form*** *of an equation of the line with slope* m *and* y*-intercept* b.

EXAMPLE 4 **Slope-Intercept Form**

Find an equation of the line with slope 3 and y-intercept -4.

Solution: Using the slope-intercept form $y = mx + b$ with $m = 3$ and $b = -4$ gives

$$y = 3x + (-4)$$
$$y = 3x - 4$$

Now Work Problem 17 ◁

APPLY IT ▷

4. One formula for the recommended dosage (in milligrams) of medication for a child t years old is

$$y = \frac{1}{24}(t + 1)a$$

where a is the adult dosage. For an over-the-counter pain reliever, $a = 1000$. Find the slope and y-intercept of this equation.

EXAMPLE 5 **Find the Slope and y-Intercept of a Line**

Find the slope and y-intercept of the line with equation $y = 5(3 - 2x)$.

Solution:

Strategy We will rewrite the equation so it has the slope-intercept form $y = mx + b$. Then the slope is the coefficient of x and the y-intercept is the constant term.

We have

$$y = 5(3 - 2x)$$
$$y = 15 - 10x$$
$$y = -10x + 15$$

Thus, $m = -10$ and $b = 15$, so the slope is -10 and the y-intercept is 15.

Now Work Problem 25 ◁

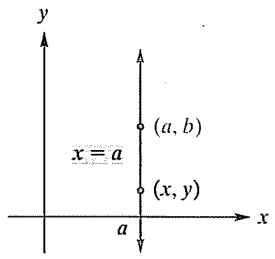


FIGURE 3.8 Vertical line through (a, b).

If a *vertical* line passes through (a, b) (see Figure 3.8), then any other point (x, y) lies on the line if and only if $x = a$. The y-coordinate can have any value. Hence, an equation of the line is $x = a$. Similarly, an equation of the *horizontal* line passing through (a, b) is $y = b$. (See Figure 3.9.) Here the x-coordinate can have any value.

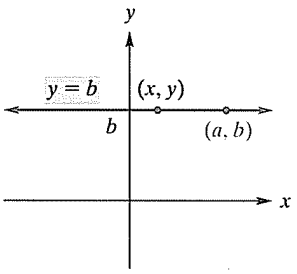


FIGURE 3.9 Horizontal line through (a, b).

EXAMPLE 6 **Equations of Horizontal and Vertical Lines**

a. An equation of the vertical line through $(-2, 3)$ is $x = -2$. An equation of the horizontal line through $(-2, 3)$ is $y = 3$.

b. The x-axis and y-axis are horizontal and vertical lines, respectively. Because $(0, 0)$ lies on both axes, an equation of the x-axis is $y = 0$, and an equation of the y-axis is $x = 0$.

Now Work Problems 21 and 23 ◁

From our discussions, we can show that every straight line is the graph of an equation of the form $Ax + By + C = 0$, where A, B, and C are constants and A and B are not both zero. We call this a **general linear equation** (or an *equation of the first degree*) ***in the variables x and y,*** and x and y are said to be **linearly related.** For example, a general linear equation for $y = 7x - 2$ is $(-7)x + (1)y + (2) = 0$. Conversely, the graph of a general linear equation is a straight line. Table 3.1 gives the various forms of equations of straight lines.

Table 3.1 Forms of Equations of Straight Lines

Point-slope form	$y - y_1 = m(x - x_1)$
Slope-intercept form	$y = mx + b$
General linear form	$Ax + By + C = 0$
Vertical line	$x = a$
Horizontal line	$y = b$

CAUTION!

Do not confuse the forms of equations of horizontal and vertical lines. Remember which one has the form $x =$ constant and which one has the form $y =$ constant.

Example 3 suggests that we could add another entry to the table. For if we know that points (x_1, y_1) and (x_2, y_2) are points on a line, then the slope of that line is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and we could say that $y - y_1 = \dfrac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ *is a* ***two-point form*** *for an equation of a line passing through points* (x_1, y_1) *and* (x_2, y_2). Whether one chooses to remember many formulas or a few problem-solving principles is very much a matter of individual taste.

APPLY IT ▶

5. Find a general linear form of the Fahrenheit–Celsius conversion equation whose slope-intercept form is $F = \frac{9}{5}C + 32$.

EXAMPLE 7 Converting Forms of Equations of Lines

a. Find a general linear form of the line whose slope-intercept form is

$$y = -\frac{2}{3}x + 4$$

Solution: Getting one side to be 0, we obtain

$$\frac{2}{3}x + y - 4 = 0$$

which is a general linear form with $A = \frac{2}{3}$, $B = 1$, and $C = -4$. An alternative general form can be obtained by clearing fractions:

$$2x + 3y - 12 = 0$$

CAUTION!

This illustrates that a general linear form of a line is not unique.

b. Find the slope-intercept form of the line having a general linear form $3x + 4y - 2 = 0$.

Solution: We want the form $y = mx + b$, so we solve the given equation for y. We have

$$\begin{aligned} 3x + 4y - 2 &= 0 \\ 4y &= -3x + 2 \\ y &= -\frac{3}{4}x + \frac{1}{2} \end{aligned}$$

which is the slope-intercept form. Note that the line has slope $-\frac{3}{4}$ and y-intercept $\frac{1}{2}$.

APPLY IT ▶

6. Sketch the graph of the Fahrenheit–Celsius conversion equation that you found in the preceding Apply it. How could you use this graph to convert a Celsius temperature to Fahrenheit?

Now Work Problem 37 ◁

EXAMPLE 8 Graphing a General Linear Equation

Sketch the graph of $2x - 3y + 6 = 0$.

Solution:

Strategy Since this is a general linear equation, its graph is a straight line. Thus, we need only determine two different points on the graph in order to sketch it. We will find the intercepts.

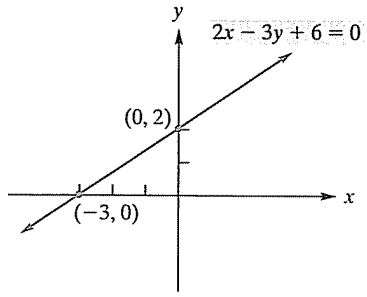


FIGURE 3.10 Graph of $2x - 3y + 6 = 0$.

If $x = 0$, then $-3y + 6 = 0$, so the y-intercept is 2. If $y = 0$, then $2x + 6 = 0$, so the x-intercept is -3. We now draw the line passing through (0, 2) and (−3, 0). (See Figure 3.10.)

Now Work Problem 27 ◁

TECHNOLOGY

To graph the equation of Example 8 with a graphing calculator, we first express y in terms of x:

$$2x - 3y + 6 = 0$$
$$3y = 2x + 6$$
$$y = \frac{1}{3}(2x + 6)$$

Essentially, y is expressed as a function of x; the graph is shown in Figure 3.11.

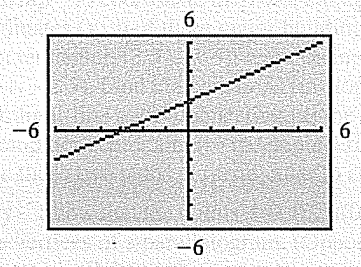


FIGURE 3.11 Calculator graph of $2x - 3y + 6 = 0$.

Parallel and Perpendicular Lines

As stated previously, there is a rule for parallel lines:

Parallel Lines *Two lines are parallel if and only if they have the same slope or are both vertical.*

It follows that any line is parallel to itself.

There is also a rule for perpendicular lines. Look back to Figure 3.5 and observe that the line with slope $-\frac{1}{2}$ is perpendicular to the line with slope 2. The fact that the slope of either of these lines is the negative reciprocal of the slope of the other line is not a coincidence, as the following rule states.

Perpendicular Lines *Two lines with slopes m_1 and m_2 are perpendicular to each other if and only if*

$$m_1 = -\frac{1}{m_2}$$

Moreover, any horizontal line and any vertical line are perpendicular to each other.

Rather than simply remembering this equation for the perpendicularity condition, observe why it makes sense. For if two lines are perpendicular, with neither vertical, then one will necessarily rise from left to right while the other will fall from left to right. Thus the slopes must have different signs. Also, if one is steep, then the other is relatively flat, which suggests a relationship such as is provided by reciprocals.

APPLY IT ▶

7. Show that a triangle with vertices at $A(0, 0)$, $B(6, 0)$, and $C(7, 7)$ is not a right triangle.

EXAMPLE 9 Parallel and Perpendicular Lines

Figure 3.12 shows two lines passing through $(3, -2)$. One is parallel to the line $y = 3x + 1$, and the other is perpendicular to it. Find equations of these lines.

Solution: The slope of $y = 3x + 1$ is 3. Thus, the line through $(3, -2)$ that is *parallel* to $y = 3x + 1$ also has slope 3. Using a point-slope form, we get

$$y - (-2) = 3(x - 3)$$
$$y + 2 = 3x - 9$$
$$y = 3x - 11$$

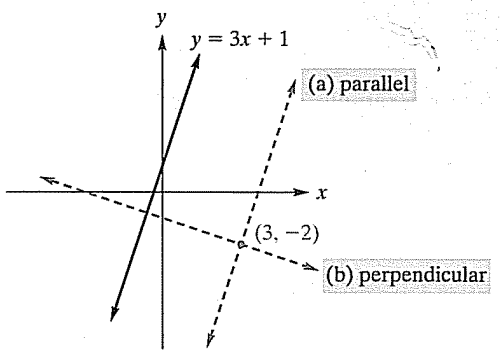


FIGURE 3.12 Lines parallel and perpendicular to $y = 3x + 1$ (Example 9).

The slope of a line *perpendicular* to $y = 3x + 1$ must be $-\frac{1}{3}$ (the negative reciprocal of 3). Using a point-slope form, we get

$$y - (-2) = -\frac{1}{3}(x - 3)$$

$$y + 2 = -\frac{1}{3}x + 1$$

$$y = -\frac{1}{3}x - 1$$

Now Work Problem 55 ◁

PROBLEMS 3.1

In Problems 1–8, find the slope of the straight line that passes through the given points.

1. (3, 2), (7, 10)
2. (−2, 10), (5, 3)
3. (6, −2), (8, −3)
4. (2, −4), (3, −4)
5. (5, 3), (5, −8)
6. (0, −4), (3, 6)
7. (5, −2), (4, −2)
8. (1, −7), (9, 0)

In Problems 9–24, find a general linear equation ($Ax + By + C = 0$) of the straight line that has the indicated properties, and sketch each line.

9. Passes through (−1, 7) and has slope −5
10. Passes through the origin and has slope 75
11. Passes through (−5, 5) and has slope $-\frac{1}{2}$
12. Passes through $(-\frac{5}{2}, 5)$ and has slope $\frac{1}{3}$
13. Passes through (−6, 1) and (1, 4)
14. Passes through (5, 2) and (6, −4)
15. Passes through (−3, −4) and (−2, −8)
16. Passes through (0, 0) and (−3, −2)
17. Has slope 2 and y-intercept 4
18. Has slope 5 and y-intercept −7
19. Has slope $-\frac{1}{2}$ and y-intercept −3
20. Has slope 0 and y-intercept $-\frac{1}{2}$
21. Is horizontal and passes through (−2, −5)
22. Is vertical and passes through (−1, −1)
23. Passes through (2, −3) and is vertical
24. Passes through the origin and is horizontal

In Problems 25–34, find, if possible, the slope and y-intercept of the straight line determined by the equation, and sketch the graph.

25. $y = 4x - 6$
26. $x + 9 = 2$
27. $3x + 5y - 9 = 0$
28. $y + 4 = 7$
29. $x = -5$
30. $x - 9 = 5y + 3$
31. $y = -2x$
32. $y - 7 = 3(x - 4)$
33. $y = 3$
34. $6y - 24 = 0$

In Problems 35–40, find a general linear form and the slope-intercept form of the given equation.

35. $2x = 5 - 3y$
36. $5x - 2y = 10$
37. $4x + 9y - 5 = 0$
38. $3(x - 4) - 7(y + 1) = 2$
39. $-\frac{x}{2} + \frac{2y}{3} = -4\frac{3}{4}$
40. $y = \frac{1}{300}x + 8$

In Problems 41–50, determine whether the lines are parallel, perpendicular, or neither.

41. $y = -5x + 7,\ y = -5x - 3$
42. $y = 4x + 3,\ y = 5 + 4x$
43. $y = 5x + 2,\ -5x + y - 3 = 0$
44. $y = x,\ y = -x$
45. $x + 3y + 5 = 0,\ y = -3x$
46. $x + 2y = 0, x + 4y - 4 = 0$
47. $y = 3,\ x = -\frac{1}{3}$
48. $x = 3,\ x = -3$
49. $3x + y = 4,\ x - 3y + 1 = 0$
50. $x - 2 = 3,\ y = 2$

In Problems 51–60, find an equation of the line satisfying the given conditions. Give the answer in slope-intercept form if possible.

51. Passing through (2, 3) and parallel to $y = 4x + 3$

52. Passing through (2, −8) and parallel to $x = -4$

53. Passing through (2, 1) and parallel to $y = 2$

54. Passing through (3, −4) and parallel to $y = 3 + 2x$

55. Perpendicular to $y = 3x - 5$ and passing through (3, 4)

56. Perpendicular to $3x + 2y - 4 = 0$ and passing through (3, 1)

57. Passing through (5, 2) and perpendicular to $y = -3$

58. Passing through (4, −5) and perpendicular to the line $3y = -\frac{2x}{5} + 3$

59. Passing through (−7, −5) and parallel to the line $2x + 3y + 6 = 0$

60. Passing through (−4, 10) and parallel to the y-axis

61. A straight line passes through (−1, −2) and (4, 1). Find the point on it that has an x-coordinate of 3.

62. A straight line has slope 3 and y-intercept (0, 1). Does the point (−1, −2) lie on the line?

63. Stock In 1996, the stock in a computer hardware company traded for \$37 per share. However, the company was in trouble and the stock price dropped steadily, to \$8 per share in 2006. Draw a line showing the relationship between the price per share and the year in which it traded for the time interval [1996, 2006], with years on the x-axis and price on the y-axis. Find and interpret the slope.

In Problems 64–65, find an equation of the line describing the following information.

64. Home Runs In one season, a major league baseball player has hit 14 home runs by the end of the third month and 20 home runs by the end of the fifth month.

65. Business A delicatessen owner starts her business with debts of \$100,000. After operating for five years, she has accumulated a profit of \$40,000.

66. Due Date The length, L, of a human fetus more than 12 weeks old can be estimated by the formula $L = 1.53t - 6.7$, where L is in centimeters and t is in weeks from conception. An obstetrician uses the length of a fetus, measured by ultrasound, to determine the approximate age of the fetus and establish a due date for the mother. The formula must be rewritten to result in an age, t, given a fetal length, L. Find the slope and L-intercept of the equation.

67. Discus Throw A mathematical model can approximate the winning distance for the Olympic discus throw by the formula $d = 184 + t$, where d is in feet and $t = 0$ corresponds to the year 1948. Find a general linear form of this equation.

68. Campus Map A coordinate map of a college campus gives the coordinates (x, y) of three major buildings as follows: computer center, (3.5, −1.5); engineering lab, (0.5, 0.5); and library (−1, −2.5). Find the equations (in slope-intercept form) of the straight-line paths connecting (a) the engineering lab with the computer center and (b) the engineering lab with the library. Are these two paths perpendicular to each other?

69. Geometry Show that the points $A(0, 0)$, $B(0, 4)$, $C(2, 3)$, and $D(2, 7)$ are the vertices of a parallelogram. (Opposite sides of a parallelogram are parallel.)

70. Approach Angle A small plane is landing at an airport with an approach angle of 45 degrees, or slope of −1. The plane begins its descent when it has an elevation of 3600 feet. Find the equation that describes the relationship between the craft's altitude and distance traveled, assuming that at distance 0 it starts the approach angle. Graph your equation on a graphing calculator. What does the graph tell you about the approach if the airport is 3800 feet from where the plane starts its landing?

71. Cost Equation The average daily cost, C, for a room at a city hospital has risen by \$59.82 per year for the years 1990 through 2000. If the average cost in 1996 was \$1128.50, what is an equation which describes the average cost during this decade, as a function of the number of years, T, since 1990?

72. Revenue Equation A small business predicts its revenue growth by a straight-line method with a slope of \$50,000 per year. In its fifth year, it had revenues of \$330,000. Find an equation that describes the relationship between revenues, R, and the number of years, T, since it opened for business.

73. Graph $y = -0.9x + 7.3$ and verify that the y-intercept is 7.3.

74. Graph the lines whose equations are

$$y = 1.5x + 1$$

$$y = 1.5x - 1$$

and

$$y = 1.5x + 2.5$$

What do you observe about the orientation of these lines? Why would you expect this result from the equations of the lines themselves?

75. Graph the line $y = 7.1x + 5.4$. Find the coordinates of any two points on the line, and use them to estimate the slope. What is the actual slope of the line?

76. Show that if a line has x-intercept a and y-intercept b, both different from 0, then $\frac{x}{a} + \frac{y}{b} = 1$ is an equation of the line.

3.2 Applications and Linear Functions

Objective

To develop the notion of demand and supply curves and to introduce linear functions.

Many situations in economics can be described by using straight lines, as evidenced by Example 1.

EXAMPLE 1 Production Levels

Suppose that a manufacturer uses 100 lb of material to produce products A and B, which require 4 lb and 2 lb of material per unit, respectively. If x and y denote the number of units produced of A and B, respectively, then all levels of production are given by the

APPLY IT ▷

8. A sporting-goods manufacturer allocates 1000 units of time per day to make skis and ski boots. If it takes 8 units of time to make a ski and 14 units of time to make a boot, find an equation to describe all possible production levels of the two products.

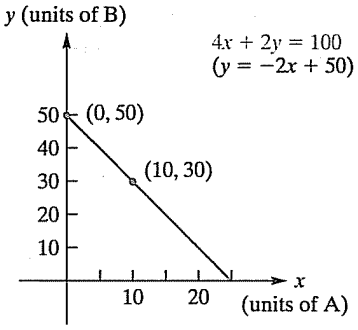


FIGURE 3.13 Linearly related production levels.

CAUTION!

Typically, a demand curve falls from left to right and a supply curve rises from left to right. However, there are exceptions. For example, the demand for insulin could be represented by a vertical line, since this demand can remain constant regardless of price.

combinations of x and y that satisfy the equation

$$4x + 2y = 100 \qquad \text{where } x,\ y \geq 0$$

Thus, the levels of production of A and B are linearly related. Solving for y gives

$$y = -2x + 50 \qquad \text{slope-intercept form}$$

so the slope is -2. The slope reflects the rate of change of the level of production of B with respect to the level of production of A. For example, if 1 more unit of A is to be produced, it will require 4 more pounds of material, resulting in $\frac{4}{2} = 2$ *fewer* units of B. Accordingly, as x increases by 1 unit, the corresponding value of y decreases by 2 units. To sketch the graph of $y = -2x + 50$, we can use the y-intercept (0, 50) and the fact that when $x = 10, y = 30$. (See Figure 3.13.)

Now Work Problem 21 ◁

Demand and Supply Curves

For each price level of a product, there is a corresponding quantity of that product that consumers will demand (that is, purchase) during some time period. Usually, the higher the price, the smaller is the quantity demanded; as the price falls, the quantity demanded increases. If the price per unit of the product is given by p and the corresponding quantity (in units) is given by q, then an equation relating p and q is called a **demand equation.** Its graph is called a **demand curve.** Figure 3.14(a) shows a demand curve. In keeping with the practice of most economists, the horizontal axis is the q-axis and the vertical axis is the p-axis. We will assume that the price per unit is given in dollars and the period is one week. Thus, the point (a, b) in Figure 3.14(a) indicates that, at a price of b dollars per unit, consumers will demand a units per week. Since negative prices or quantities are not meaningful, both a and b must be nonnegative. For most products, an increase in the quantity demanded corresponds to a decrease in price. Thus, a demand curve typically falls from left to right, as in Figure 3.14(a).

In response to various prices, there is a corresponding quantity of product that *producers* are willing to supply to the market during some time period. Usually, the higher the price per unit, the larger is the quantity that producers are willing to supply; as the price falls, so will the quantity supplied. If p denotes the price per unit and q denotes the corresponding quantity, then an equation relating p and q is called a **supply equation,** and its graph is called a **supply curve.** Figure 3.14(b) shows a supply curve. If p is in dollars and the period is one week, then the point (c, d) indicates that, at a price of d dollars each, producers will supply c units per week. As before, c and d are nonnegative. A supply curve usually rises from left to right, as in Figure 3.14(b). This indicates that a producer will supply more of a product at higher prices.

Observe that a function whose graph either falls from left to right or rises from left to right *throughout its entire domain* will pass the horizontal line test of Section 2.5. Certainly, the demand curve and the supply curve in Figure 3.15 are each cut at most once by any horizontal line. Thus if the demand curve is the graph of a function $p = D(q)$,

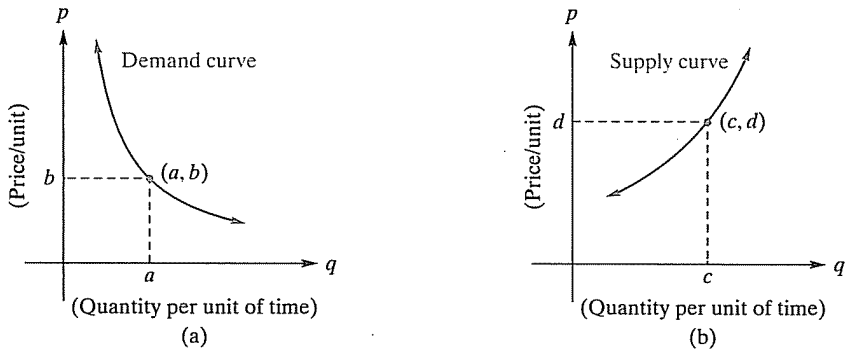


FIGURE 3.14 Demand and supply curves.

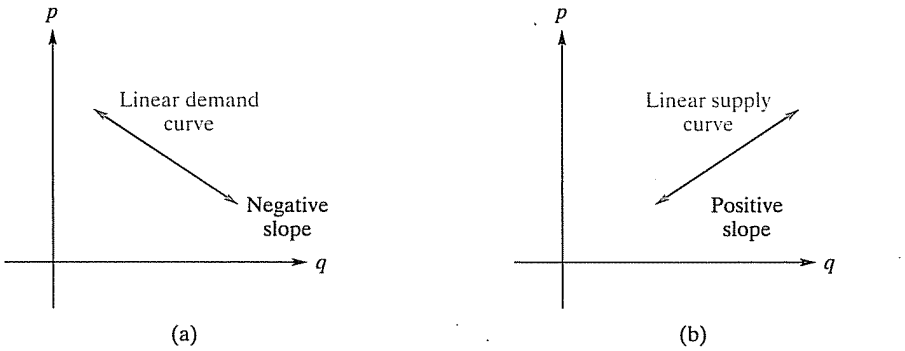


FIGURE 3.15 Linear demand and supply curves.

then D will have an inverse and we can solve for q uniquely to get $q = D^{-1}(p)$. Similarly, if the supply curve is the graph of a function $p = S(q)$, then S is also one-to-one, has an inverse S^{-1}, and we can write $q = S^{-1}(p)$.

We will now focus on demand and supply curves that are straight lines (Figure 3.15). They are called *linear* demand and *linear* supply curves. Such curves have equations in which p and q are linearly related. Because a demand curve typically falls from left to right, a linear demand curve has a negative slope. [See Figure 3.15(a).] However, the slope of a linear supply curve is positive, because the curve rises from left to right. [See Figure 3.15(b).]

APPLY IT ▶

9. The demand per week for 26-inch television sets is 1200 units when the price is \$575 each and 800 units when the price is \$725 each. Find the demand equation for the sets, assuming that it is linear.

EXAMPLE 2 Finding a Demand Equation

Suppose the demand per week for a product is 100 units when the price is \$58 per unit and 200 units at \$51 each. Determine the demand equation, assuming that it is linear.

Solution:

Strategy Since the demand equation is linear, the demand curve must be a straight line. We are given that quantity q and price p are linearly related such that $p = 58$ when $q = 100$ and $p = 51$ when $q = 200$. Thus, the given data can be represented in a q,p-coordinate plane [see Figure 3.15(a)] by points (100, 58) and (200, 51). With these points, we can find an equation of the line—that is, the demand equation.

The slope of the line passing through (100, 58) and (200, 51) is

$$m = \frac{51 - 58}{200 - 100} = -\frac{7}{100}$$

An equation of the line (point-slope form) is

$$p - p_1 = m(q - q_1)$$

$$p - 58 = -\frac{7}{100}(q - 100)$$

Simplifying gives the demand equation

$$p = -\frac{7}{100}q + 65 \tag{1}$$

Customarily, a demand equation (as well as a supply equation) expresses p, in terms of q and actually defines a function of q. For example, Equation (1) defines p as a function of q and is called the *demand function* for the product. (See Figure 3.16.)

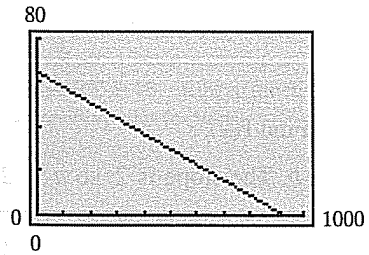
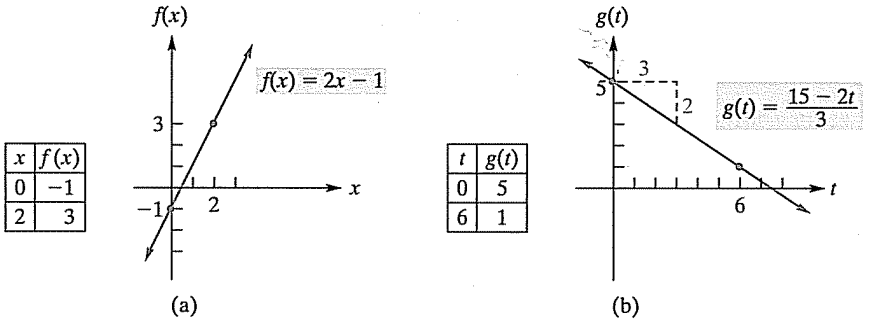


FIGURE 3.16 Graph of demand function $p = -\frac{7}{100}q + 65$.

Now Work Problem 15 ◁

Linear Functions

A *linear function* was defined in Section 2.2 to be a polynomial function of degree 1. Somewhat more explicitly,



x	$f(x)$
0	−1
2	3

t	$g(t)$
0	5
6	1

FIGURE 3.17 Graphs of linear functions.

Definition

A function f is a *linear function* if and only if $f(x)$ can be written in the form $f(x) = ax + b$, where a and b are constants and $a \neq 0$.

Suppose that $f(x) = ax + b$ is a linear function, and let $y = f(x)$. Then $y = ax + b$, which is an equation of a straight line with slope a and y-intercept b. Thus, **the graph of a linear function is a straight line that is neither vertical nor horizontal.** We say that the function $f(x) = ax + b$ has slope a.

APPLY IT ▷

10. A computer repair company charges a fixed amount plus an hourly rate for a service call. If x is the number of hours needed for a service call, the total cost of a call is described by the function $f(x) = 40x + 60$. Graph the function by finding and plotting two points.

EXAMPLE 3 Graphing Linear Functions

a. Graph $f(x) = 2x - 1$.

Solution: Here f is a linear function (with slope 2), so its graph is a straight line. Since two points determine a straight line, we need only plot two points and then draw a line through them. [See Figure 3.17(a).] Note that one of the points plotted is the vertical-axis intercept, -1, which occurs when $x = 0$.

b. Graph $g(t) = \dfrac{15 - 2t}{3}$.

Solution: Notice that g is a linear function, because we can express it in the form $g(t) = at + b$.

$$g(t) = \frac{15 - 2t}{3} = \frac{15}{3} - \frac{2t}{3} = -\frac{2}{3}t + 5$$

The graph of g is shown in Figure 3.17(b). Since the slope is $-\frac{2}{3}$, observe that as t increases by 3 units, $g(t)$ *decreases* by 2.

Now Work Problem 3 ◁

APPLY IT ▷

11. The height of children between the ages of 6 years and 10 years can be modeled by a linear function of age t in years. The height of one child changes by 2.3 inches per year, and she is 50.6 inches tall at age 8. Find a function that describes the height of this child at age t.

EXAMPLE 4 Determining a Linear Function

Suppose f is a linear function with slope 2 and $f(4) = 8$. Find f(x).

Solution: Since f is linear, it has the form $f(x) = ax + b$. The slope is 2, so $a = 2$, and we have

$$f(x) = 2x + b \tag{2}$$

Now we determine b. Since $f(4) = 8$, we replace x by 4 in Equation (2) and solve for b:

$$f(4) = 2(4) + b$$
$$8 = 8 + b$$
$$0 = b$$

Hence, $f(x) = 2x$.

Now Work Problem 7 ◁

APPLY IT ▶

12. An antique necklace is expected to be worth \$360 after 3 years and \$640 after 7 years. Find a function that describes the value of the necklace after x years.

EXAMPLE 5 **Determining a Linear Function**

If $y = f(x)$ is a linear function such that $f(-2) = 6$ and $f(1) = -3$, find f(x).

Solution:

Strategy The function values correspond to points on the graph of f. With these points we can determine an equation of the line and hence the linear function.

The condition that $f(-2) = 6$ means that when $x = -2$, then $y = 6$. Thus, $(-2, 6)$ lies on the graph of f, which is a straight line. Similarly, $f(1) = -3$ implies that $(1, -3)$ also lies on the line. If we set $(x_1, y_1) = (-2, 6)$ and $(x_2, y_2) = (1, -3)$, the slope of the line is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-3 - 6}{1 - (-2)} = \frac{-9}{3} = -3$$

We can find an equation of the line by using a point-slope form:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 6 &= -3[x - (-2)] \\ y - 6 &= -3x - 6 \\ y &= -3x \end{aligned}$$

Because $y = f(x)$, $f(x) = -3x$. Of course, the same result is obtained if we set $(x_1, y_1) = (1, -3)$.

Now Work Problem 9 ◁

In many studies, data are collected and plotted on a coordinate system. An analysis of the results may indicate a functional relationship between the variables involved. For example, the data points may be approximated by points on a straight line. This would indicate a linear functional relationship, such as the one in the next example.

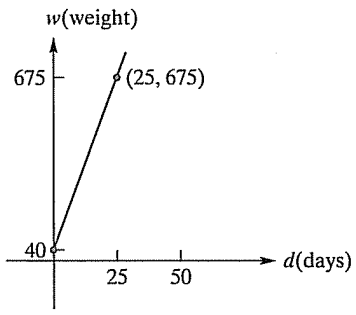


FIGURE 3.18 Linear function describing diet for hens.

EXAMPLE 6 **Diet for Hens**

In testing an experimental diet for hens, it was determined that the average live weight w (in grams) of a hen was statistically a linear function of the number of days d after the diet began, where $0 \le d \le 50$. Suppose the average weight of a hen beginning the diet was 40 grams and 25 days later it was 675 grams.

a. Determine w as a linear function of d.

Solution: Since w is a linear function of d, its graph is a straight line. When $d = 0$ (the beginning of the diet), $w = 40$. Thus, (0, 40) lies on the graph. (See Figure 3.18.) Similarly, (25, 675) lies on the graph. If we set $(d_1, w_1) = (0, 40)$ and $(d_2, w_2) = (25, 675)$, the slope of the line is

$$m = \frac{w_2 - w_1}{d_2 - d_1} = \frac{675 - 40}{25 - 0} = \frac{635}{25} = \frac{127}{5}$$

Using a point-slope form, we have

$$\begin{aligned} w - w_1 &= m(d - d_1) \\ w - 40 &= \frac{127}{5}(d - 0) \\ w - 40 &= \frac{127}{5}d \\ w &= \frac{127}{5}d + 40 \end{aligned}$$

which expresses w as a linear function of d.

b. Find the average weight of a hen when $d = 10$.

Solution: When $d = 10, w = \frac{127}{5}(10) + 40 = 254 + 40 = 294$. Thus, the average weight of a hen 10 days after the beginning of the diet is 294 grams.

Now Work Problem 19 ◁

PROBLEMS 3.2

In Problems 1–6, find the slope and vertical-axis intercept of the linear function, and sketch the graph.

1. $y = f(x) = -4x$
2. $y = f(x) = x + 1$
3. $h(t) = 5t - 7$
4. $f(s) = 3(5 - s)$
5. $p(q) = \dfrac{5 - q}{3}$
6. $h(q) = 0.5q + 0.25$

In Problems 7–14, find f(x) if f is a linear function that has the given properties.

7. slope $= 4, f(2) = 8$
8. $f(0) = 3, f(4) = -5$
9. $f(1) = 2, f(-2) = 8$
10. slope $= -5, f(\frac{1}{4}) = 9$
11. slope $= -\frac{2}{3}, f(-\frac{2}{3}) = -\frac{2}{3}$
12. $f(1) = 1, f(2) = 2$
13. $f(-2) = -1, f(-4) = -3$
14. slope $= 0.01, f(0.1) = 0.01$

15. **Demand Equation** Suppose consumers will demand 60 units of a product when the price is \$15.30 per unit and 35 units when the price is \$19.30 each. Find the demand equation, assuming that it is linear. Find the price per unit when 40 units are demanded.

16. **Demand Equation** The demand per week for a CD is 26,000 copies when the price is \$12 each, and 10,000 copies when the price is \$18 each. Find the demand equation for the CD, assuming that it is linear.

17. **Supply Equation** A refrigerator manufacturer will produce 3000 units when the price is \$940, and 2200 units when the price is \$740. Assume that price, p, and quantity, q, produced are linearly related and find the supply equation.

18. **Supply Equation** Suppose a manufacturer of shoes will place on the market 50 (thousand pairs) when the price is 35 (dollars per pair) and 35 when the price is 30. Find the supply equation, assuming that price p and quantity q are linearly related.

19. **Cost Equation** Suppose the cost to produce 10 units of a product is \$40 and the cost of 20 units is \$70. If cost, c, is linearly related to output, q, find a linear equation relating c and q. Find the cost to produce 35 units.

20. **Cost Equation** An advertiser goes to a printer and is charged \$89 for 100 copies of one flyer and \$93 for 200 copies of another flyer. This printer charges a fixed setup cost plus a charge for every copy of single-page flyers. Find a function that describes the cost of a printing job, if x is the number of copies made.

21. **Electric Rates** An electric utility company charges residential customers 12.5 cents per kilowatt-hour plus a base charge each month. One customer's monthly bill comes to \$51.65 for 380 kilowatt-hours. Find a linear function that describes the total monthly charges for electricity if x is the number of kilowatt-hours used in a month.

22. **Radiation Therapy** A cancer patient is to receive drug and radiation therapies. Each cubic centimeter of the drug to be used contains 210 curative units, and each minute of radiation exposure gives 305 curative units. The patient requires 2410 curative units. If d cubic centimeters of the drug and r minutes of radiation are administered, determine an equation relating d and r. Graph the equation for $d \geq 0$, and $r \geq 0$; label the horizontal axis as d.

23. **Depreciation** Suppose the value of a mountain bike decreases each year by 10% of its original value. If the original value is \$1800, find an equation that expresses the value v of the bike t years after purchase, where $0 \leq t \leq 10$. Sketch the equation, choosing t as the horizontal axis and v as the vertical axis. What is the slope of the resulting line? This method of considering the value of equipment is called *straight-line depreciation.*

24. **Depreciation** A new television depreciates \$120 per year, and it is worth \$340 after four years. Find a function that describes the value of this television, if x is the age of the television in years.

25. **Appreciation** A new house was sold for \$1,183,000 six years after it was built and purchased. The original owners calculated that the house appreciated \$53,000 per year while they owned it. Find a linear function that describes the appreciation of the building, in thousands of dollars, if x is the number of years since the original purchase.

26. **Appreciation** A house purchased for \$245,000 is expected to double in value in 15 years. Find a linear equation that describes the house's value after t years.

27. **Repair Charges** A business-copier repair company charges a fixed amount plus an hourly rate for a service call. If a customer is billed \$159 for a one-hour service call and \$287 for a three-hour service call, find a linear function that describes the price of a service call, where x is the number of hours of service.

28. **Sheep Wool Length** For sheep maintained at high environmental temperatures, respiratory rate, r (per minute), increases as wool length, l (in centimeters), decreases.[2] Suppose sheep with a wool length of 2 cm have an (average) respiratory rate of 160, and those with a wool length of 4 cm have a respiratory rate of 125. Assume that r and l are linearly related. (a) Find an equation that gives r in terms of l. (b) Find the respiratory rate of sheep with a wool length of 1 cm.

29. **Isocost Line** In production analysis, an *isocost line* is a line whose points represent all combinations of two factors of production that can be purchased for the same amount. Suppose a farmer has allocated \$20,000 for the purchase of x tons of fertilizer (costing \$200 per ton) and y acres of land (costing \$2000 per acre). Find an equation of the isocost line that describes the

[2]Adapted from G. E. Folk, Jr., *Textbook of Environmental Physiology*, 2nd ed. (Philadelphia: Lea & Febiger, 1974).

various combinations that can be purchased for \$20,000. Observe that neither x nor y can be negative.

30. Isoprofit Line A manufacturer produces products X and Y for which the profits per unit are \$7 and \$8, respectively. If x units of X and y units of Y are sold, then the total profit P is given by $P = P(x, y) = 7x + 8y$, where $x, y \geq 0$. (a) Sketch the graph of this equation for $P = 260$. The result is called an *isoprofit line,* and its points represent all combinations of sales that produce a profit of \$260. [It is an example of a *level curve* for the function $P(x, y) = 7x + 8y$ of two variables as introduced in Section 2.8.] (b) Determine the slope for $P = 260$. (c) For $P = 860$, determine the slope. (d) Are isoprofit lines always parallel?

31. Grade Scaling For reasons of comparison, a professor wants to rescale the scores on a set of test papers so that the maximum score is still 100, but the average is 65 instead of 56. (a) Find a linear equation that will do this. [*Hint:* You want 56 to become 65 and 100 to remain 100. Consider the points (56, 65) and (100, 100) and, more generally, (x, y), where x is the old score and y is the new score. Find the slope and use a point-slope form. Express y in terms of x.] (b) If 62 on the new scale is the lowest passing score, what was the lowest passing score on the original scale?

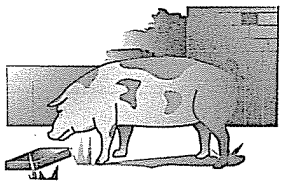
32. Psychology The result of Sternberg's psychological experiment[3] on information retrieval is that a person's reaction time R, in milliseconds, is statistically a linear function of memory set size N as follows:

$$R = 38N + 397$$

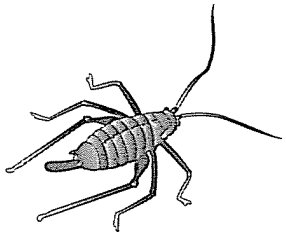
Sketch the graph for $1 \leq N \leq 5$. What is the slope?

33. Psychology In a certain learning experiment involving repetition and memory,[4] the proportion, p, of items recalled was estimated to be linearly related to the effective study time, t (in seconds), where t is between 5 and 9. For an effective study time of 5 seconds, the proportion of items recalled was 0.32. For each 1-second increase in study time, the proportion recalled increased by 0.059. (a) Find an equation that gives p in terms of t. (b) What proportion of items was recalled with 9 seconds of effective study time?

34. Diet for Pigs In testing an experimental diet for pigs, it was determined that the (average) live weight, w (in kilograms), of a pig was statistically a linear function of the number of days, d, after the diet was initiated, where $0 \leq d \leq 100$. If the weight of a pig beginning the diet was 21 kg, and thereafter the pig gained 6.3 kg every 10 days, determine w as a function of d, and find the weight of a pig 55 days after the beginning of the diet.



35. Cricket Chirps Biologists have found that the number of chirps made per minute by crickets of a certain species is related to the temperature. The relationship is very close to being linear. At 68°F, the crickets chirp about 124 times a minute. At 80°F, they chirp about 172 times a minute. (a) Find an equation that gives Fahrenheit temperature, t, in terms of the number of chirps, c, per minute. (b) If you count chirps for only 15 seconds, how can you quickly estimate the temperature?



Objective

To sketch parabolas arising from quadratic functions.

3.3 Quadratic Functions

In Section 2.2, a *quadratic function* was defined as a polynomial function of degree 2. In other words,

Definition

A function f is a ***quadratic function*** if and only if $f(x)$ can be written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are constants and $a \neq 0$.

For example, the functions $f(x) = x^2 - 3x + 2$ and $F(t) = -3t^2$ are quadratic. However, $g(x) = \dfrac{1}{x^2}$ is *not* quadratic, because it cannot be written in the form $g(x) = ax^2 + bx + c$.

[3] G. R. Loftus and E. F. Loftus, *Human Memory: The Processing of Information* (New York: Lawrence Erlbaum Associates, Inc., distributed by the Halsted Press, Division of John Wiley & Sons, Inc., 1976).

[4] D. L. Hintzman, "Repetition and Learning," in *The Psychology of Learning,* Vol. 10, ed. G. H. Bower (New York: Academic Press, Inc., 1976), p. 77.

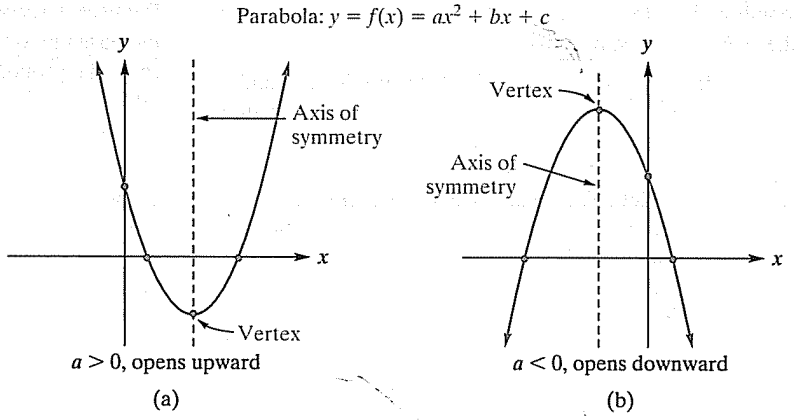


FIGURE 3.19 Parabolas.

The graph of the quadratic function $y = f(x) = ax^2 + bx + c$ is called a **parabola** and has a shape like the curves in Figure 3.19. If $a > 0$, the graph extends upward indefinitely, and we say that the parabola *opens upward* [Figure 3.19(a)]. If $a < 0$, the parabola *opens downward* [Figure 3.19(b)].

Each parabola in Figure 3.19 is *symmetric* about a vertical line, called the **axis of symmetry** of the parabola. That is, if the page were folded on one of these lines, then the two halves of the corresponding parabola would coincide. The axis (of symmetry) is *not* part of the parabola, but is a useful aid in sketching the parabola.

Each part of Figure 3.19 shows a point labeled **vertex,** where the axis cuts the parabola. If $a > 0$, the vertex is the "lowest" point on the parabola. This means that $f(x)$ has a minimum value at this point. By performing algebraic manipulations on $ax^2 + bx + c$ (referred to as *completing the square*), we can determine not only this minimum value, but also where it occurs. We have

$$f(x) = ax^2 + bx + c = (ax^2 + bx) + c$$

Adding and subtracting $\dfrac{b^2}{4a}$ gives

$$f(x) = \left(ax^2 + bx + \frac{b^2}{4a}\right) + c - \frac{b^2}{4a}$$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

so that

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Since $\left(x + \dfrac{b}{2a}\right)^2 \geq 0$ and $a > 0$, it follows that $f(x)$ has a minimum value when $x + \dfrac{b}{2a} = 0$; that is, when $x = -\dfrac{b}{2a}$. The y-coordinate corresponding to this value of x is $f\left(-\dfrac{b}{2a}\right)$. Thus, the vertex is given by

$$\text{vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

This is also the vertex of a parabola that opens downward ($a < 0$), but in this case $f\left(-\dfrac{b}{2a}\right)$ is the maximum value of $f(x)$. [See Figure 3.19(b).]

Observe that a function whose graph is a parabola is not one-to-one, in either the opening upward or opening downward case, since many horizontal lines will cut the graph twice. However, if we restrict the domain of a quadratic function to either

$\left[-\frac{b}{2a}, \infty\right)$ or $\left(-\infty, -\frac{b}{2a}\right]$, then the restricted function will pass the horizontal line test and therefore be one-to-one. (There are many other restrictions of a quadratic function that are one-to-one; however, their domains consist of more than one interval.) It follows that such restricted quadratic functions have inverse functions.

The point where the parabola $y = ax^2 + bx + c$ intersects the y-axis (that is, the y-intercept) occurs when $x = 0$. The y-coordinate of this point is c, so the y-intercept is c. In summary, we have the following.

Graph of Quadratic Function

The graph of the quadratic function $y = f(x) = ax^2 + bx + c$ is a parabola.

1. If $a > 0$, the parabola opens upward. If $a < 0$, it opens downward.
2. The vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.
3. The y-intercept is c.

We can quickly sketch the graph of a quadratic function by first locating the vertex, the y-intercept, and a few other points, such as those where the parabola intersects the x-axis. These *x-intercepts* are found by setting $y = 0$ and solving for x. Once the intercepts and vertex are found, it is then relatively easy to pass the appropriate parabola through these points. In the event that the x-intercepts are very close to the vertex or that no x-intercepts exist, we find a point on each side of the vertex, so that we can give a reasonable sketch of the parabola. Keep in mind that passing a (dashed) vertical line through the vertex gives the axis of symmetry. By plotting points to one side of the axis, we can use symmetry and obtain corresponding points on the other side.

APPLY IT ▷

13. A car dealership believes that the daily profit from the sale of minivans is given by $P(x) = -x^2 + 2x + 399$, where x is the number of minivans sold. Find the function's vertex and intercepts, and graph the function. If their model is correct, comment on the viability of dealing in minivans.

EXAMPLE 1 **Graphing a Quadratic Function**

Graph the quadratic function $y = f(x) = -x^2 - 4x + 12$.

Solution: Here $a = -1$, $b = -4$, and $c = 12$. Since $a < 0$, the parabola opens downward and thus has a highest point. The x-coordinate of the vertex is

$$-\frac{b}{2a} = -\frac{-4}{2(-1)} = -2$$

The y-coordinate is $f(-2) = -(-2)^2 - 4(-2) + 12 = 16$. Thus, the vertex is $(-2, 16)$, so the maximum value of $f(x)$ is 16. Since $c = 12$, the y-intercept is 12. To find the x-intercepts, we let y be 0 in $y = -x^2 - 4x + 12$ and solve for x:

$$0 = -x^2 - 4x + 12$$
$$0 = -(x^2 + 4x - 12)$$
$$0 = -(x + 6)(x - 2)$$

Hence, $x = -6$ or $x = 2$, so the x-intercepts are -6 and 2. Now we plot the vertex, axis of symmetry, and intercepts. [See Figure 3.20(a).] Since (0, 12) is *two* units to the *right* of the axis of symmetry, there is a corresponding point *two* units to the *left* of the axis with the same y-coordinate. Thus, we get the point $(-4, 12)$. Through all points, we draw a parabola opening downward. [See Figure 3.20(b).]

Now Work Problem 15 ◁

EXAMPLE 2 **Graphing a Quadratic Function**

Graph $p = 2q^2$.

Solution: Here p is a quadratic function of q, where $a = 2, b = 0$, and $c = 0$. Since $a > 0$, the parabola opens upward and thus has a lowest point. The q-coordinate of the

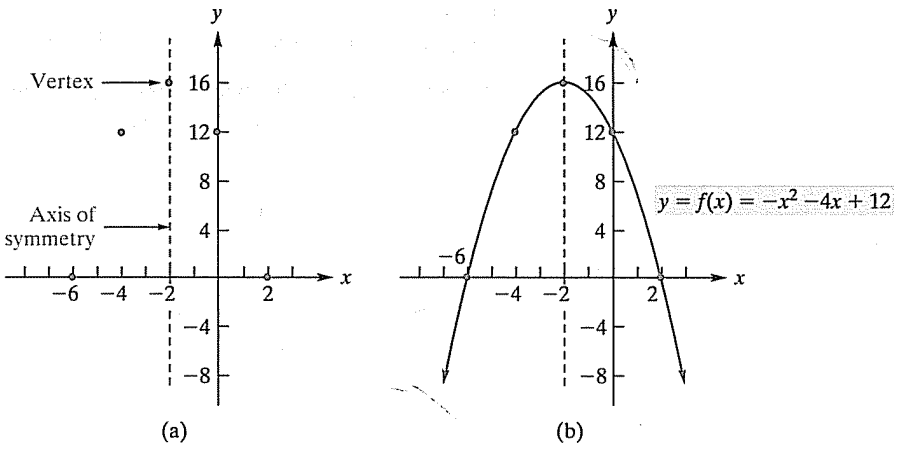


FIGURE 3.20 Graph of parabola $y = f(x) = -x^2 - 4x + 12$.

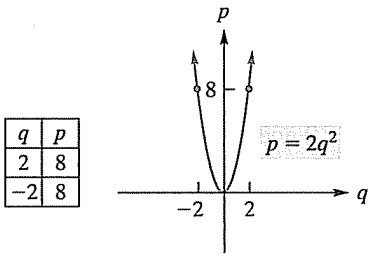


FIGURE 3.21 Graph of parabola $p = 2q^2$.

vertex is

$$-\frac{b}{2a} = -\frac{0}{2(2)} = 0$$

and the p-coordinate is $2(0)^2 = 0$. Consequently, the *minimum* value of p is 0 and the vertex is (0, 0). In this case, the p-axis is the axis of symmetry. A parabola opening upward with vertex at (0, 0) cannot have any other intercepts. Hence, to draw a reasonable graph, we plot a point on each side of the vertex. If $q = 2$, then $p = 8$. This gives the point (2, 8) and, by symmetry, the point (−2, 8). (See Figure 3.21.)

Now Work Problem 13 ◁

Example 3 illustrates that finding intercepts may require use of the quadratic formula.

APPLY IT ▷

14. A man standing on a pitcher's mound throws a ball straight up with an initial velocity of 32 feet per second. The height h of the ball in feet t seconds after it was thrown is described by the function $h(t) = -16t^2 + 32t + 8$, for $t \geq 0$. Find the function's vertex and intercepts, and graph the function.

EXAMPLE 3 Graphing a Quadratic Function

Graph $g(x) = x^2 - 6x + 7$.

Solution: Here g is a quadratic function, where $a = 1, b = -6$, and $c = 7$. The parabola opens upward, because $a > 0$. The x-coordinate of the vertex (lowest point) is

$$-\frac{b}{2a} = -\frac{-6}{2(1)} = 3$$

and $g(3) = 3^2 - 6(3) + 7 = -2$, which is the minimum value of $g(x)$. Thus, the vertex is (3, −2). Since $c = 7$, the vertical-axis intercept is 7. To find x-intercepts, we set $g(x) = 0$.

$$0 = x^2 - 6x + 7$$

The right side does not factor easily, so we will use the quadratic formula to solve for x:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)}$$
$$= \frac{6 \pm \sqrt{8}}{2} = \frac{6 \pm \sqrt{4 \cdot 2}}{2} = \frac{6 \pm 2\sqrt{2}}{2}$$
$$= \frac{6}{2} \pm \frac{2\sqrt{2}}{2} = 3 \pm \sqrt{2}$$

Therefore, the x-intercepts are $3 + \sqrt{2}$ and $3 - \sqrt{2}$. After plotting the vertex, intercepts, and (by symmetry) the point (6, 7), we draw a parabola opening upward in Figure 3.22.

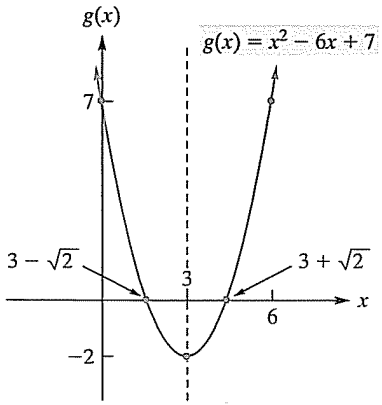
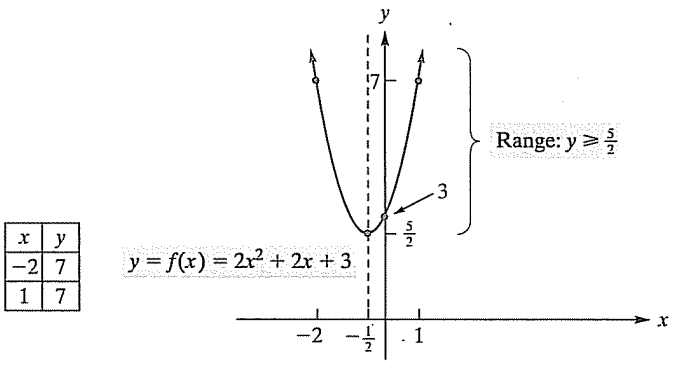


FIGURE 3.22 Graph of parabola $g(x) = x^2 - 6x + 7$.

Now Work Problem 17 ◁



x	y
−2	7
1	7

FIGURE 3.23 Graph of $y = f(x) = 2x^2 + 2x + 3$.

EXAMPLE 4 **Graphing a Quadratic Function**

Graph $y = f(x) = 2x^2 + 2x + 3$ and find the range of f.

Solution: This function is quadratic with $a = 2, b = 2$, and $c = 3$. Since $a > 0$, the graph is a parabola opening upward. The x-coordinate of the vertex is

$$-\frac{b}{2a} = -\frac{2}{2(2)} = -\frac{1}{2}$$

and the y-coordinate is $2(-\frac{1}{2})^2 + 2(-\frac{1}{2}) + 3 = \frac{5}{2}$. Thus, the vertex is $(-\frac{1}{2}, \frac{5}{2})$. Since $c = 3$, the y-intercept is 3. A parabola opening upward with its vertex above the x-axis has no x-intercepts. In Figure 3.23 we plotted the y-intercept, the vertex, and an additional point $(-2, 7)$ to the left of the vertex. By symmetry, we also get the point $(1, 7)$. Passing a parabola through these points gives the desired graph. From the figure, we see that the range of f is all $y \geq \frac{5}{2}$, that is, the interval $[\frac{5}{2}, \infty)$.

Now Work Problem 21 ◁

EXAMPLE 5 **Finding and Graphing an Inverse**

For the parabola given by the function

$$y = f(x) = ax^2 + bx + c$$

determine the inverse of the restricted function given by $g(x) = ax^2 + bx + c$, for $x \geq -\dfrac{b}{2a}$. Graph g and g^{-1} in the same plane, in the case where $a = 2$, $b = 2$, and $c = 3$.

Solution: Following the procedure described in Example 5 of Section 2.4, we begin by solving $y = ax^2 + bx + c$, where $x \geq -\dfrac{b}{2a}$, for x in terms of y. We do this by applying the quadratic formula to $ax^2 + bx + c - y = 0$, which gives $x = \dfrac{-b \pm \sqrt{b^2 - 4a(c-y)}}{2a} = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2 - 4a(c-y)}}{2a}$. Whenever $\sqrt{b^2 - 4a(c-y)}$ is defined (as a real number) it is nonnegative. Therefore, the sign of $\dfrac{\sqrt{b^2 - 4a(c-y)}}{2a}$ depends on a. It is nonnegative when a is positive, that is, when the parabola opens upward, and nonpositive when a is negative, that is, when the parabola opens downward. Thus, in order to satisfy $x \geq -\dfrac{b}{2a}$ we must take the $+$ in $\pm$ when $a > 0$ and the parabola opens upward and the $-$ in $\pm$ when $a < 0$ and the parabola opens downward. For definiteness now, let us deal with the case of $a > 0$. It follows, returning to the procedure of Example 5 of 2.4, that $g^{-1}(x) = \dfrac{-b + \sqrt{b^2 - 4a(c-x)}}{2a}$. The vertex of any paraola has y-coordinate given by $f\left(-\dfrac{b}{2a}\right) = a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c = -\dfrac{b^2 - 4ac}{4a}$.

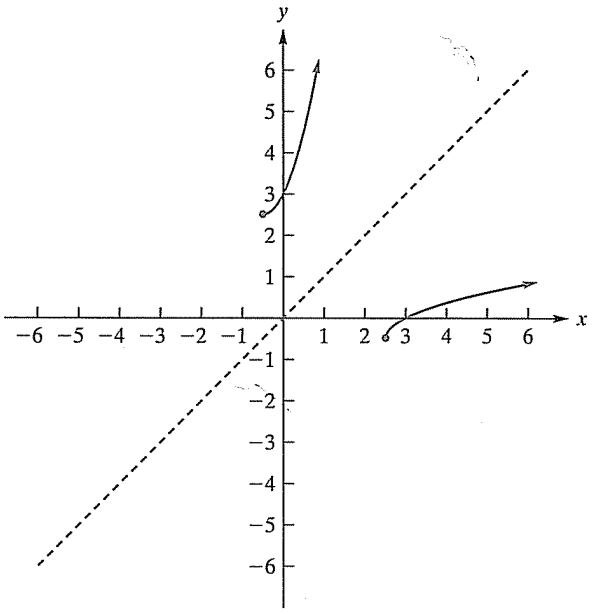


FIGURE 3.24 Graph of g and g^{-1}.

The domain of g is by definition $\left[-\frac{b}{2a}, \infty\right)$. It is now apparent that in the upward-opening case, the range of g is $\left[-\frac{b^2-4ac}{4a}, \infty\right)$. As was stated in Section 2.4, it is a general fact that the domain of g^{-1} is the range of g. Let us verify that claim in this situation by considering the domain of $\frac{-b+\sqrt{b^2-4a(c-x)}}{2a}$ directly. The domain is the set of all x for which $b^2-4a(c-x) \geq 0$. Evidently, this inequality is equivalent to $b^2-4ac+4ax \geq 0$, which in turn is equivalent to $4ax \geq -(b^2-4ac)$. In other words, $x \geq -\frac{b^2-4ac}{4a}$ as required.

To complete the exercise, observe that in Figure 3.23 we have provided the graph of $y = 2x^2 + 2x + 3$. For the task at hand, we redraw that part of the curve which lies to the right of the axis of symmetry. This provides the graph of g. Next we provide a dotted copy of the line $y = x$. Finally, we draw the mirror image of g in the line $y = x$ to obtain the graph of g^{-1} as in Figure 3.24.

Now Work Problem 27 ◁

EXAMPLE 6 **Maximum Revenue**

APPLY IT ▷

15. The demand function for a publisher's line of cookbooks is $p = 6 - 0.003q$, where p is the price (in dollars) per unit when q units are demanded (per day) by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

The formula for total revenue is part of the repertoire of relationships in business and economics.

The demand function for a manufacturer's product is $p = 1000 - 2q$, where p is the price (in dollars) per unit when q units are demanded (per week) by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

Solution:

Strategy To maximize revenue, we must determine the revenue function, $r = f(q)$. Using the relation

$$\textbf{total revenue} = \textbf{(price)(quantity)}$$

we have

$$r = pq$$

Using the demand equation, we can express p in terms of q, so r will be a function of q.

We have

$$r = pq$$
$$= (1000 - 2q)q$$
$$r = 1000q - 2q^2$$

Note that r is a quadratic function of q, with $a = -2, b = 1000$, and $c = 0$. Since $a < 0$ (the parabola opens downward), r is maximum at the vertex (q, r), where

$$q = -\frac{b}{2a} = -\frac{1000}{2(-2)} = 250$$

The maximum value of r is given by

$$r = 1000(250) - 2(250)^2$$
$$= 250{,}000 - 125{,}000 = 125{,}000$$

Thus, the maximum revenue that the manufacturer can receive is \$125,000, which occurs at a production level of 250 units. Figure 3.25 shows the graph of the revenue function. Only that portion for which $q \geq 0$ and $r \geq 0$ is drawn, since quantity and revenue cannot be negative.

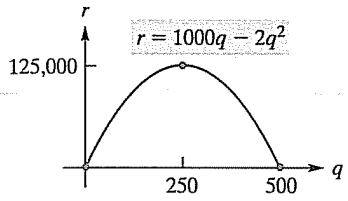


FIGURE 3.25 Graph of revenue function.

Now Work Problem 29 ◁

PROBLEMS 3.3

In Problems 1–8, state whether the function is quadratic.

1. $f(x) = 5x^2$

2. $g(x) = \dfrac{1}{2x^2 - 4}$

3. $g(x) = 7 - 6x$

4. $k(v) = 2v^2(2v^2 + 2)$

5. $h(q) = (3 - q)^2$

6. $f(t) = 2t(3 - t) + 4t$

7. $f(s) = \dfrac{s^2 - 9}{2}$

8. $g(t) = (t^2 - 1)^2$

In Problems 9–12, do not include a graph.

9. (a) For the parabola $y = f(x) = 3x^2 + 5x + 1$, find the vertex. (b) Does the vertex correspond to the highest point or the lowest point on the graph?

10. Repeat Problem 9 if $y = f(x) = 8x^2 + 4x - 1$.

11. For the parabola $y = f(x) = x^2 + x - 6$, find (a) the y-intercept, (b) the x-intercepts, and (c) the vertex.

12. Repeat Problem 11 if $y = f(x) = 5 - x - 3x^2$.

In Problems 13–22, graph each function. Give the vertex and intercepts, and state the range.

13. $y = f(x) = x^2 - 6x + 5$

14. $y = f(x) = 9x^2$

15. $y = g(x) = -2x^2 - 6x$

16. $y = f(x) = x^2 - 4$

17. $s = h(t) = t^2 + 6t + 9$

18. $s = h(t) = 2t^2 + 3t - 2$

19. $y = f(x) = -5 + 3x - 3x^2$

20. $y = H(x) = 1 - x - x^2$

21. $t = f(s) = s^2 - 8s + 14$

22. $t = f(s) = s^2 + 6s + 11$

In Problems 23–26, state whether f(x) has a maximum value or a minimum value, and find that value.

23. $f(x) = 49x^2 - 10x + 17$

24. $f(x) = -7x^2 - 2x + 6$

25. $f(x) = 4x - 50 - 0.1x^2$

26. $f(x) = x(x + 3) - 12$

In Problems 27 and 28, restrict the quadratic function to those x satisfying $x \geq v$, where v is the x-coordinate of the vertex of the parabola. Determine the inverse of the restricted function. Graph the restricted function and its inverse in the same plane.

27. $f(x) = x^2 - 2x + 4$

28. $f(x) = -x^2 + 4x - 3$

29. Revenue The demand function for a manufacturer's product is $p = f(q) = 100 - 10q$, where p is the price (in dollars) per unit when q units are demanded (per day). Find the level of production that maximizes the manufacturer's total revenue and determine this revenue.

30. Revenue The demand function for an office supply company's line of plastic rulers is $p = 0.85 - 0.00045q$, where p is the price (in dollars) per unit when q units are demanded (per day) by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

31. Revenue The demand function for an electronics company's laptop computer line is $p = 2400 - 6q$, where p is the price (in dollars) per unit when q units are demanded (per week) by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

32. Marketing A marketing firm estimates that n months after the introduction of a client's new product, $f(n)$ thousand households will use it, where

$$f(n) = \frac{10}{9}n(12 - n), \quad 0 \leq n \leq 12$$

Estimate the maximum number of households that will use the product.

33. Profit The daily profit for the garden department of a store from the sale of trees is given by $P(x) = -x^2 + 18x + 144$, where x is the number of trees sold. Find the function's vertex and intercepts, and graph the function.

34. Psychology A prediction made by early psychology relating the magnitude of a stimulus, x, to the magnitude of a response, y, is expressed by the equation $y = kx^2$, where k is

a constant of the experiment. In an experiment on pattern recognition, $k = 3$. Find the function's vertex and graph the equation. (Assume no restriction on x.)

35. Biology Biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[5] The protein consisted of yeast and corn flour. By varying the percentage, P, of yeast in the protein mix, the group estimated that the average weight gain (in grams) of a rat over a period of time was

$$f(P) = -\frac{1}{50}P^2 + 2P + 20, \quad 0 \le P \le 100$$

Find the maximum weight gain.

36. Height of Ball Suppose that the height, s, of a ball thrown vertically upward is given by

$$s = -4.9t^2 + 62.3t + 1.8$$

where s is in meters and t is elapsed time in seconds. (See Figure 3.26.) After how many seconds will the ball reach its maximum height? What is the maximum height?

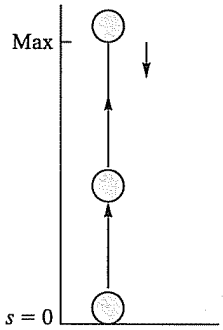


FIGURE 3.26 Ball thrown upward (Problem 36).

37. Archery A boy standing on a hill shoots an arrow straight up with an initial velocity of 85 feet per second. The height, h, of the arrow in feet, t seconds after it was released, is described by the function $h(t) = -16t^2 + 85t + 22$. What is the maximum height reached by the arrow? How many seconds after release does it take to reach this height?

38. Toy Toss A 6-year-old girl standing on a toy chest throws a doll straight up with an initial velocity of 16 feet per second. The height h of the doll in feet t seconds after it was released is described by the function $h(t) = -16t^2 + 16t + 4$. How long does it take the doll to reach its maximum height? What is the maximum height?

39. Rocket Launch A toy rocket is launched straight up from the roof of a garage with an initial velocity of 90 feet per second. The height, h, of the rocket in feet, t seconds after it was released, is described by the function $h(t) = -16t^2 + 90t + 14$. Find the function's vertex and intercepts, and graph the function.

40. Area Express the area of the rectangle shown in Figure 3.27 as a quadratic function of x. For what value of x will the area be a maximum?

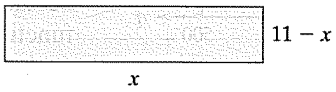


FIGURE 3.27 Diagram for Problem 40.

41. Enclosing Plot A building contractor wants to fence in a rectangular plot adjacent to a straight highway using the highway for one side, which will be left unfenced. (See Figure 3.28.) If the contractor has 500 feet of fence, find the dimensions of the maximum enclosed area.

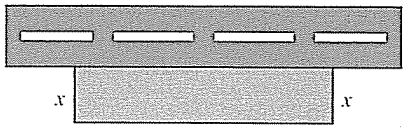


FIGURE 3.28 Diagram for Problem 41.

42. Find two numbers whose sum is 78 and whose product is a maximum.

3.4 Systems of Linear Equations

Objective

To solve systems of linear equations in both two and three variables by using the technique of elimination by addition or by substitution. (In Chapter 6, other methods are shown.)

Two-Variable Systems

When a situation must be described mathematically, it is not unusual for a *set* of equations to arise. For example, suppose that the manager of a factory is setting up a production schedule for two models of a new product. Model A requires 4 resistors and 9 transistors. Model B requires 5 resistors and 14 transistors. From its suppliers, the factory gets 335 resistors and 850 transistors each day. How many of each model should the manager plan to make each day so that all the resistors and transistors are used?

It's a good idea to construct a table that summarizes the important information. Table 3.2 shows the number of resistors and transistors required for each model, as well as the total number available.

Suppose we let x be the number of model A made each day and y be the number of model B. Then these require a total of $4x + 5y$ resistors and $9x + 14y$ transistors. Since

[5] Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein*, ed. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

Table 3.2

	Model A	Model B	Total Available
Resistors	4	5	335
Transistors	9	14	850

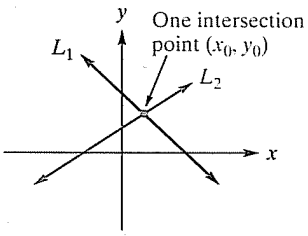


FIGURE 3.29 Linear system (one solution).

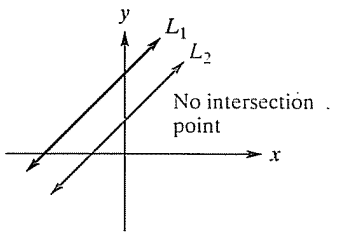


FIGURE 3.30 Linear system (no solution).

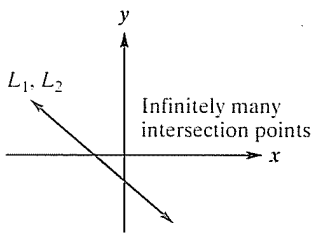


FIGURE 3.31 Linear system (infinitely many solutions).

335 resistors and 850 transistors are available, we have

$$\begin{cases} 4x + 5y = 335 & (1) \\ 9x + 14y = 850 & (2) \end{cases}$$

We call this set of equations a **system** of two linear equations in the variables x and y. The problem is to find values of x and y for which *both* equations are true *simultaneously*. A pair (x, y) of such values is called a *solution* of the system.

CAUTION!

Note that a *single* solution is given by an *ordered pair* of values.

Since Equations (1) and (2) are linear, their graphs are straight lines; call these lines L_1 and L_2. Now, the coordinates of any point on a line satisfy the equation of that line; that is, they make the equation true. Thus, the coordinates of any point of intersection of L_1 and L_2 will satisfy both equations. This means that a point of intersection gives a solution of the system.

If L_1 and L_2 are drawn on the same plane, there are three situations that could occur:

1. L_1 and L_2 may intersect at exactly one point, say, (a, b). (See Figure 3.29.) Thus, the system has the solution $x = a$ and $y = b$.
2. L_1 and L_2 may be parallel and have no points in common. (See Figure 3.30.) In this case, there is no solution.
3. L_1 and L_2 may be the same line. (See Figure 3.31.) Here the coordinates of any point on the line are a solution of the system. Consequently, there are infinitely many solutions.

Our main concern in this section is algebraic methods of solving a system of linear equations. We will successively replace the system by other systems that have the same solutions. Generalizing the terminology of Section 0.7, in the subsection titled "Equivalent Equations," we say that two systems are *equivalent* if their sets of solutions are equal. The replacement systems have progressively more desirable forms for determining the solution. More precisely, we seek an equivalent system containing an equation in which one of the variables does not appear. (In this case we say that the variable has been *eliminated*.) In dealing with systems of *linear* equations, our passage from a system to an equivalent system will always be accomplished by one of the following procedures:

1. Interchanging two equations
2. Multiplying one equation by a nonzero constant
3. Replacing an equation by itself plus a multiple of another equation

We will return to these procedures in more detail in Chapter 6. For the moment, since we will also consider nonlinear systems in this chapter, it is convenient to express

our solutions in terms of the very general principles of Section 0.7 that guarantee equivalence of equations.

We will illustrate the elimination procedure for the system in the problem originally posed:

$$\begin{cases} 4x + 5y = 335 & (3) \\ 9x + 14y = 850 & (4) \end{cases}$$

To begin, we will obtain an equivalent system in which x does not appear in one equation. First we find an equivalent system in which the coefficients of the x-terms in each equation are the same except for their sign. Multiplying Equation (3) by 9 [that is, multiplying both sides of Equation (3) by 9] and multiplying Equation (4) by -4 gives

$$\begin{cases} 36x + 45y = 3015 & (5) \\ -36x - 56y = -3400 & (6) \end{cases}$$

The left and right sides of Equation (5) are equal, so each side can be *added* to the corresponding side of Equation (6). This results in

$$-11y = -385$$

which has only one variable, as planned. Solving gives

$$y = 35$$

so we obtain the equivalent system

$$\begin{cases} 36x + 45y = 3015 & (7) \\ y = 35 & (8) \end{cases}$$

Replacing y in Equation (7) by 35, we get

$$\begin{aligned} 36x + 45(35) &= 3015 \\ 36x + 1575 &= 3015 \\ 36x &= 1440 \\ x &= 40 \end{aligned}$$

Thus, the original system is equivalent to

$$\begin{cases} x = 40 \\ y = 35 \end{cases}$$

We can check our answer by substituting $x = 40$ and $y = 35$ into *both* of the original equations. In Equation (3), we get $4(40) + 5(35) = 335$, or $335 = 335$. In Equation (4), we get $9(40) + 14(35) = 850$, or $850 = 850$. Hence, the solution is

$$x = 40 \quad \text{and} \quad y = 35$$

Each day the manager should plan to make 40 of model A and 35 of model B. Our procedure is referred to as **elimination by addition.** Although we chose to eliminate x first, we could have done the same for y by a similar procedure.

APPLY IT ▶

16. A computer consultant has \$200,000 invested for retirement, part at 9% and part at 8%. If the total yearly income from the investments is \$17,200, how much is invested at each rate?

EXAMPLE 1 **Elimination-by-Addition Method**

Use elimination by addition to solve the system.

$$\begin{cases} 3x - 4y = 13 \\ 3y + 2x = 3 \end{cases}$$

Solution: Aligning the x- and y-terms for convenience gives

$$\begin{cases} 3x - 4y = 13 & (9) \\ 2x + 3y = 3 & (10) \end{cases}$$

To eliminate y, we multiply Equation (9) by 3 and Equation (10) by 4:

$$\begin{cases} 9x - 12y = 39 & (11) \\ 8x + 12y = 12 & (12) \end{cases}$$

Adding Equation (11) to Equation (12) gives $17x = 51$, from which $x = 3$. We have the equivalent system

$$\begin{cases} 9x - 12y = 39 & (13) \\ x = 3 & (14) \end{cases}$$

Replacing x by 3 in Equation (13) results in

$$\begin{aligned} 9(3) - 12y &= 39 \\ -12y &= 12 \\ y &= -1 \end{aligned}$$

so the original system is equivalent to

$$\begin{cases} y = -1 \\ x = 3 \end{cases}$$

The solution is $x = 3$ and $y = -1$. Figure 3.32 shows a graph of the system.

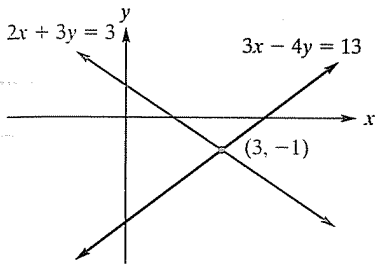


FIGURE 3.32 Linear system of Example 1: one solution.

Now Work Problem 1 ◁

The system in Example 1,

$$\begin{cases} 3x - 4y = 13 & (15) \\ 2x + 3y = 3 & (16) \end{cases}$$

can be solved another way. We first choose one of the equations—for example, Equation (15)—and solve it for one variable in terms of the other, say x in terms of y. Hence Equation (15) is equivalent to $3x = 4y + 13$, which is equivalent to

$$x = \frac{4}{3}y + \frac{13}{3}$$

and we obtain

$$\begin{cases} x = \frac{4}{3}y + \frac{13}{3} & (17) \\ 2x + 3y = 3 & (18) \end{cases}$$

Substituting the right side of Equation (17) for x in Equation (18) gives

$$2\left(\frac{4}{3}y + \frac{13}{3}\right) + 3y = 3 \qquad (19)$$

Thus, x has been eliminated. Solving Equation (19), we have

$$\begin{aligned} \frac{8}{3}y + \frac{26}{3} + 3y &= 3 \\ 8y + 26 + 9y &= 9 \qquad \text{clearing fractions} \\ 17y &= -17 \\ y &= -1 \end{aligned}$$

Replacing y in Equation (17) by -1 gives $x = 3$, and the original system is equivalent to

$$\begin{cases} x = 3 \\ y = -1 \end{cases}$$

as before. This method is called **elimination by substitution.**

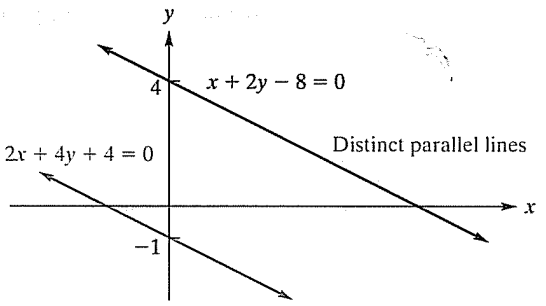


FIGURE 3.33 Linear system of Example 2: no solution.

APPLY IT ▶

17. Two species of deer, A and B, living in a wildlife refuge are given extra food in the winter. Each week, they receive 2 tons of food pellets and 4.75 tons of hay. Each deer of species A requires 4 pounds of the pellets and 5 pounds of hay. Each deer of species B requires 2 pounds of the pellets and 7 pounds of hay. How many of each species of deer will the food support so that all of the food is consumed each week?

EXAMPLE 2 Method of Elimination by Substitution

Use elimination by substitution to solve the system

$$\begin{cases} x + 2y - 8 = 0 \\ 2x + 4y + 4 = 0 \end{cases}$$

Solution: It is easy to solve the first equation for x. Doing so gives the equivalent system

$$\begin{cases} x = -2y + 8 & (20) \\ 2x + 4y + 4 = 0 & (21) \end{cases}$$

Substituting $-2y + 8$ for x in Equation (21) yields

$$2(-2y + 8) + 4y + 4 = 0$$
$$-4y + 16 + 4y + 4 = 0$$

The latter equation simplifies to $20 = 0$. Thus, we have the system

$$\begin{cases} x = -2y + 8 & (22) \\ 20 = 0 & (23) \end{cases}$$

Since Equation (23) is *never* true, there is **no solution** of the original system. The reason is clear if we observe that the original equations can be written in slope-intercept form as

$$y = -\frac{1}{2}x + 4$$

and

$$y = -\frac{1}{2}x - 1$$

These equations represent straight lines having slopes of $-\frac{1}{2}$, but different y-intercepts, namely, 4 and -1. That is, they determine different parallel lines. (See Figure 3.33.)

Now Work Problem 9 ◁

APPLY IT ▶

18. Two species of fish, A and B, are raised in one pond at a fish farm where they are fed two vitamin supplements. Each day, they receive 100 grams of the first supplement and 200 grams of the second supplement. Each fish of species A requires 15 mg of the first supplement and 30 mg of the second supplement. Each fish of species B requires 20 mg of the first supplement and 40 mg of the second supplement. How many of each species of fish will the pond support so that all of the supplements are consumed each day?

EXAMPLE 3 A Linear System with Infinitely Many Solutions

Solve

$$\begin{cases} x + 5y = 2 & (24) \\ \frac{1}{2}x + \frac{5}{2}y = 1 & (25) \end{cases}$$

Solution: We begin by eliminating x from the second equation. Multiplying Equation (25) by -2, we have

$$\begin{cases} x + 5y = 2 & (26) \\ -x - 5y = -2 & (27) \end{cases}$$

Adding Equation (26) to Equation (27) gives

$$\begin{cases} x + 5y = 2 & (28) \\ 0 = 0 & (29) \end{cases}$$

Because Equation (29) is *always* true, any solution of Equation (28) is a solution of the system. Now let us see how we can express our answer. From Equation (28), we have $x = 2 - 5y$, where y can be any real number, say, r. Thus, we can write $x = 2 - 5r$. The complete solution is

$$x = 2 - 5r$$
$$y = r$$

where r is any real number. In this situation r is called a **parameter,** and we say that we have a one-parameter family of solutions. Each value of r determines a particular solution. For example, if $r = 0$, then $x = 2$ and $y = 0$ is a solution; if $r = 5$, then $x = -23$ and $y = 5$ is another solution. Clearly, the given system has infinitely many solutions.

It is worthwhile to note that by writing Equations (24) and (25) in their slope-intercept forms, we get the equivalent system

$$\begin{cases} y = -\frac{1}{5}x + \frac{2}{5} \\ y = -\frac{1}{5}x + \frac{2}{5} \end{cases}$$

in which both equations represent the same line. Hence, the lines coincide (Figure 3.34), and Equations (24) and (25) are equivalent. The solution of the system consists of the coordinate pairs of all points on the line $x + 5y = 2$, and these points are given by our parametric solution.

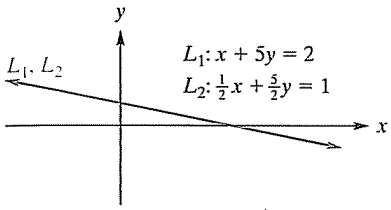


FIGURE 3.34 Linear system of Example 3: infinitely many solutions.

Now Work Problem 13 ◁

TECHNOLOGY

Graphically solve the system

$$\begin{cases} 9x + 4.1y = 7 \\ 2.6x - 3y = 18 \end{cases}$$

Solution: First we solve each equation for y, so that each equation has the form $y = f(x)$:

$$y = \frac{1}{4.1}(7 - 9x)$$
$$y = -\frac{1}{3}(18 - 2.6x)$$

Next we enter these functions as Y_1 and Y_2 and display them on the same viewing rectangle. (See Figure 3.35.)

Finally, either using trace and zoom or using the intersection feature, we estimate the solution to be $x = 2.52$, $y = -3.82$.

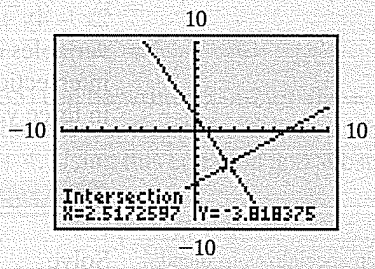


FIGURE 3.35 Graphical solution of system.

EXAMPLE 4 Mixture

A chemical manufacturer wishes to fill an order for 500 liters of a 25% acid solution. (Twenty-five percent by volume is acid.) If solutions of 30% and 18% are available in stock, how many liters of each must be mixed to fill the order?

Solution: Let x and y be the number of liters of the 30% and 18%, solutions respectively, that should be mixed. Then

$$x + y = 500$$

To help visualize the situation, we draw the diagram in Figure 3.36. In 500 liters of a 25% solution, there will be $0.25(500) = 125$ liters of acid. This acid comes from two sources: $0.30x$ liters of it come from the 30% solution, and $0.18y$ liters of it come from the 18% solution. Hence,

$$0.30x + 0.18y = 125$$

These two equations form a system of two linear equations in two unknowns. Solving the first for x gives $x = 500 - y$. Substituting in the second gives

$$0.30(500 - y) + 0.18y = 125$$

Solving this equation for y, we find that $y = 208\frac{1}{3}$ liters. Thus, $x = 500 - 208\frac{1}{3} = 291\frac{2}{3}$ liters. (See Figure 3.37.)

Now Work Problem 25 ◁

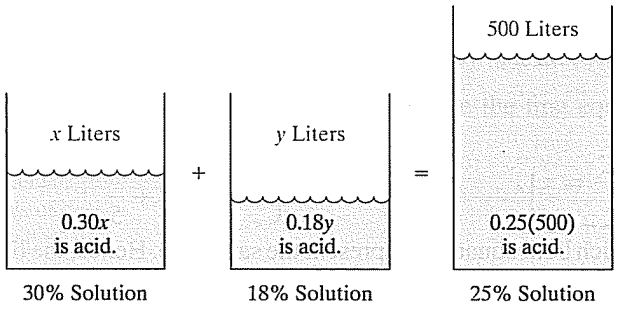


FIGURE 3.36 Mixture problem.

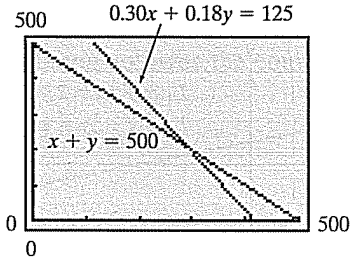


FIGURE 3.37 Graph for Example 4.

Three-Variable Systems

The methods used in solving a two-variable system of linear equations can be used to solve a three-variable system of linear equations. A **general linear equation in the three variables** x, y, **and** z is an equation having the form

$$Ax + By + Cz = D$$

where A, B, C, and D are constants and A, B, and C are not all zero. For example, $2x - 4y + z = 2$ is such an equation. Geometrically, a general linear equation in three variables represents a *plane* in space, and a solution to a system of such equations is the intersection of planes. Example 5 shows how to solve a system of three linear equations in three variables.

APPLY IT ▷

19. A coffee shop specializes in blending gourmet coffees. From type A, type B, and type C coffees, the owner wants to prepare a blend that will sell for \$8.50 for a 1-pound bag. The cost per pound of these coffees is \$12, \$9, and \$7, respectively. The amount of type B is to be twice the amount of type A. How much of each type of coffee will be in the final blend?

EXAMPLE 5 **Solving a Three-Variable Linear System**

Solve

$$\begin{cases} 2x + y + z = 3 & (30) \\ -x + 2y + 2z = 1 & (31) \\ x - y - 3z = -6 & (32) \end{cases}$$

Solution: This system consists of three linear equations in three variables. From Equation (32), $x = y + 3z - 6$. By substituting for x in Equations (30) and (31), we obtain

$$\begin{cases} 2(y + 3z - 6) + y + z = 3 \\ -(y + 3z - 6) + 2y + 2z = 1 \\ x = y + 3z - 6 \end{cases}$$

Simplifying gives

$$\begin{cases} 3y + 7z = 15 & (33) \\ y - z = -5 & (34) \\ x = y + 3z - 6 & (35) \end{cases}$$

Note that x does not appear in Equations (33) and (34). Since any solution of the original system must satisfy Equations (33) and (34), we will consider their solution first:

$$\begin{cases} 3y + 7z = 15 & (33) \\ y - z = -5 & (34) \end{cases}$$

From Equation (34), $y = z - 5$. This means that we can replace Equation (33) by

$$3(z - 5) + 7z = 15 \quad \text{that is,} \quad z = 3$$

Since z is 3, we can replace Equation (34) with $y = -2$. Hence, the previous system is equivalent to

$$\begin{cases} z = 3 \\ y = -2 \end{cases}$$

The original system becomes

$$\begin{cases} z = 3 \\ y = -2 \\ x = y + 3z - 6 \end{cases}$$

from which $x = 1$. The solution is $x = 1, y = -2$, and $z = 3$, which you should verify.

Now Work Problem 15 ◁

Just as a two-variable system may have a one-parameter family of solutions, a three-variable system may have a one-parameter or a two-parameter family of solutions. The next two examples illustrate.

EXAMPLE 6 **One-Parameter Family of Solutions**

Solve

$$\begin{cases} x - 2y = 4 & (35) \\ 2x - 3y + 2z = -2 & (36) \\ 4x - 7y + 2z = 6 & (37) \end{cases}$$

Solution: Note that since Equation (35) can be written $x - 2y + 0z = 4$, we can view Equations (35) to (37) as a system of three linear equations in the variables x, y, and z. From Equation (35), we have $x = 2y + 4$. Using this equation and substitution, we can eliminate x from Equations (36) and (37):

$$\begin{cases} x = 2y + 4 \\ 2(2y + 4) - 3y + 2z = -2 \\ 4(2y + 4) - 7y + 2z = 6 \end{cases}$$

which simplifies to give

$$\begin{cases} x = 2y + 4 & (38) \\ y + 2z = -10 & (39) \\ y + 2z = -10 & (40) \end{cases}$$

Multiplying Equation (40) by -1 gives

$$\begin{cases} x = 2y + 4 \\ y + 2z = -10 \\ -y - 2z = 10 \end{cases}$$

Adding the second equation to the third yields

$$\begin{cases} x = 2y + 4 \\ y + 2z = -10 \\ 0 = 0 \end{cases}$$

Since the equation $0 = 0$ is always true, the system is equivalent to

$$\begin{cases} x = 2y + 4 & (41) \\ y + 2z = -10 & (42) \end{cases}$$

Solving Equation (42) for y, we have

$$y = -10 - 2z$$

which expresses y in terms of z. We can also express x in terms of z. From Equation (41),

$$\begin{aligned} x &= 2y + 4 \\ &= 2(-10 - 2z) + 4 \\ &= -16 - 4z \end{aligned}$$

Thus, we have

$$\begin{cases} x = -16 - 4z \\ y = -10 - 2z \end{cases}$$

Since no restriction is placed on z, this suggests a parametric family of solutions. Setting $z = r$, we have the following family of solutions of the given system:

$$\begin{aligned} x &= -16 - 4r \\ y &= -10 - 2r \\ z &= r \end{aligned}$$

Other parametric representations of the solution are possible.

where r can be any real number. We see, then, that the given system has infinitely many solutions. For example, setting $r = 1$ gives the particular solution $x = -20$, $y = -12$, and $z = 1$. There is nothing special about the name of the parameter. In fact, since $z = r$, we could consider z to be the parameter.

Now Work Problem 19 ◁

EXAMPLE 7 **Two-Parameter Family of Solutions**

Solve the system

$$\begin{cases} x + 2y + z = 4 \\ 2x + 4y + 2z = 8 \end{cases}$$

Solution: This is a system of two linear equations in three variables. We will eliminate x from the second equation by first multiplying that equation by $-\frac{1}{2}$:

$$\begin{cases} x + 2y + z = 4 \\ -x - 2y - z = -4 \end{cases}$$

Adding the first equation to the second gives

$$\begin{cases} x + 2y + z = 4 \\ 0 = 0 \end{cases}$$

From the first equation, we obtain

$$x = 4 - 2y - z$$

Since no restriction is placed on either y or z, they can be arbitrary real numbers, giving us a two-parameter family of solutions. Setting $y = r$ and $z = s$, we find that the

solution of the given system is

$$x = 4 - 2r - s$$
$$y = r$$
$$z = s$$

where r and s can be any real numbers. Each assignment of values to r and s results in a solution of the given system, so there are infinitely many solutions. For example, letting $r = 1$ and $s = 2$ gives the particular solution $x = 0, y = 1$, and $z = 2$. As in the last example, there is nothing special about the names of the parameters. In particular, since $y = r$ and $z = s$, we could consider y and z to be the two parameters.

Now Work Problem 23 ◁

PROBLEMS 3.4

In Problems 1–24, solve the systems algebraically.

1. $\begin{cases} x + 4y = 3 \\ 3x - 2y = -5 \end{cases}$

2. $\begin{cases} 4x + 2y = 9 \\ 5y - 4x = 5 \end{cases}$

3. $\begin{cases} 2x + 3y = 1 \\ x + 2y = 0 \end{cases}$

4. $\begin{cases} 2x - y = 1 \\ -x + 2y = 7 \end{cases}$

5. $\begin{cases} u + v = 5 \\ u - v = 7 \end{cases}$

6. $\begin{cases} 2p + q = 16 \\ 3p + 3q = 33 \end{cases}$

7. $\begin{cases} x - 2y = -7 \\ 5x + 3y = -9 \end{cases}$

8. $\begin{cases} 4x + 12y = 12 \\ 2x + 4y = 12 \end{cases}$

9. $\begin{cases} 4x - 3y - 2 = 3x - 7y \\ x + 5y - 2 = y + 4 \end{cases}$

10. $\begin{cases} 5x + 7y + 2 = 9y - 4x + 6 \\ \frac{21}{2}x - \frac{4}{3}y - \frac{11}{4} = \frac{3}{2}x + \frac{2}{3}y + \frac{5}{4} \end{cases}$

11. $\begin{cases} \frac{2}{3}x + \frac{1}{2}y = 2 \\ \frac{3}{8}x + \frac{5}{6}y = -\frac{11}{2} \end{cases}$

12. $\begin{cases} \frac{1}{2}z - \frac{1}{4}w = \frac{1}{6} \\ \frac{1}{2}z + \frac{1}{4}w = \frac{1}{6} \end{cases}$

13. $\begin{cases} 2p + 3q = 5 \\ 10p + 15q = 25 \end{cases}$

14. $\begin{cases} 5x - 3y = 2 \\ -10x + 6y = 4 \end{cases}$

15. $\begin{cases} 2x + y + 6z = 3 \\ x - y + 4z = 1 \\ 3x + 2y - 2z = 2 \end{cases}$

16. $\begin{cases} x + y + z = -1 \\ 3x + y + z = 1 \\ 4x - 2y + 2z = 0 \end{cases}$

17. $\begin{cases} x + 4y + 3z = 10 \\ 4x + 2y - 2z = -2 \\ 3x - y + z = 11 \end{cases}$

18. $\begin{cases} x + 2y + z = 4 \\ 2x - 4y - 5z = 26 \\ 2x + 3y + z = 10 \end{cases}$

19. $\begin{cases} x - 2z = 1 \\ y + z = 3 \end{cases}$

20. $\begin{cases} 2y + 3z = 1 \\ 3x - 4z = 0 \end{cases}$

21. $\begin{cases} x - y + 2z = 0 \\ 2x + y - z = 0 \\ x + 2y - 3z = 0 \end{cases}$

22. $\begin{cases} x - 2y - z = 0 \\ 2x - 4y - 2z = 0 \\ -x + 2y + z = 0 \end{cases}$

23. $\begin{cases} x - 3y + z = 5 \\ -2x + 6y - 2z = -10 \end{cases}$

24. $\begin{cases} 5x + y + z = 17 \\ 4x + y + z = 14 \end{cases}$

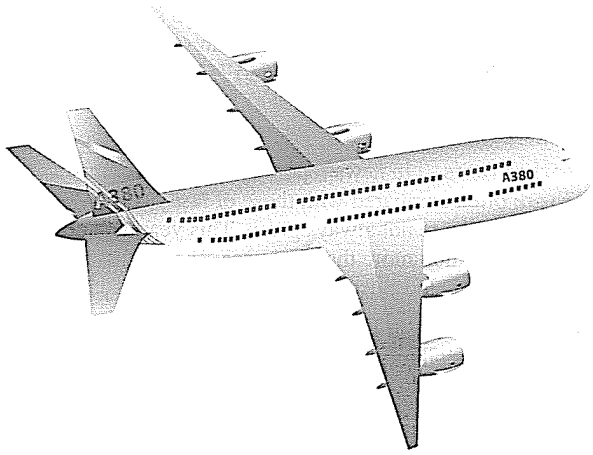
25. Mixture A chemical manufacturer wishes to fill an order for 800 gallons of a 25% acid solution. Solutions of 20% and 35% are in stock. How many gallons of each solution must be mixed to fill the order?

26. Mixture A gardener has two fertilizers that contain different concentrations of nitrogen. One is 3% nitrogen and the other is 11% nitrogen. How many pounds of each should she mix to obtain 20 pounds of a 9% concentration?

27. Fabric A textile mill produces fabric made from different fibers. From cotton, polyester, and nylon, the owners want to produce a fabric blend that will cost \$3.25 per pound to make. The cost per pound of these fibers is \$4.00, \$3.00, and \$2.00, respectively. The amount of nylon is to be the same as the amount of polyester. How much of each fiber will be in the final fabric?

28. Taxes A company has taxable income of \$758,000. The federal tax is 35% of that portion left after the state tax has been paid. The state tax is 15% of that portion left after the federal tax has been paid. Find the federal and state taxes.

29. Airplane Speed An airplane travels 900 mi in 2 h, 55 min, with the aid of a tailwind. It takes 3 h, 26 min, for the return trip, flying against the same wind. Find the speed of the airplane in still air and the speed of the wind.



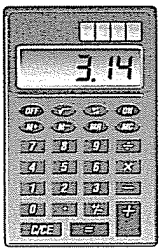
30. Speed of Raft On a trip on a raft, it took $\frac{1}{2}$ hour to travel 10 miles downstream. The return trip took $\frac{3}{4}$ hour. Find the speed of the raft in still water and the speed of the current.

31. Furniture Sales A manufacturer of dining-room sets produces two styles: early American and contemporary. From past experience, management has determined that 20% more of the early American styles can be sold than the contemporary styles. A profit of \$250 is made on each early American set sold, whereas a profit of \$350 is made on each contemporary set. If, in the forthcoming year, management desires a total profit of \$130,000, how many units of each style must be sold?

32. Survey National Surveys was awarded a contract to perform a product-rating survey for Crispy Crackers. A total of

250 people were interviewed. National Surveys reported that 62.5% more people liked Crispy Crackers than disliked them. However, the report did not indicate that 16% of those interviewed had no comment. How many of those surveyed liked Crispy Crackers? How many disliked them? How many had no comment?

33. Equalizing Cost United Products Co. manufactures calculators and has plants in the cities of Exton and Whyton. At the Exton plant, fixed costs are \$5000 per month, and the cost of producing each calculator is \$5.50. At the Whyton plant, fixed costs are \$6000 per month, and each calculator costs \$4.50 to produce. Next month, United Products must produce 1000 calculators. How many must be made at each plant if the total cost at each plant is to be the same?



34. Coffee Blending A coffee wholesaler blends together three types of coffee that sell for \$2.20, \$2.30, and \$2.60 per pound, so as to obtain 100 lb of coffee worth \$2.40 per pound. If the wholesaler uses the same amount of the two higher-priced coffees, how much of each type must be used in the blend?

35. Commissions A company pays its salespeople on a basis of a certain percentage of the first \$100,000 in sales, plus a certain percentage of any amount over \$100,000 in sales. If one salesperson earned \$8500 on sales of \$175,000 and another salesperson earned \$14,800 on sales of \$280,000, find the two percentages.

36. Yearly Profits In news reports, profits of a company this year (T) are often compared with those of last year (L), but actual values of T and L are not always given. This year, a company had profits of \$25 million more than last year. The profits were up 30%. Determine T and L from these data.

37. Fruit Packaging The `Ilovetiny.com` Organic Produce Company has 3600 lb of Donut Peaches that it is going to package in boxes. Half of the boxes will be loose filled, each containing 20 lb of peaches, and the others will be packed with 8-lb clamshells (flip-top plastic containers), each containing 2.2 lb of peaches. Determine the number of boxes and the number of clamshells that are required.

38. Investments A person made two investments, and the percentage return per year on each was the same. Of the total amount invested, 40% of it minus \$1000 was invested in one venture, and at the end of 1 year the person received a return of \$400 from that venture. If the total return after 1 year was \$1200, find the total amount invested.

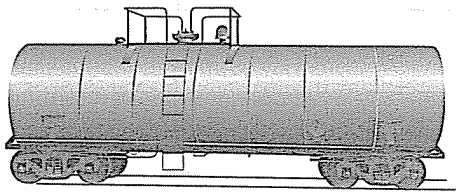
39. Production Run A company makes three types of patio furniture: chairs, rockers, and chaise lounges. Each requires wood, plastic, and aluminum, in the amounts shown in the following table. The company has in stock 400 units of wood, 600 units of plastic, and 1500 units of aluminum. For its end-of-the-season production run, the company wants to use up all the stock. To do this, how many chairs, rockers, and chaise lounges should it make?

	Wood	Plastic	Aluminum
Chair	1 unit	1 unit	2 units
Rocker	1 unit	1 unit	3 units
Chaise lounge	1 unit	2 units	5 units

40. Investments A total of \$35,000 was invested at three interest rates: 7, 8, and 9%. The interest for the first year was \$2830, which was not reinvested. The second year the amount originally invested at 9% earned 10% instead, and the other rates remained the same. The total interest the second year was \$2960. How much was invested at each rate?

41. Hiring Workers A company pays skilled workers in its assembly department \$16 per hour. Semiskilled workers in that department are paid \$9.50 per hour. Shipping clerks are paid \$10 per hour. Because of an increase in orders, the company needs to hire a total of 70 workers in the assembly and shipping departments. It will pay a total of \$725 per hour to these employees. Because of a union contract, twice as many semiskilled workers as skilled workers must be employed. How many semiskilled workers, skilled workers, and shipping clerks should the company hire?

42. Solvent Storage A 10,000-gallon railroad tank car is to be filled with solvent from two storage tanks, A and B. Solvent from A is pumped at the rate of 25 gal/min. Solvent from B is pumped at 35 gal/min. Usually, both pumps operate at the same time. However, because of a blown fuse, the pump on A is delayed 5 minutes. Both pumps finish operating at the same time. How many gallons from each storage tank will be used to fill the car?



3.5 Nonlinear Systems

Objective

To use substitution to solve nonlinear systems.

A system of equations in which at least one equation is not linear is called a **nonlinear system.** We can often solve a nonlinear system by substitution, as was done with linear systems. The following examples illustrate.

EXAMPLE 1 **Solving a Nonlinear System**

Solve

$$\begin{cases} x^2 - 2x + y - 7 = 0 & (1) \\ 3x - y + 1 = 0 & (2) \end{cases}$$

Solution:

Strategy If a nonlinear system contains a linear equation, we usually solve the linear equation for one variable and substitute for that variable in the other equation.

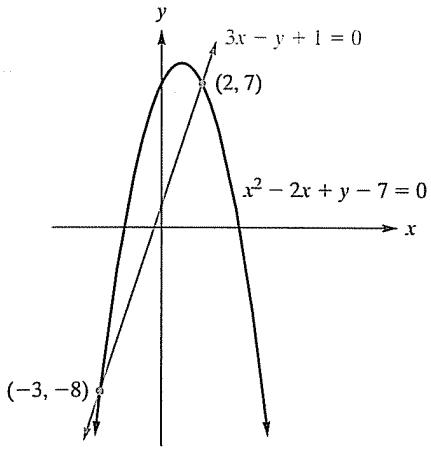


FIGURE 3.38 Nonlinear system of equations.

Solving Equation (2) for y gives

$$y = 3x + 1 \tag{3}$$

Substituting into Equation (1) and simplifying, we have

$$\begin{aligned} x^2 - 2x + (3x + 1) - 7 &= 0 \\ x^2 + x - 6 &= 0 \\ (x + 3)(x - 2) &= 0 \\ x = -3 \text{ or } x &= 2 \end{aligned}$$

If $x = -3$, then Equation (3) implies that $y = -8$; if $x = 2$, then $y = 7$. You should verify that each pair of values satisfies the given system. Hence, the solutions are $x = -3$, $y = -8$ and $x = 2$, $y = 7$. These solutions can be seen geometrically in the graph of the system in Figure 3.38. Notice that the graph of Equation (1) is a parabola and the graph of Equation (2) is a line. The solutions correspond to the intersection points $(-3, -8)$ and $(2, 7)$.

Now Work Problem 1 ◁

EXAMPLE 2 **Solving a Nonlinear System**

CAUTION!

This example illustrates the need for checking all "solutions".

Solve

$$\begin{cases} y = \sqrt{x + 2} \\ x + y = 4 \end{cases}$$

Solution: Solving the second equation, which is linear, for y gives

$$y = 4 - x \tag{4}$$

Substituting into the first equation yields

$$\begin{aligned} 4 - x &= \sqrt{x + 2} \\ 16 - 8x + x^2 &= x + 2 \qquad \text{squaring both sides} \\ x^2 - 9x + 14 &= 0 \\ (x - 2)(x - 7) &= 0 \end{aligned}$$

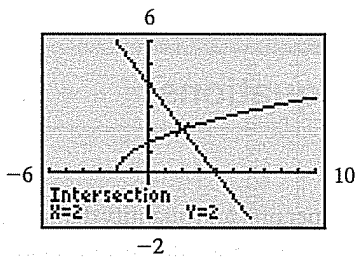


FIGURE 3.39 Nonlinear system of Example 2.

Thus, $x = 2$ or $x = 7$. From Equation (4), if $x = 2$, then $y = 2$; if $x = 7$, then $y = -3$. Since we performed the operation of squaring both sides, we must check our results. Although the pair $x = 2$, $y = 2$ satisfies both of the original equations, this is not the case for $x = 7$ and $y = -3$. Thus, the solution is $x = 2$, $y = 2$. (See Figure 3.39.)

Now Work Problem 13 ◁

TECHNOLOGY

Graphically solve the equation $0.5x^2+x=3$, where $x \geq 0$.

Solution: To solve the equation, we could find zeros of the function $f(x) = 0.5x^2 + x - 3$. Alternatively, we can think of this problem as solving the nonlinear system

$$y = 0.5x^2 + x$$
$$y = 3$$

In Figure 3.40, the intersection point is estimated to be $x = 1.65, y = 3$. Note that the graph of $y = 3$ is a horizontal line. The solution of the given equation is $x = 1.65$.

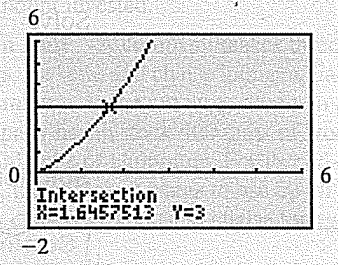


FIGURE 3.40 Solution of $0.5x^2 + x = 3$.

PROBLEMS 3.5

In Problems 1–14, solve the given nonlinear system.

1. $\begin{cases} y = x^2 - 9 \\ 2x + y = 3 \end{cases}$

2. $\begin{cases} y = x^3 \\ x - 2y = 0 \end{cases}$

3. $\begin{cases} p^2 = 5 - q \\ p = q + 1 \end{cases}$

4. $\begin{cases} y^2 - x^2 = 28 \\ x - y = 14 \end{cases}$

5. $\begin{cases} x = y^2 \\ y = x^2 \end{cases}$

6. $\begin{cases} p^2 - q + 1 = 0 \\ 5q - 3p - 2 = 0 \end{cases}$

7. $\begin{cases} y = 4 + 2x - x^2 \\ y = x^2 + 1 \end{cases}$

8. $\begin{cases} x^2 + 4x - y = -4 \\ y - x^2 - 4x + 3 = 0 \end{cases}$

9. $\begin{cases} p = \sqrt{q} \\ p = q^2 \end{cases}$

10. $\begin{cases} z = 4/w \\ 3z = 2w + 2 \end{cases}$

11. $\begin{cases} x^2 = y^2 + 13 \\ y = x^2 - 15 \end{cases}$

12. $\begin{cases} x^2 + y^2 + 2xy = 1 \\ 2x - y = 2 \end{cases}$

13. $\begin{cases} x = y + 1 \\ y = 2\sqrt{x+2} \end{cases}$

14. $\begin{cases} y = \dfrac{x^2}{x-1} + 1 \\ y = \dfrac{1}{x-1} \end{cases}$

15. Decorations The shape of a paper streamer suspended above a dance floor can be described by the function $y = 0.01x^2 + 0.01x + 7$, where y is the height of the streamer (in feet) above the floor and x is the horizontal distance (in feet) from the center of the room. A rope holding up other decorations touches the streamer and is described by the function $y = 0.01x + 8.0$. Where does the rope touch the paper streamer?

16. Awning The shape of a decorative awning over a storefront can be described by the function $y = 0.06x^2 + 0.012x + 8$, where y is the height of the edge of the awning (in feet) above the sidewalk and x is the distance (in feet) from the center of the store's doorway. A vandal pokes a stick through the awning, piercing it in two places. The position of the stick can be described by the function $y = 0.912x + 5$. Where are the holes in the awning caused by the vandal?

17. Graphically determine how many solutions there are to the system

$$\begin{cases} y = \dfrac{1}{x^2} \\ y = 2 - x^2 \end{cases}$$

18. Graphically solve the system

$$\begin{cases} 2y = x^3 \\ y = 8 - x^2 \end{cases}$$

to one-decimal-place accuracy.

19. Graphically solve the system

$$\begin{cases} y = x^2 - 2x + 1 \\ y = x^3 + x^2 - 2x + 3 \end{cases}$$

to one-decimal-place accuracy.

20. Graphically solve the system

$$\begin{cases} y = x^3 - x + 1 \\ y = 3x + 2 \end{cases}$$

to one-decimal-place accuracy.

In Problems 21–23, graphically solve the equation by treating it as a system. Round answers to two decimal places.

21. $0.8x^2 + 2x = 6$ where $x \geq 0$

22. $-\sqrt{x+3} = 1 - x$

23. $x^3 - 3x^2 = x - 8$

Objective

To solve systems describing equilibrium and break-even points.

3.6 Applications of Systems of Equations

Equilibrium

Recall from Section 3.2 that an equation that relates price per unit and quantity demanded (supplied) is called a *demand equation (supply equation)*. Suppose that, for product Z, the demand equation is

$$p = -\frac{1}{180}q + 12 \qquad (1)$$

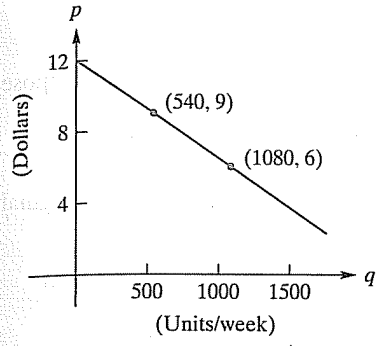


FIGURE 3.41 Demand curve.

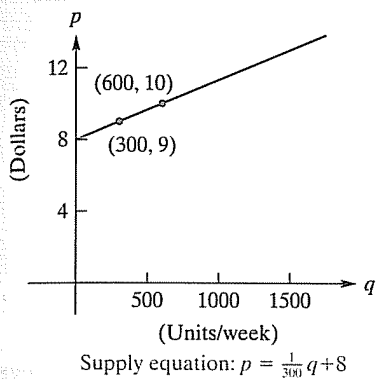


FIGURE 3.42 Supply curve.

and the supply equation is

$$p = \frac{1}{300}q + 8 \tag{2}$$

where q, $p \geq 0$. The corresponding demand and supply curves are the lines in Figures 3.41 and 3.42, respectively. In analyzing Figure 3.41, we see that consumers will purchase 540 units per week when the price is \$9 per unit, 1080 units when the price is \$6, and so on. Figure 3.42 shows that when the price is \$9 per unit producers will place 300 units per week on the market, at \$10 they will supply 600 units, and so on.

When the demand and supply curves of a product are represented on the same coordinate plane, the point (m, n) where the curves intersect is called the **point of equilibrium.** (See Figure 3.43.) The price n, called the **equilibrium price,** is the price at which consumers will purchase the same quantity of a product that producers wish to sell at that price. In short, n is the price at which stability in the producer–consumer relationship occurs. The quantity m is called the **equilibrium quantity.**

To determine precisely the equilibrium point, we solve the system formed by the supply and demand equations. Let us do this for our previous data, namely, the system

$$\begin{cases} p = -\frac{1}{180}q + 12 & \text{demand equation} \\ p = \frac{1}{300}q + 8 & \text{supply equation} \end{cases}$$

By substituting $\frac{1}{300}q + 8$ for p in the demand equation, we get

$$\frac{1}{300}q + 8 = -\frac{1}{180}q + 12$$

$$\left(\frac{1}{300} + \frac{1}{180}\right)q = 4$$

$$q = 450 \qquad \text{equilibrium quantity}$$

Thus,

$$p = \frac{1}{300}(450) + 8$$

$$= 9.50 \qquad \text{equilibrium price}$$

and the equilibrium point is (450, 9.50). Therefore, at the price of \$9.50 per unit, manufacturers will produce exactly the quantity (450) of units per week that consumers will purchase at that price. (See Figure 3.44.)

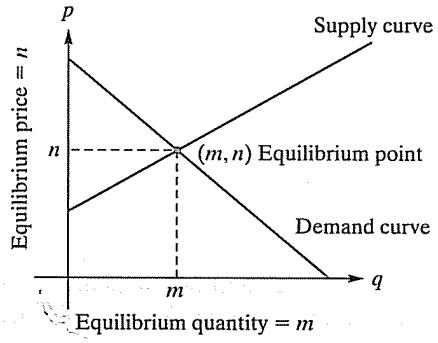


FIGURE 3.43 Equilibrium.

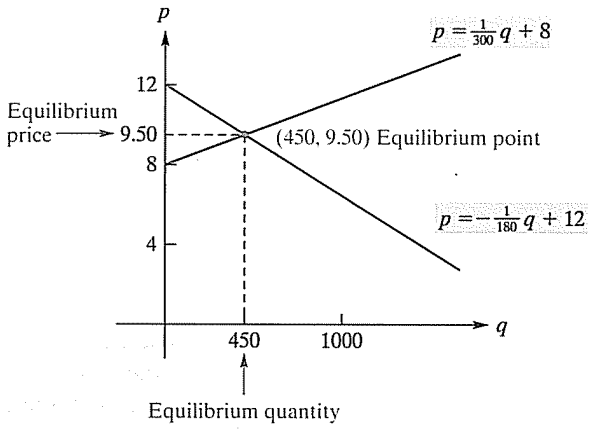


FIGURE 3.44 Equilibrium.

EXAMPLE 1 **Tax Effect on Equilibrium**

Let $p = \frac{8}{100}q + 50$ be the supply equation for a manufacturer's product, and suppose the demand equation is $p = -\frac{7}{100}q + 65$.

a. If a tax of \$1.50 per unit is to be imposed on the manufacturer, how will the original equilibrium price be affected if the demand remains the same?

Solution: Before the tax, the equilibrium price is obtained by solving the system

$$\begin{cases} p = \frac{8}{100}q + 50 \\ p = -\frac{7}{100}q + 65 \end{cases}$$

By substitution,

$$\begin{aligned} -\frac{7}{100}q + 65 &= \frac{8}{100}q + 50 \\ 15 &= \frac{15}{100}q \\ 100 &= q \end{aligned}$$

and

$$p = \frac{8}{100}(100) + 50 = 58$$

Thus, \$58 is the original equilibrium price. Before the tax, the manufacturer supplies q units at a price of $p = \frac{8}{100}q + 50$ per unit. After the tax, he will sell the same q units for an additional \$1.50 per unit. The price per unit will be $\left(\frac{8}{100}q + 50\right) + 1.50$, so the new supply equation is

$$p = \frac{8}{100}q + 51.50$$

Solving the system

$$\begin{cases} p = \frac{8}{100}q + 51.50 \\ p = -\frac{7}{100}q + 65 \end{cases}$$

will give the new equilibrium price:

$$\begin{aligned} \frac{8}{100}q + 51.50 &= -\frac{7}{100}q + 65 \\ \frac{15}{100}q &= 13.50 \\ q &= 90 \\ p &= \frac{8}{100}(90) + 51.50 = 58.70 \end{aligned}$$

The tax of \$1.50 per unit increases the equilibrium price by \$0.70. (See Figure 3.45.) Note that there is also a decrease in the equilibrium quantity from $q = 100$ to $q = 90$, because of the change in the equilibrium price. (In the problems, you are asked to find the effect of a subsidy given to the manufacturer, which will reduce the price of the product.)

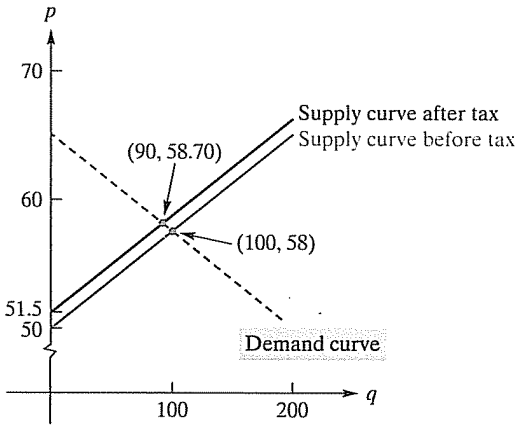


FIGURE 3.45 Equilibrium before and after tax.

Now Work Problem 15 ◁

b. Determine the total revenue obtained by the manufacturer at the equilibrium point both before and after the tax.

Solution: If q units of a product are sold at a price of p dollars each, then the total revenue is given by

$$y_{TR} = pq$$

Before the tax, the revenue at (100, 58) is (in dollars)

$$y_{TR} = (58)(100) = 5800$$

After the tax, it is

$$y_{TR} = (58.70)(90) = 5283$$

which is a decrease.

◁

EXAMPLE 2 Equilibrium with Nonlinear Demand

Find the equilibrium point if the supply and demand equations of a product are $p = \frac{q}{40} + 10$ and $p = \frac{8000}{q}$, respectively.

Solution: Here the demand equation is not linear. Solving the system

$$\begin{cases} p = \dfrac{q}{40} + 10 \\ p = \dfrac{8000}{q} \end{cases}$$

by substitution gives

$$\frac{8000}{q} = \frac{q}{40} + 10$$

$$320{,}000 = q^2 + 400q \qquad \text{multiplying both sides by } 40q$$

$$q^2 + 400q - 320{,}000 = 0$$

$$(q + 800)(q - 400) = 0$$

$$q = -800 \quad \text{or} \quad q = 400$$

We disregard $q = -800$, since q represents quantity. Choosing $q = 400$, we have $p = (8000/400) = 20$, so the equilibrium point is (400, 20). (See Figure 3.46.)

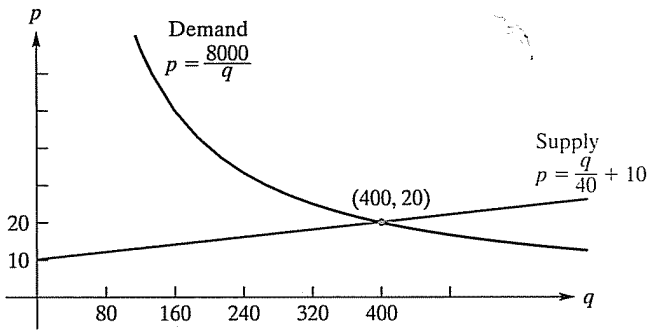


FIGURE 3.46 Equilibrium with nonlinear demand.

◁

Break-Even Points

Suppose a manufacturer produces product A and sells it at \$8 per unit. Then the total revenue y_{TR} received (in dollars) from selling q units is

$$y_{TR} = 8q \qquad \text{total revenue}$$

The difference between the total revenue received for q units and the total cost of q units is the manufacturer's profit:

$$\textbf{profit} = \textbf{total revenue} - \textbf{total cost}$$

(If profit is negative, then we have a loss.) **Total cost,** y_{TC}, is the sum of total variable costs y_{VC} and total fixed costs y_{FC}:

$$y_{TC} = y_{VC} + y_{FC}$$

Fixed costs are those costs that, under normal conditions, do not depend on the level of production; that is, over some period of time they remain constant at all levels of output. (Examples are rent, officers' salaries, and normal maintenance.) **Variable costs** are those costs that vary with the level of production (such as the cost of materials, labor, maintenance due to wear and tear, etc.). For q units of product A, suppose that

$$y_{FC} = 5000 \qquad \text{fixed cost}$$

$$\text{and } y_{VC} = \frac{22}{9}q \qquad \text{variable cost}$$

Then

$$y_{TC} = \frac{22}{9}q + 5000 \qquad \text{total cost}$$

The graphs of total cost and total revenue appear in Figure 3.47. The horizontal axis represents the level of production, q, and the vertical axis represents the total dollar value, be it revenue or cost. The **break-even point** is the point at which total revenue equals total cost (TR = TC). It occurs when the levels of production and sales result in neither a profit nor a loss to the manufacturer. In the diagram, called a *break-even chart,* the break-even point is the point (m, n) at which the graphs of $y_{TR} = 8q$ and $y_{TC} = \frac{22}{9}q + 5000$ intersect. We call m the **break-even quantity** and n the **break-even revenue.** When total cost and revenue are linearly related to output, as in this case, for any production level greater than m, total revenue is greater than total cost, resulting in a profit. However, at any level less than m units, total revenue is less than total cost, resulting in a loss. At an output of m units, the profit is zero. In the following example, we will examine our data in more detail.

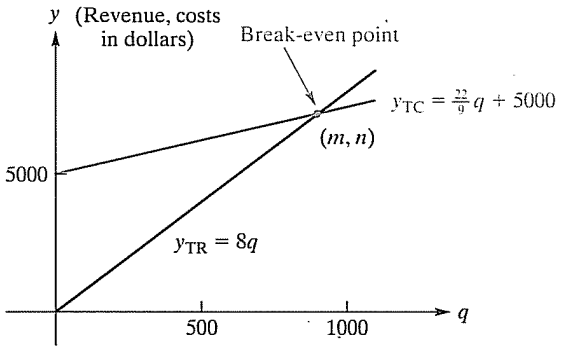


FIGURE 3.47 Break-even chart.

EXAMPLE 3 Break-Even Point, Profit, and Loss

A manufacturer sells a product at \$8 per unit, selling all that is produced. Fixed cost is \$5000 and variable cost per unit is $\frac{22}{9}$ (dollars).

a. Find the total output and revenue at the break-even point.

Solution: At an output level of q units, the variable cost is $y_{VC} = \frac{22}{9}q$ and the total revenue is $y_{TR} = 8q$. Hence,

$$y_{TR} = 8q$$
$$y_{TC} = y_{VC} + y_{FC} = \frac{22}{9}q + 5000$$

At the break-even point, total revenue equals total cost. Thus, we solve the system formed by the foregoing equations. Since

$$y_{TR} = y_{TC}$$

we have

$$8q = \frac{22}{9}q + 5000$$
$$\frac{50}{9}q = 5000$$
$$q = 900$$

Hence, the desired output is 900 units, resulting in a total revenue (in dollars) of

$$y_{TR} = 8(900) = 7200$$

(See Figure 3.48.)

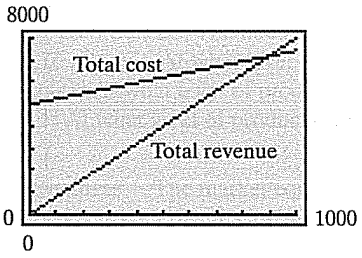


FIGURE 3.48 Equilibrium point (900, 7200).

b. Find the profit when 1800 units are produced.

Solution: Since profit = total revenue − total cost, when $q = 1800$ we have

$$y_{TR} - y_{TC} = 8(1800) - \left[\frac{22}{9}(1800) + 5000\right]$$
$$= 5000$$

The profit when 1800 units are produced and sold is \$5000.

c. Find the loss when 450 units are produced.

Solution: When $q = 450$,

$$y_{TR} - y_{TC} = 8(450) - \left[\frac{22}{9}(450) + 5000\right] = -2500$$

A loss of \$2500 occurs when the level of production is 450 units.

d. Find the output required to obtain a profit of \$10,000.

Solution: In order to obtain a profit of \$10,000, we have

$$\text{profit} = \text{total revenue} - \text{total cost}$$

$$10{,}000 = 8q - \left(\frac{22}{9}q + 5000\right)$$

$$15{,}000 = \frac{50}{9}q$$

$$q = 2700$$

Thus, 2700 units must be produced.

Now Work Problem 9 ◁

EXAMPLE 4 Break-Even Quantity

Determine the break-even quantity of XYZ Manufacturing Co., given the following data: total fixed cost, \$1200; variable cost per unit, \$2; total revenue for selling q units, $y_{TR} = 100\sqrt{q}$.

Solution: For q units of output,

$$y_{TR} = 100\sqrt{q}$$

$$y_{TC} = 2q + 1200$$

Equating total revenue to total cost gives

$$100\sqrt{q} = 2q + 1200$$

$$50\sqrt{q} = q + 600 \qquad \text{dividing both sides by 2}$$

Squaring both sides, we have

$$2500q = q^2 + 1200q + (600)^2$$

$$0 = q^2 - 1300q + 360{,}000$$

By the quadratic formula,

$$q = \frac{1300 \pm \sqrt{250{,}000}}{2}$$

$$q = \frac{1300 \pm 500}{2}$$

$$q = 400 \quad \text{or} \quad q = 900$$

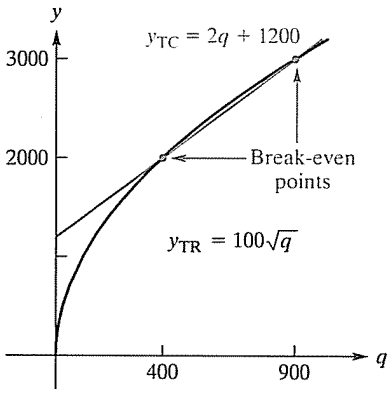


FIGURE 3.49 Two break-even points.

Although both $q = 400$ and $q = 900$ are break-even quantities, observe in Figure 3.49 that when $q > 900$, total cost is greater than total revenue, so there will always be a loss. This occurs because here total revenue is not linearly related to output. Thus, producing more than the break-even quantity does not necessarily guarantee a profit.

Now Work Problem 21 ◁

PROBLEMS 3.6

In Problems 1–8, you are given a supply equation and a demand equation for a product. If p represents price per unit in dollars and q represents the number of units per unit of time, find the equilibrium point. In Problems 1 and 2, sketch the system.

1. Supply: $p = \frac{2}{100}q + 3$, Demand: $p = -\frac{3}{100}q + 11$
2. Supply: $p = \frac{1}{1500}q + 4$, Demand: $p = -\frac{1}{2000}q + 9$
3. Supply: $35q - 2p + 250 = 0$, Demand: $65q + p - 537.5 = 0$
4. Supply: $246p - 3.25q - 2460 = 0$, Demand: $410p + 3q - 14{,}452.5 = 0$
5. Supply: $p = 2q + 20$, Demand: $p = 200 - 2q^2$
6. Supply: $p = (q + 12)^2$, Demand: $p = 644 - 6q - q^2$
7. Supply: $p = \sqrt{q + 10}$, Demand: $p = 20 - q$
8. Supply: $p = \frac{1}{4}q + 6$, Demand: $p = \dfrac{2240}{q + 12}$

In Problems 9–14, y_{TR} represents total revenue in dollars and y_{TC} represents total cost in dollars for a manufacturer. If q represents both the number of units produced and the number of units sold, find the break-even quantity. Sketch a break-even chart in Problems 9 and 10.

9. $y_{TR} = 4q$
$y_{TC} = 2q + 5000$

10. $y_{TR} = 14q$
$y_{TC} = \frac{40}{3}q + 1200$

11. $y_{TR} = 0.02q$
$y_{TC} = 0.5q + 30$

12. $y_{TR} = 0.25q$
$y_{TC} = 0.16q + 360$

13. $y_{TR} = 90 - \dfrac{900}{q+3}$
$y_{TC} = 1.1q + 37.3$

14. $y_{TR} = 0.1q^2 + 9q$
$y_{TC} = 3q + 400$

15. Business Supply and demand equations for a certain product are

$$3q - 200p + 1800 = 0$$

and

$$3q + 100p - 1800 = 0$$

respectively, where p represents the price per unit in dollars and q represents the number of units sold per time period.
(a) Find the equilibrium price algebraically, and derive it graphically.
(b) Find the equilibrium price when a tax of 27 cents per unit is imposed on the supplier.

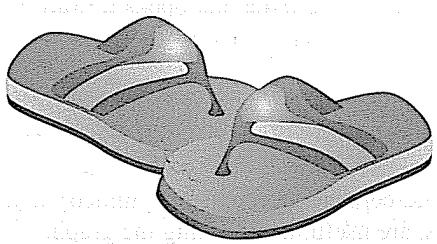
16. Business A manufacturer of a product sells all that is produced. The total revenue is given by $y_{TR} = 8q$, and the total cost is given by $y_{TC} = 7q + 500$, where q represents the number of units produced and sold.
(a) Find the level of production at the break-even point, and draw the break-even chart.
(b) Find the level of production at the break-even point if the total cost increases by 4%.

17. Business A manufacturer sells a product at \$8.35 per unit, selling all produced. The fixed cost is \$2116 and the variable cost is \$7.20 per unit. At what level of production will there be a profit of \$4600? At what level of production will there be a loss of \$1150? At what level of production will the break-even point occur?

18. Business The market equilibrium point for a product occurs when 13,500 units are produced at a price of \$4.50 per unit. The producer will supply no units at \$1, and the consumers will demand no units at \$20. Find the supply and demand equations if they are both linear.

19. Business A manufacturer of a children's toy will break even at a sales volume of \$200,000. Fixed costs are \$40,000, and each unit of output sells for \$5. Determine the variable cost per unit.

20. Business The Bigfoot Sandal Co. manufactures sandals for which the material cost is \$0.85 per pair and the labor cost is \$0.96 per pair. Additional variable costs amount to \$0.32 per pair. Fixed costs are \$70,500. If each pair sells for \$2.63, how many pairs must be sold for the company to break even?



21. Business **(a)** Find the break-even points for company X, which sells all it produces, if the variable cost per unit is \$3, fixed costs are \$2, and $y_{TR} = 5\sqrt{q}$, where q is the number of thousands of units of output produced.
(b) Graph the total revenue curve and the total cost curve in the same plane.
(c) Use your answer in **(a)** to report the quantity interval in which maximum profit occurs.

22. Business A company has determined that the demand equation for its product is $p = 1000/q$, where p is the price per unit for q units produced and sold in some period. Determine the quantity demanded when the price per unit is (a) \$4, (b) \$2, and (c) \$0.50. For each of these prices, determine the total revenue that the company will receive. What will be the revenue regardless of the price? [*Hint:* Find the revenue when the price is p dollars.]

23. Business Using the data in Example 1, determine how the original equilibrium price will be affected if the company is given a government subsidy of \$1.50 per unit.

24. Business The Monroe Forging Company sells a corrugated steel product to the Standard Manufacturing Company and is in competition on such sales with other suppliers of the Standard Manufacturing Co. The vice president of sales of Monroe Forging Co. believes that by reducing the price of the product, a 40% increase in the volume of units sold to the Standard Manufacturing Co. could be secured. As the manager of the cost and analysis department, you have been asked to analyze the proposal of the vice president and submit your recommendations as to whether it is financially beneficial to the Monroe Forging Co. You are specifically requested to determine the following:
(a) Net profit or loss based on the pricing proposal
(b) Unit sales volume under the proposed price that is required to make the same \$40,000 profit that is now earned at the current price and unit sales volume
Use the following data in your analysis:

	Current Operations	Proposal of Vice President of Sales
Unit price	\$2.50	\$2.00
Unit sales volume	200,000 units	280,000 units
Variable cost		
Total	\$350,000	\$490,000
Per unit	\$1.75	\$1.75
Fixed cost	\$110,000	\$110,000
Profit	\$40,000	?

25. Business Suppose products A and B have demand and supply equations that are related to each other. If q_A and q_B are the quantities produced and sold of A and B, respectively, and p_A and p_B are their respective prices, the demand equations are

$$q_A = 7 - p_A + p_B$$

and

$$q_B = 24 + p_A - p_B$$

and the supply equations are

$$q_A = -3 + 4p_A - 2p_B$$

and

$$q_B = -5 - 2p_A + 4p_B$$

Eliminate q_A and q_B to get the equilibrium prices.

26. Business The supply equation for a product is

$$p = q^2 - 10$$

and the demand equation is

$$p = \frac{20}{q-2}$$

Here p represents price per unit in dollars and $q > 3.2$ represents number of units (in thousands) per unit time. Graph both equations, and, from your graph, determine the equilibrium price and equilibrium quantity to one decimal place.

27. Business For a manufacturer, the total-revenue equation is

$$y_{TR} = 20.5\sqrt{q+4} - 41$$

and the total-cost equation is

$$y_{TC} = 0.02q^3 + 10.4,$$

where q represents (in thousands) both the number of units produced and the number of units sold. Graph a break-even chart and find the break-even quantity.

Chapter 3 Review

Important Terms and Symbols

Examples

Section 3.1 Lines

slope of a line — point-slope form — slope-intercept form — Ex. 1, p. 129
general linear equation in x and y — linearly related — Ex. 7, p. 132

Section 3.2 Applications and Linear Functions

demand equation — demand curve — supply equation — supply curve — Ex. 2, p. 137
linear function — Ex. 3, p. 138

Section 3.3 Quadratic Functions

quadratic function — parabola — axis of symmetry — vertex — Ex. 1, p. 143

Section 3.4 Systems of Linear Equations

system of equations — equivalent systems — elimination by addition — Ex. 1, p. 150
elimination by substitution — parameter — Ex. 3, p. 152
general linear equation in x, y, and z — Ex. 5, p. 154

Section 3.5 Nonlinear Systems

nonlinear system — Ex. 1, p. 159

Section 3.6 Applications of Systems of Equations

point of equilibrium — equilibrium price — equilibrium quantity — Ex. 1, p. 162
break-even point — break-even quantity — break-even revenue — Ex. 3, p. 165

Summary

The orientation of a nonvertical line is characterized by the slope of the line given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are two different points on the line. The slope of a vertical line is not defined, and the slope of a horizontal line is zero. Lines rising from left to right have positive slopes; lines falling from left to right have negative slopes. Two lines are parallel if and only if they have the same slope or both are vertical. Two lines with non-zero slopes m_1 and m_2 are perpendicular to each other if and only if $m_1 = -\frac{1}{m_2}$. Any horizontal line and any vertical line are perpendicular to each other.

Basic forms of equations of lines are as follows:

$y - y_1 = m(x - x_1)$	point-slope form
$y = mx + b$	slope-intercept form
$x = a$	vertical line
$y = b$	horizontal line
$Ax + By + C = 0$	general

The linear function

$$f(x) = ax + b \quad (a \neq 0)$$

has a straight line for its graph.

In economics, supply functions and demand functions have the form $p = f(q)$ and play an important role. Each gives a correspondence between the price p of a product and the number of units q of the product that manufacturers (or consumers) will supply (or purchase) at that price during some time period.

A quadratic function has the form

$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$

The graph of f is a parabola that opens upward if $a > 0$ and downward if $a < 0$. The vertex is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

and the y-intercept is c. The axis of symmetry and the x- and y-intercepts, are useful in sketching the graph.

A system of linear equations can be solved with the method of elimination by addition or elimination by substitution. A solution may involve one or more parameters. Substitution is also useful in solving nonlinear systems.

Solving a system formed by the supply and demand equations for a product gives the equilibrium point, which indicates the price at which consumers will purchase the same quantity of a product that producers wish to sell at that price.

Profit is total revenue minus total cost, where total cost is the sum of fixed costs and variable costs. The break-even points are the points where total revenue equals total cost.

Review Problems

1. The slope of the line through (2, 5) and (3, k) is 4. Find k.

2. The slope of the line through (5, 4) and (k, 4) is 0. Find k.

In Problems 3–9, determine the slope-intercept form and a general linear form of an equation of the straight line that has the indicated properties.

3. Passes through $(-2, 3)$ and has y-intercept -1

4. Passes through $(-1, -1)$ and is parallel to the line $y = 3x - 4$

5. Passes through $(8, 3)$ and has slope 3

6. Passes through (3, 5) and is vertical

7. Passes through $(-2, 4)$ and is horizontal

8. Passes through (1, 2) and is perpendicular to the line $-3y + 5x = 7$

9. Has y-intercept -3 and is perpendicular to $2y + 5x = 2$

10. Determine whether the point (3, 11) lies on the line through (2, 7) and (4, 13).

In Problems 11–16, determine whether the lines are parallel, perpendicular, or neither.

11. $x + 4y + 2 = 0$, $8x - 2y - 2 = 0$

12. $y - 2 = 2(x - 1)$, $2x + 4y - 3 = 0$

13. $x - 3 = 2(y + 4)$, $y = 4x + 2$

14. $2x + 7y - 4 = 0$, $6x + 21y = 90$

15. $y = 5x + 2$, $10x - 2y = 3$

16. $y = 7x$, $y = 7$

In Problems 17–20, write each line in slope-intercept form, and sketch. What is the slope of the line?

17. $3x - 2y = 4$

18. $x = -3y + 4$

19. $4 - 3y = 0$

20. $3x - 5y = 0$

In Problems 21–30, graph each function. For those that are linear, give the slope and the vertical-axis intercept. For those that are quadratic, give all intercepts and the vertex.

21. $y = f(x) = 17 - 5x$

22. $s = g(t) = 5 - 3t + t^2$

23. $y = f(x) = 9 - x^2$

24. $y = f(x) = 3x - 7$

25. $y = h(t) = 3 + 2t + t^2$

26. $y = k(t) = -3 - 3t$

27. $p = g(t) = -7t$

28. $y = F(x) = (2x - 1)^2$

29. $y = F(x) = -(x^2 + 2x + 3)$

30. $y = f(x) = 5x + 2$

In Problems 31–44, solve the given system.

31. $\begin{cases} 2x - y = 6 \\ 3x + 2y = 5 \end{cases}$

32. $\begin{cases} 8x - 4y = 7 \\ y = 2x - 4 \end{cases}$

33. $\begin{cases} 7x + 5y = 5 \\ 6x + 5y = 3 \end{cases}$

34. $\begin{cases} 2x + 4y = 8 \\ 3x + 6y = 12 \end{cases}$

35. $\begin{cases} \frac{1}{2}x - \frac{1}{3}y = 2 \\ \frac{3}{4}x + \frac{1}{2}y = 3 \end{cases}$

36. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = \frac{1}{12} \\ \frac{4}{3}x + 3y = \frac{5}{3} \end{cases}$

37. $\begin{cases} 3x - 2y + z = -2 \\ 2x + y + z = 1 \\ x + 3y - z = 3 \end{cases}$

38. $\begin{cases} 2x + \frac{3y + x}{3} = 9 \\ y + \frac{5x + 2y}{4} = 7 \end{cases}$

39. $\begin{cases} x^2 - y + 5x = 2 \\ x^2 + y = 3 \end{cases}$

40. $\begin{cases} y = \frac{3}{x + 2} \\ x + y - 2 = 0 \end{cases}$

41. $\begin{cases} x + 2z = -2 \\ x + y + z = 5 \end{cases}$

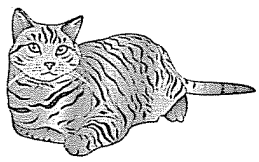
42. $\begin{cases} x + y + z = 0 \\ x - y + z = 0 \\ x + z = 0 \end{cases}$

43. $\begin{cases} x - y - z = 0 \\ 2x - 2y + 3z = 0 \end{cases}$

44. $\begin{cases} 2x - 5y + 6z = 1 \\ 4x - 10y + 12z = 2 \end{cases}$

45. Suppose a and b are linearly related so that $a = 0$ when $b = -3$ and $a = 3$ when $b = -5$. Find a general linear form of an equation that relates a and b. Also, find a when $b = 3$.

46. Temperature and Heart Rate When the temperature, T (in degrees Celsius), of a cat is reduced, the cat's heart rate, r (in beats per minute), decreases. Under laboratory conditions, a cat at a temperature of 36°C had a heart rate of 206, and at a temperature of 30°C its heart rate was 122.
If r is linearly related to T, where T is between 26 and 38, (a) determine an equation for r in terms of T, and (b) determine the cat's heart rate at a temperature of 27°C.



47. Suppose f is a linear function such that $f(1) = 5$ and $f(x)$ decreases by four units for every three-unit increase in x. Find $f(x)$.

48. If f is a linear function such that $f(-1) = 8$ and $f(2) = 5$, find $f(x)$.

49. Maximum Revenue The demand function for a manufacturer's product is $p = f(q) = 200 - 2q$, where p is the price (in dollars) per unit when q units are demanded. Find the level of production that maximizes the manufacturer's total revenue, and determine this revenue.

50. Sales Tax The difference in price of two items before a 7% sales tax is imposed is \$2.00. The difference in price after the sales tax is imposed is allegedly \$3.10. Show that this scenario is not possible.

51. Equilibrium Price If the supply and demand equations of a certain product are $120p - q - 240 = 0$ and $100p + q - 1200 = 0$, respectively, find the equilibrium price.

4 Exponential and Logarithmic Functions

4.1 Exponential Functions

4.2 Logarithmic Functions

4.3 Properties of Logarithms

4.4 Logarithmic and Exponential Equations

Chapter 4 Review

EXPLORE & EXTEND
Drug Dosages

Just as biological viruses spread through contact between organisms, so computer viruses spread when computers interact via the Internet. Computer scientists study how to fight computer viruses, which cause a lot of damage in the form of deleted and corrupted files. One thing computer scientists do is devise mathematical models of how quickly viruses spread. For instance, on Friday, March 26, 1999, the first cases of the virus known as Melissa were reported; by Monday, March 29, Melissa had reached over 100,000 computers.

Exponential functions, which this chapter discusses in detail, provide one plausible model. Consider a computer virus that hides in an e-mail attachment and, once the attachment is downloaded, automatically sends a message with a similar attachment to every address in the host computer's e-mail address book. If the typical address book contains 20 addresses, and if the typical computer user retrieves his or her e-mail once a day, then a virus on a single machine will have infected 20 machines after one day, $20^2 = 400$ machines after two days, $20^3 = 8000$ after three days, and, in general, after t days, the number N of infected computers will be given by the exponential function $N(t) = 20^t$.

This model assumes that all the computers involved are linked, via their address book lists, into a single, well-connected group. Exponential models are most accurate for small values of t; this model, in particular, ignores the slowdown that occurs when most e-mails start going to computers already infected, which happens as several days pass. For example, our model tells us that after eight days, the virus will infect $20^8 = 25.6$ billion computers—more computers than actually exist! But despite its limitations, the exponential model does explain why new viruses often infect many thousands of machines before antivirus experts have had time to react.

Objective

To study exponential functions and their applications to such areas as compound interest, population growth, and radioactive decay.

4.1 Exponential Functions

The functions of the form $f(x) = b^x$, for constant b, are important in mathematics, business, economics, science and other areas of study. An excellent example is $f(x) = 2^x$. Such functions are called *exponential functions*. More precisely,

Definition

The function f defined by

$$f(x) = b^x$$

where $b > 0, b \neq 1$, and the exponent x is any real number, is called an ***exponential function*** with base b.[1]

CAUTION

Do not confuse the exponential function $y = 2^x$ with the *power function* $y = x^2$, which has a variable base and a constant exponent.

Since the exponent in b^x can be any real number, you may wonder how we assign a value to something like $2^{\sqrt{2}}$, where the exponent is an irrational number. Stated simply, we use approximations. Because $\sqrt{2} = 1.41421\ldots$, $2^{\sqrt{2}}$ is approximately $2^{1.4} = 2^{7/5} = \sqrt[5]{2^7}$, which *is* defined. Better approximations are $2^{1.41} = 2^{141/100} = \sqrt[100]{2^{141}}$, and so on. In this way, the meaning of $2^{\sqrt{2}}$ becomes clear. A calculator value of $2^{\sqrt{2}}$ is (approximately) 2.66514.

When you work with exponential functions, it may be necessary to apply rules for exponents. These rules are as follows, where x and y are real numbers and b and c are positive.

TO REVIEW exponents, refer to Section 0.3.

Rules for Exponents

1. $b^x b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$
4. $(bc)^x = b^x c^x$
5. $\left(\dfrac{b}{c}\right)^x = \dfrac{b^x}{c^x}$
6. $b^1 = b$
7. $b^0 = 1$
8. $b^{-x} = \dfrac{1}{b^x}$

Some functions that do not appear to have the exponential form b^x can be put in that form by applying the preceding rules. For example, $2^{-x} = 1/(2^x) = (\frac{1}{2})^x$ and $3^{2x} = (3^2)^x = 9^x$.

EXAMPLE 1 **Bacteria Growth**

APPLY IT ▶

1. The number of bacteria in a culture that doubles every hour is given by $N(t) = A \cdot 2^t$, where A is the number originally present and t is the number of hours the bacteria have been doubling. Use a graphing calculator to plot this function for various values of $A > 1$. How are the graphs similar? How does the value of A alter the graph?

The number of bacteria present in a culture after t minutes is given by

$$N(t) = 300\left(\frac{4}{3}\right)^t$$

Note that $N(t)$ is a constant multiple of the exponential function $\left(\frac{4}{3}\right)^t$.

[1] If $b = 1$, then $f(x) = 1^x = 1$. This function is already known to us as a constant function.

a. How many bacteria are present initially?

Solution: Here we want to find $N(t)$ when $t = 0$. We have

$$N(0) = 300\left(\frac{4}{3}\right)^0 = 300(1) = 300$$

Thus, 300 bacteria are initially present.

b. Approximately how many bacteria are present after 3 minutes?

Solution:

$$N(3) = 300\left(\frac{4}{3}\right)^3 = 300\left(\frac{64}{27}\right) = \frac{6400}{9} \approx 711$$

Hence, approximately 711 bacteria are present after 3 minutes.

Now Work Problem 31 ◁

Graphs of Exponential Functions

APPLY IT ▶

2. Suppose an investment increases by 10% every year. Make a table of the factor by which the investment increases from the original amount for 0 to 4 years. For each year, write an expression for the increase as a power of some base. What base did you use? How does that base relate to the problem? Use your table to graph the multiplicative increase as a function of the number of years. Use your graph to determine when the investment will double.

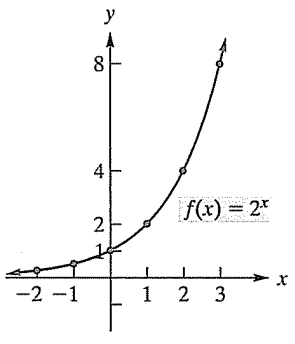
EXAMPLE 2 **Graphing Exponential Functions with $b > 1$**

Graph the exponential functions $f(x) = 2^x$ and $f(x) = 5^x$.

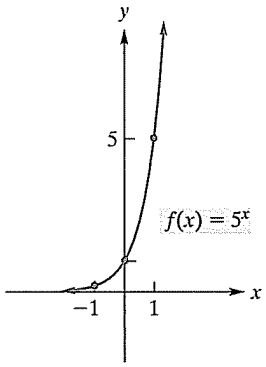
Solution: By plotting points and connecting them, we obtain the graphs in Figure 4.1. For the graph of $f(x) = 5^x$, because of the unit distance chosen on the y-axis, the points $(-2, \frac{1}{25})$, (2, 25), and (3, 125) are not shown.

We can make some observations about these graphs. The domain of each function consists of all real numbers, and the range consists of all positive real numbers. Each graph has y-intercept (0, 1). Moreover, the graphs have the same general shape. Each *rises* from left to right. As x increases, $f(x)$ also increases. In fact, $f(x)$ increases without bound. However, in quadrant I, the graph of $f(x) = 5^x$ rises more quickly than that of $f(x) = 2^x$ because the base in 5^x is *greater* than the base in 2^x (that is, $5 > 2$). Looking at quadrant II, we see that as x becomes very negative, the graphs of both functions approach the x-axis.[2] This implies that the function values get very close to 0.

x	2^x
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



(a)



(b)

x	5^x
-2	$\frac{1}{25}$
-1	$\frac{1}{5}$
0	1
1	5
2	25
3	125

FIGURE 4.1 Graphs of $f(x) = 2^x$ and $f(x) = 5^x$.

Now Work Problem 1 ◁

[2]We say that the x-axis is an *asymptote* for each graph.

The observations made in Example 2 are true for all exponential functions whose base b is greater than 1. Example 3 will examine the case for a base between 0 and 1 ($0 < b < 1$).

APPLY IT ▶

3. Suppose the value of a car depreciates by 15% every year. Make a table of the factor by which the value decreases from the original amount for 0 to 3 years. For each year, write an expression for the decrease as a power of some base. What base did you use? How does that base relate to the problem? Use your table to graph the multiplicative decrease as a function of the number of years. Use your graph to determine when the car will be worth half as much as its original price.

EXAMPLE 3 **Graphing Exponential Functions with $0 < b < 1$**

Graph the exponential function $f(x) = \left(\frac{1}{2}\right)^x$.

Solution: By plotting points and connecting them, we obtain the graph in Figure 4.2. Notice that the domain consists of all real numbers, and the range consists of all positive real numbers. The graph has y-intercept (0, 1). Compared to the graphs in Example 2, the graph here *falls* from left to right. That is, as x increases, $f(x)$ decreases. Notice that as x becomes very positive, $f(x)$ takes on values close to 0 and the graph approaches the x-axis. However, as x becomes very negative, the function values are unbounded.

Now Work Problem 3 ◁

x	$\left(\frac{1}{2}\right)^x$
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

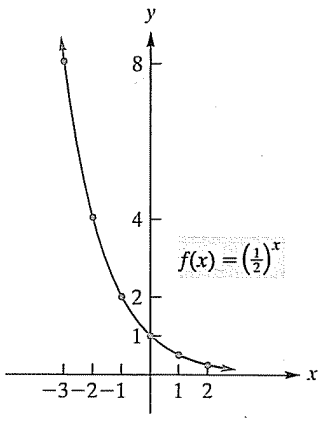


FIGURE 4.2 Graph of $f(x) = \left(\frac{1}{2}\right)^x$.

In general, the graph of an exponential function has one of two shapes, depending on the value of the base b. This is illustrated in Figure 4.3. It is important to observe that in either case the graph passes the horizontal line test. Thus all exponential functions are one-to-one. The basic properties of an exponential function and its graph are summarized in Table 4.1.

Recall from Section 2.7 that the graph of one function may be related to that of another by means of a certain transformation. Our next example pertains to this concept.

There are two basic shapes for the graphs of exponential functions, and they depend on the base involved.

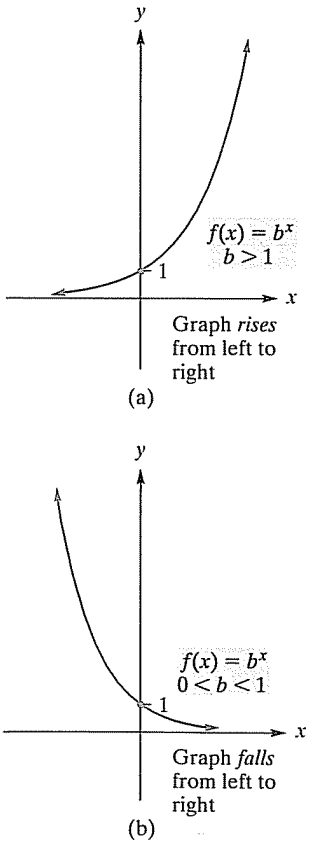


FIGURE 4.3 General shapes of $f(x) = b^x$.

Table 4.1 Properties of the Exponential Function $f(x) = b^x$

1. The domain of any exponential function is $(-\infty, \infty)$.	The range of any exponential function is $(0, \infty)$.
2. The graph of $f(x) = b^x$ has y-intercept (0, 1).	There is no x-intercept.
3. If $b > 1$, the graph *rises* from left to right.	If $0 < b < 1$, the graph *falls* from left to right.
4. If $b > 1$, the graph approaches the x-axis as x becomes more and more negative.	If $0 < b < 1$, the graph approaches the x-axis as x becomes more and more positive.

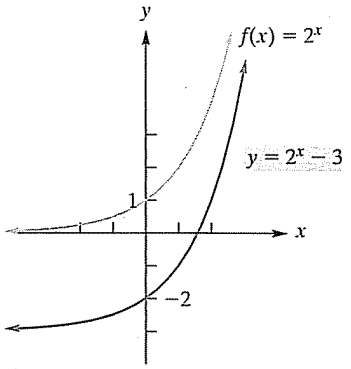


FIGURE 4.4 Graph of $y = 2^x - 3$.

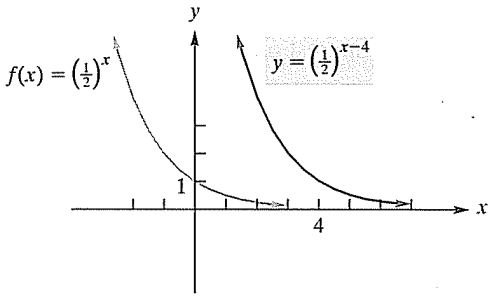
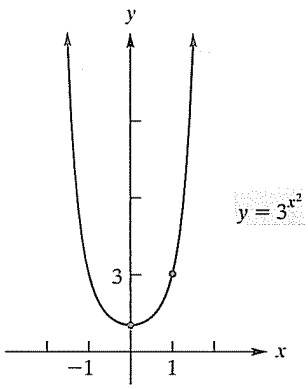


FIGURE 4.5 Graph of $y = \left(\frac{1}{2}\right)^{x-4}$.



x	0	1	2
y	1	3	81

FIGURE 4.6 Graph of $y = 3^{x^2}$.

Example 4 makes use of transformations from Table 2.2 of Section 2.7.

APPLY IT ▶

4. After watching his sister's money grow for three years in a plan with an 8% yearly return, George started a savings account with the same plan. If $y = 1.08^t$ represents the multiplicative increase in his sister's account, write an equation that will represent the multiplicative increase in George's account, using the same time reference. If George has a graph of the multiplicative increase in his sister's money at time t years since she started saving, how could he use the graph to project the increase in his money?

EXAMPLE 4 **Transformations of Exponential Functions**

a. Use the graph of $y = 2^x$ to plot $y = 2^x - 3$.

Solution: The function has the form $f(x) - c$, where $f(x) = 2^x$ and $c = 3$. Thus its graph is obtained by shifting the graph of $f(x) = 2^x$ three units downward. (See Figure 4.4.)

b. Use the graph of $y = \left(\frac{1}{2}\right)^x$ to graph $y = \left(\frac{1}{2}\right)^{x-4}$.

Solution: The function has the form $f(x - c)$, where $f(x) = \left(\frac{1}{2}\right)^x$ and $c = 4$. Hence, its graph is obtained by shifting the graph of $f(x) = \left(\frac{1}{2}\right)^x$ four units to the right. (See Figure 4.5.)

Now Work Problem 7 ◁

EXAMPLE 5 **Graph of a Function with a Constant Base**

Graph $y = 3^{x^2}$.

Solution: Although this is not an exponential function, it does have a constant base. We see that replacing x by $-x$ results in the same equation. Thus, the graph is symmetric about the y-axis. Plotting some points and using symmetry gives the graph in Figure 4.6.

Now Work Problem 5 ◁

TECHNOLOGY

If $y = 4^x$, consider the problem of finding x when $y = 6$. One way to solve this equation is to find the intersection of the graphs of $y = 6$ and $y = 4^x$. Figure 4.7 shows that x is approximately 1.29.

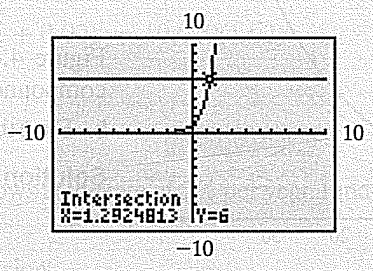


FIGURE 4.7 Solving the equation $6 = 4^x$.

Compound Interest

Exponential functions are involved in **compound interest,** whereby the interest earned by an invested amount of money (or **principal**) is reinvested so that it, too, earns interest. That is, the interest is converted (or *compounded*) into principal, and hence, there is "interest on interest."

For example, suppose that \$100 is invested at the rate of 5% compounded annually. At the end of the first year, the value of the investment is the original principal (\$100), plus the interest on the principal [100(0.05)]:

$$100 + 100(0.05) = \$105$$

This is the amount on which interest is earned for the second year. At the end of the second year, the value of the investment is the principal at the end of the first year (\$105), plus the interest on that sum [105(0.05)]:

$$105 + 105(0.05) = \$110.25$$

Thus, each year the principal increases by 5%. The \$110.25 represents the original principal, plus all accrued interest; it is called the **accumulated amount** or **compound amount.** The difference between the compound amount and the original principal is called the **compound interest.** Here the compound interest is $110.25 - 100 = 10.25$.

More generally, if a principal of P dollars is invested at a rate of $100r$ percent compounded annually (for example, at 5%, r is 0.05), the compound amount after 1 year is $P + Pr$, or, by factoring, $P(1 + r)$. At the end of the second year, the compound amount is

$$P(1 + r) + [P(1 + r)]r = P(1 + r)[1 + r] \qquad \text{factoring}$$
$$= P(1 + r)^2$$

Actually, the calculation above using factoring is not necessary to show that the compounded amount after two years is $P(1+r)^2$. Since *any* amount P is worth $P(1+r)$ a year later, it follows that the amount of $P(1 + r)$ is worth $P(1 + r)(1 + r) = P(1 + r)^2$ a year later and one year later still the amount of $P(1+r)^2$ is worth $P(1+r)^2(1+r) = P(1+r)^3$.

This pattern continues. After four years, the compound amount is $P(1 + r)^4$. In general, **the compound amount S of the principal P at the end of n years at the rate of r compounded annually** is given by

$$S = P(1 + r)^n \tag{1}$$

APPLY IT ▷

5. Suppose \$2000 is invested at 13% compounded annually. Find the value of the investment after five years. Find the interest earned over the first five years.

Notice from Equation (1) that, for a given principal and rate, S is a function of n. In fact, S is a constant multiple of the exponential function with base $1 + r$.

EXAMPLE 6 **Compound Amount and Compound Interest**

Suppose \$1000 is invested for 10 years at 6% compounded annually.

a. Find the compound amount.

Solution: We use Equation (1) with $P = 1000$, $r = 0.06$, and $n = 10$:

$$S = 1000(1 + 0.06)^{10} = 1000(1.06)^{10} \approx \$1790.85$$

Figure 4.8 shows the graph of $S = 1000(1.06)^n$. Notice that as time goes on, the compound amount grows dramatically.

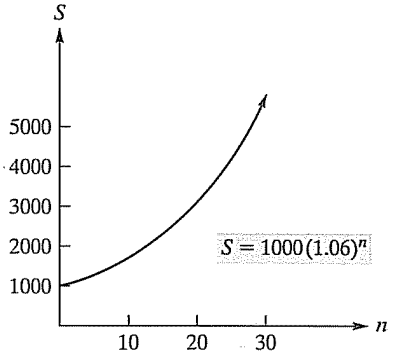


FIGURE 4.8 Graph of $S = 1000(1.06)^n$.

b. Find the compound interest.

Solution: Using the results from part (a), we have

$$\text{compound interest} = S - P$$
$$= 1790.85 - 1000 = \$790.85$$

Now Work Problem 19 ◁

Suppose the principal of \$1000 in Example 6 is invested for 10 years as before, but this time the compounding takes place every three months (that is, *quarterly*) at the rate of $1\frac{1}{2}\%$ *per quarter*. Then there are four **interest periods** per year, and in 10 years there are $10(4) = 40$ interest periods. Thus, the compound amount with $r = 0.015$ is now

$$1000(1.015)^{40} \approx \$1814.02$$

and the compound interest is \$814.02. Usually, the interest rate per interest period is stated as an annual rate. Here we would speak of an annual rate of 6% compounded quarterly, so that the rate per interest period, or the **periodic rate,** is $6\%/4 = 1.5\%$. This *quoted* annual rate of 6% is called the **nominal rate** or the **annual percentage rate (APR).** Unless otherwise stated, all interest rates will be assumed to be annual (nominal) rates. Thus a rate of 15% compounded monthly corresponds to a periodic rate of $15\%/12 = 1.25\%$.

The abbreviation APR is a common one and is found on credit card statements and in advertising.

On the basis of our discussion, we can generalize Equation (1). The formula

$$S = P(1 + r)^n \tag{2}$$

gives **the compound amount S of a principal P at the end of n interest periods at the periodic rate of r.**

CAUTION!

A nominal rate of 6% does not necessarily mean that an investment increases in value by 6% in a year's time. The increase depends on the frequency of compounding.

We have seen that for a principal of \$1000 at a nominal rate of 6% over a period of 10 years, annual compounding results in a compound interest of \$790.85, and with quarterly compounding the compound interest is \$814.02. It is typical that for a given nominal rate, the more frequent the compounding, the greater is the compound interest. However, while increasing the compounding frequency always increases the amount of interest earned, the effect is not unbounded. For example, with weekly compounding the compound interest is

$$1000\left(1 + \frac{0.06}{52}\right)^{10(52)} - 1000 \approx \$821.49$$

and with daily compounding it is

$$1000\left(1 + \frac{0.06}{365}\right)^{10(365)} - 1000 \approx \$822.03$$

Sometimes the phrase "money is worth" is used to express an annual interest rate. Thus, saying that money is worth 6% compounded quarterly refers to an annual (nominal) rate of 6% compounded quarterly.

Population Growth

Equation (2) can be applied not only to the growth of money, but also to other types of growth, such as that of population. For example, suppose the population P of a town of 10,000 is increasing at the rate of 2% per year. Then P is a function of time t, in years. It is common to indicate this functional dependence by writing

$$P = P(t)$$

Here the letter P is used in two ways: On the right side, P represents the function; on the left side, P represents the dependent variable. From Equation (2), we have

$$P(t) = 10{,}000(1 + 0.02)^t = 10{,}000(1.02)^t$$

APPLY IT ▶

6. A new company with five employees expects the number of employees to grow at the rate of 120% per year. Find the number of employees in four years.

EXAMPLE 7 Population Growth

The population of a town of 10,000 grows at the rate of 2% per year. Find the population three years from now.

Solution: From the preceding discussion,

$$P(t) = 10{,}000(1.02)^t$$

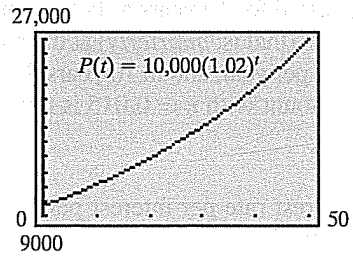


FIGURE 4.9 Graph of population function $P(t) = 10{,}000(1.02)^t$.

For $t = 3$, we have

$$P(3) = 10{,}000(1.02)^3 \approx 10,612$$

Thus, the population three years from now will be 10,612. (See Figure 4.9.)

Now Work Problem 15 ◁

The Number e

It is useful to conduct a "thought experiment," based on the discussion following Example 6, to introduce an important number. Suppose that a single dollar is invested for one year with an APR of 100% (remember, this is a thought experiment!) compounded annually. Then the compound amount S at the end of the year is given by

$$S = 1(1 + 1)^1 = 2^1 = 2$$

Without changing any of the other data, we now consider the effect of increasing the number of interest periods per year. If there are n interest periods per year, then the compound amount is given by

$$S = 1\left(1 + \frac{1}{n}\right)^n = \left(\frac{n+1}{n}\right)^n$$

In Table 4.2 we give approximate values for $\left(\frac{n+1}{n}\right)^n$ for some values of n.

Table 4.2 Approximations of e

n	$\left(\frac{n+1}{n}\right)^n$
1	$\left(\frac{2}{1}\right)^1 = 2.00000$
2	$\left(\frac{3}{2}\right)^2 = 2.25000$
3	$\left(\frac{4}{3}\right)^3 \approx 2.37037$
4	$\left(\frac{5}{4}\right)^4 \approx 2.44141$
5	$\left(\frac{6}{5}\right)^5 = 2.48832$
10	$\left(\frac{11}{10}\right)^{10} \approx 2.59374$
100	$\left(\frac{101}{100}\right)^{100} \approx 2.70481$
1000	$\left(\frac{1001}{1000}\right)^{1000} \approx 2.71692$
10,000	$\left(\frac{10{,}001}{10{,}000}\right)^{10{,}000} \approx 2.71815$
100,000	$\left(\frac{100{,}001}{100{,}000}\right)^{100{,}000} \approx 2.71827$
1,000,000	$\left(\frac{1{,}000{,}001}{1{,}000{,}000}\right)^{1{,}000{,}000} \approx 2.71828$

Apparently, the numbers $\left(\frac{n+1}{n}\right)^n$ increase as n does. However, they do not increase without bound. For example, it is possible to show that for any positive integer n, $\left(\frac{n+1}{n}\right)^n < 3$. In terms of our thought experiment, this means that if you start with \$1.00 invested at 100%, then, no matter how many interest periods there are per year, you will always have less than \$3.00 at the end of a year. There is a smallest real number that is greater than all of the numbers $\left(\frac{n+1}{n}\right)^n$. It is denoted by the letter e, in honor of the Swiss mathematician Leonhard Euler (1707–1783). The number e is irrational so

that its decimal expansion is nonrepeating, like those of π and $\sqrt{2}$ that we mentioned in Section 0.1. However, each of the numerical values for $\left(\frac{n+1}{n}\right)^n$ can be considered to be a decimal approximation of e. The approximate value $(\frac{1{,}000{,}001}{1{,}000{,}000})^{1{,}000{,}000} \approx 2.71828$ gives an approximation of e that is correct to 5 decimal places. The approximation of e correct to 12 decimal places is $e \approx 2.718281828459$.

Exponential Function with Base e

The number e provides the most important base for an exponential function. In fact, the exponential function with base e is called the **natural exponential function** and even ***the*** **exponential function** to stress its importance.

Although e may seem to be a strange base, the natural exponential function has a remarkable property in calculus (which we will see in a later chapter) that justifies the name. It also occurs in economic analysis and problems involving growth or decay, such as population studies, compound interest, and radioactive decay. Approximate values of e^x can be found with a single key on most calculators. The graph of $y = e^x$ is shown in Figure 4.10. The accompanying table indicates y-values to two decimal places. Of course, the graph has the general shape of an exponential function with base greater than 1.

The graph of the natural exponential function in Figure 4.10 is important.

x	y
-2	0.14
-1	0.37
0	1
1	2.72
2	7.39

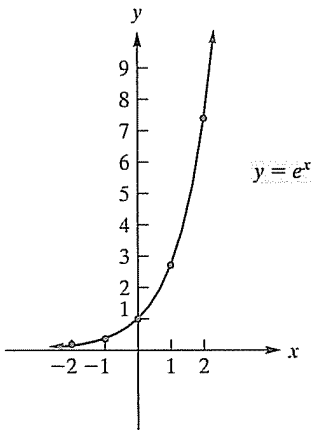


FIGURE 4.10 Graph of natural exponential function.

APPLY IT ▶

7. The multiplicative decrease in purchasing power P after t years of inflation at 6% can be modeled by $P = e^{-0.06t}$. Graph the decrease in purchasing power as a function of t years.

EXAMPLE 8 **Graphs of Functions Involving e**

a. Graph $y = e^{-x}$.

Solution: Since $e^{-x} = \left(\frac{1}{e}\right)^x$ and $0 < \frac{1}{e} < 1$, the graph is that of an exponential function falling from left to right. (See Figure 4.11.) Alternatively, we can consider the graph of $y = e^{-x}$ as a transformation of the graph of $f(x) = e^x$. Because $e^{-x} = f(-x)$, the graph of $y = e^{-x}$ is simply the reflection of the graph of f about the y-axis. (Compare the graphs in Figures 4.10 and 4.11.)

b. Graph $y = e^{x+2}$.

Solution: The graph of $y = e^{x+2}$ is related to that of $f(x) = e^x$. Since e^{x+2} is $f(x+2)$, we can obtain the graph of $y = e^{x+2}$ by horizontally shifting the graph of $f(x) = e^x$ two units to the left. (See Figure 4.12.)

◁

x	y
−2	7.39
−1	2.72
0	1
1	0.37
2	0.14

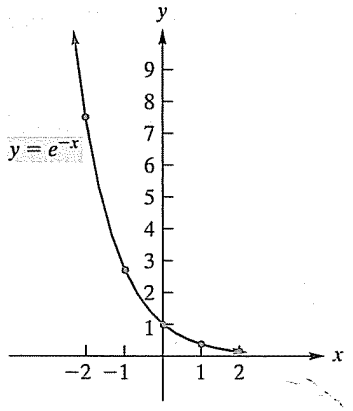


FIGURE 4.11 Graph of $y = e^{-x}$.

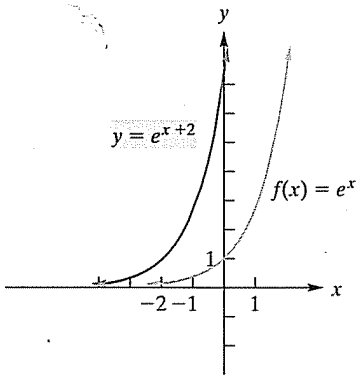


FIGURE 4.12 Graph of $y = e^{x+2}$.

EXAMPLE 9 Population Growth

The projected population P of a city is given by

$$P = 100{,}000e^{0.05t}$$

where t is the number of years after 1990. Predict the population for the year 2010.

Solution: The number of years from 1990 to 2010 is 20, so let $t = 20$. Then

$$P = 100{,}000e^{0.05(20)} = 100{,}000e^1 = 100{,}000e \approx 271{,}828$$

Now Work Problem 35 ◁

In statistics, an important function used to model certain events occurring in nature is the **Poisson distribution function:**

$$f(n) = \frac{e^{-\mu}\mu^n}{n!} \quad n = 0, 1, 2, \ldots$$

The symbol μ (read "mu") is a Greek letter. In certain situations, $f(n)$ gives the probability that exactly n events will occur in an interval of time or space. The constant μ is the average, also called *mean,* number of occurrences in the interval. The next example illustrates the Poisson distribution.

EXAMPLE 10 Hemocytometer and Cells

A hemocytometer is a counting chamber divided into squares and is used in studying the number of microscopic structures in a liquid. In a well-known experiment,[3] yeast cells were diluted and thoroughly mixed in a liquid, and the mixture was placed in a hemocytometer. With a microscope, the number of yeast cells on each square were counted. The probability that there were exactly n yeast cells on a hemocytometer square was found to fit a Poisson distribution with $\mu = 1.8$. Find the probability that there were exactly four cells on a particular square.

Solution: We use the Poisson distribution function with $\mu = 1.8$ and $n = 4$:

$$f(n) = \frac{e^{-\mu}\mu^n}{n!}$$

$$f(4) = \frac{e^{-1.8}(1.8)^4}{4!} \approx 0.072$$

[3] R. R. Sokal and F. J. Rohlf, *Introduction to Biostatistics* (San Francisco: W. H. Freeman and Company, 1973).

For example, this means that in 400 squares we would *expect* $400(0.072) \approx 29$ squares to contain exactly 4 cells. (In the experiment, in 400 squares the actual number observed was 30.)

◁

Radioactive Decay

Radioactive elements are such that the amount of the element decreases with respect to time. We say that the element *decays*. It can be shown that, if N is the amount at time t, then

$$N = N_0 e^{-\lambda t} \tag{3}$$

where N_0 and λ (a Greek letter read "lambda") are positive constants. Notice that N involves an exponential function of t. We say that N follows an **exponential law of decay.** If $t = 0$, then $N = N_0 e^0 = N_0 \cdot 1 = N_0$. Thus, the constant N_0 represents the amount of the element present at time $t = 0$ and is called the **initial amount.** The constant λ depends on the particular element involved and is called the **decay constant.**

Because N decreases as time progresses, suppose we let T be the length of time it takes for the element to decrease to half of the initial amount. Then at time $t = T$, we have $N = N_0/2$. Equation (3) implies that

$$\frac{N_0}{2} = N_0 e^{-\lambda T}$$

We will now use this fact to show that over *any* time interval of length T, half of the amount of the element decays. Consider the interval from time t to $t + T$, which has length T. At time t, the amount of the element is $N_0 e^{-\lambda t}$, and at time $t + T$ it is

$$\begin{aligned} N_0 e^{-\lambda(t+T)} &= N_0 e^{-\lambda t} e^{-\lambda T} = (N_0 e^{-\lambda T}) e^{-\lambda t} \\ &= \frac{N_0}{2} e^{-\lambda t} = \frac{1}{2}(N_0 e^{-\lambda t}) \end{aligned}$$

which is half of the amount at time t. This means that if the initial amount present, N_0, were 1 gram, then at time T, $\frac{1}{2}$ gram would remain; at time $2T$, $\frac{1}{4}$ gram would remain; and so on. The value of T is called the **half-life** of the radioactive element. Figure 4.13 shows a graph of radioactive decay.

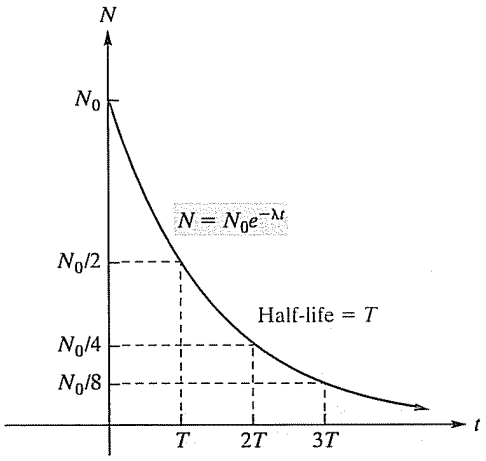


FIGURE 4.13 Radioactive decay.

EXAMPLE 11 Radioactive Decay

A radioactive element decays such that after t days the number of milligrams present is given by

$$N = 100e^{-0.062t}$$

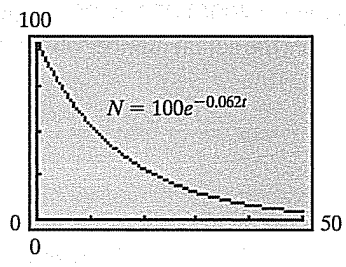


FIGURE 4.14 Graph of radioactive decay function $N = 100e^{-0.062t}$.

a. How many milligrams are initially present?

Solution: This equation has the form of Equation (3), $N = N_0e^{-\lambda t}$, where $N_0 = 100$ and $\lambda = 0.062$. N_0 is the initial amount and corresponds to $t = 0$. Thus, 100 milligrams are initially present. (See Figure 4.14.)

b. How many milligrams are present after 10 days?

Solution: When $t = 10$,

$$N = 100e^{-0.062(10)} = 100e^{-0.62} \approx 53.8$$

Therefore, approximately 53.8 milligrams are present after 10 days.

Now Work Problem 47 ◁

PROBLEMS 4.1

In Problems 1–12, graph each function.

1. $y = f(x) = 4^x$

2. $y = f(x) = 3^x$

3. $y = f(x) = \left(\frac{1}{3}\right)^x$

4. $y = f(x) = \left(\frac{1}{4}\right)^x$

5. $y = f(x) = 2^{(x-1)^2}$

6. $y = f(x) = 3(2)^x$

7. $y = f(x) = 3^{x+2}$

8. $y = f(x) = 2^{x-1}$

9. $y = f(x) = 3^x - 2$

10. $y = f(x) = 3^{x-1} - 1$

11. $y = f(x) = 3^{-x}$

12. $y = f(x) = \frac{1}{2}(2^{x/2})$

Problems 13 and 14 refer to Figure 4.15, which shows the graphs of $y = 0.4^x$, $y = 2^x$, and $y = 5^x$.

13. Of the curves A, B, and C, which is the graph of $y = 5^x$?

14. Of the curves A, B, and C, which is the graph of $y = 2^x$?

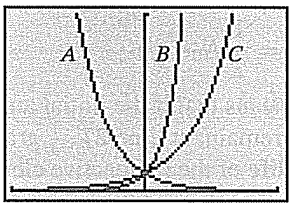


FIGURE 4.15

15. Population The projected population of a city is given by $P = 125{,}000(1.11)^{t/20}$, where t is the number of years after 1995. What is the projected population in 2015?

16. Population For a certain city, the population P grows at the rate of 1.5% per year. The formula $P = 1{,}527{,}000(1.015)^t$ gives the population t years after 1998. Find the population in (a) 1999 and (b) 2000.

17. Paired-Associate Learning In a psychological experiment involving learning,[4] subjects were asked to give particular responses after being shown certain stimuli. Each stimulus was a pair of letters, and each response was either the digit 1 or 2. After each response, the subject was told the correct answer. In this so-called *paired-associate* learning experiment, the theoretical probability P that a subject makes a correct response on the nth trial is given by

$$P = 1 - \frac{1}{2}(1 - c)^{n-1}, \quad n \geq 1, \quad 0 < c < 1$$

where c is a constant. Take $c = \frac{1}{2}$ and find P when $n = 1$, $n = 2$, and $n = 3$.

[4]D. Laming, *Mathematical Psychology* (New York: Academic Press, Inc., 1973).

18. Express $y = 2^{3x}$ as an exponential function in base 8.

In Problems 19–27, find (a) the compound amount and (b) the compound interest for the given investment and annual rate.

19. \$2000 for 5 years at 3% compounded annually

20. \$5000 for 20 years at 5% compounded annually

21. \$700 for 15 years at 7% compounded semiannually

22. \$4000 for 12 years at $7\frac{1}{2}$% compounded semiannually

23. \$3000 for 16 years at $8\frac{3}{4}$% compounded quarterly

24. \$6000 for 2 years at 8% compounded quarterly

25. \$5000 for $2\frac{1}{2}$ years at 9% compounded monthly

26. \$500 for 5 years at 11% compounded semiannually

27. \$8000 for 3 years at $6\frac{1}{4}$% compounded daily. (Assume that there are 365 days in a year.)

28. Investment Suppose \$900 is placed in a savings account that earns interest at the rate of 4.5% compounded semiannually. (a) What is the value of the account at the end of five years? (b) If the account had earned interest at the rate of 4.5% compounded annually, what would be the value after five years?

29. Investment A certificate of deposit is purchased for \$6500 and is held for three years. If the certificate earns 2% compounded quarterly, what is it worth at the end of three years?

30. Population Growth The population of a town of 5000 grows at the rate of 3% per year. (a) Determine an equation that gives the population t years from now. (b) Find the population three years from now. Give your answer to (b) to the nearest integer.

31. Bacteria Growth Bacteria are growing in a culture, and their number is increasing at the rate of 5% an hour. Initially, 400 bacteria are present. (a) Determine an equation that gives the number, N, of bacteria present after t hours. (b) How many bacteria are present after one hour? (c) After four hours? Give your answers to (b) and (c) to the nearest integer.

32. Bacteria Reduction A certain medicine reduces the bacteria present in a person by 10% each hour. Currently, 100,000 bacteria are present. Make a table of values for the number of bacteria present each hour for 0 to 4 hours. For each hour, write an expression for the number of bacteria as a product of 100,000 and a power of $\frac{9}{10}$. Use the expressions to make an entry in your table for the number of bacteria after t hours. Write a function N for the number of bacteria after t hours.

33. Recycling Suppose the amount of plastic being recycled increases by 30% every year. Make a table of the factor by which recycling increases over the original amount for 0 to 3 years. For each year, write an expression for the increase as a power of some base. What base did you use? How does that base relate to the problem? Use your table to graph the multiplicative increase as a function of years. Use your graph to determine when the recycling will triple.

34. Population Growth Cities A and B presently have populations of 270,000 and 360,000, respectively. City A grows at the rate of 6% per year, and B grows at the rate of 4% per year. Determine the larger and by how much the populations differ at the end of five years. Give your answer to the nearest integer.

Problems 35 and 36 involve a declining population. If a population declines at the rate of r per time period, then the population after t time periods is given by

$$P = P_0(1 - r)^t$$

where P_0 is the initial population (the population when $t = 0$).

35. Population Because of an economic downturn, the population of a certain urban area declines at the rate of 1.5% per year. Initially, the population is 350,000. To the nearest person, what is the population after three years?

36. Enrollment After a careful demographic analysis, a university forecasts that student enrollments will drop by 3% per year for the the next 12 years. If the university currently has 14,000 students, how many students will it have 12 years from now?

In Problems 37–40, use a calculator to find the value (rounded to four decimal places) of each expression.

37. $e^{1.5}$ **38.** $e^{3.4}$ **39.** $e^{-0.8}$ **40.** $e^{-2/3}$

In Problems 41 and 42, graph the functions.

41. $y = -e^{-(x+1)}$ **42.** $y = 2e^x$

43. Telephone Calls The probability that a telephone operator will receive exactly x calls during a certain period is given by

$$P = \frac{e^{-3}3^x}{x!}$$

Find the probability that the operator will receive exactly three calls. Round your answer to four decimal places.

44. Normal Distribution An important function used in economic and business decisions is the *normal distribution density function,* which, in standard form, is

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-(\frac{1}{2})x^2}$$

Evaluate $f(0)$, $f(1)$, and $f(2)$. Round your answers to three decimal places.

45. Express e^{kt} in the form b^t. **46.** Express $\frac{1}{e^x}$ in the form b^x.

47. Radioactive Decay A radioactive element is such that N grams remain after t hours, where

$$N = 12e^{-0.031t}$$

(a) How many grams are initially present? To the nearest tenth of a gram, how many grams remain after (b) 10 hours? (c) 44 hours? (d) Based on your answer to part (c), what is your estimate of the half-life of this element?

48. Radioactive Decay At a certain time, there are 75 milligrams of a radioactive substance. The substance decays so that after t years the number of milligrams present, N, is given by

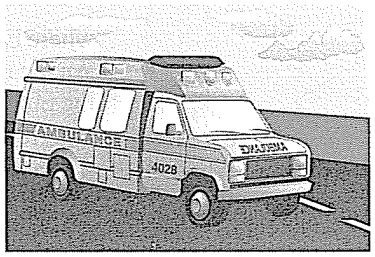
$$N = 75e^{-0.045t}$$

How many milligrams are present after 10 years? Give your answer to the nearest milligram.

49. Radioactive Decay a radioactive substance has a half-life of 9 years, how lo[illegible] does it take for 1 gram of the substance to decay to $\frac{1}{8}$ gra[illegible]?

50. Marketing A mail-o[illegible]der company advertises in a national magazine. The company finds that, of all small towns, the percentage (given as a decimal) in which exactly x people respond to an ad fits a Poisson distribution with $\mu = 0.5$. From what percentage of small towns can the company expect exactly two people to respond? Round your answer to four decimal places.

51. Emergency-Room Admissions Suppose the number of patients admitted into a hospital emergency room during a certain hour of the day has a Poisson distribution with mean 4. Find the probability that during that hour there will be exactly two emergency patients. Round your answer to four decimal places.



52. Graph $y = 17^x$ and $y = \left(\frac{1}{17}\right)^x$ on the same screen. Determine the intersection point.

53. Let $a > 0$ be a constant. Graph $y = 2^x$ and $y = 2^a \cdot 2^x$ on the same screen, for constant values $a = 2$ and $a = 3$. It appears that the graph of $y = 2^a \cdot 2^x$ is the graph of $y = 2^x$ shifted a units to the left. Prove algebraically that this is indeed true.

54. For $y = 5^x$, find x if $y = 3$. Round your answer to two decimal places.

55. For $y = 2^x$, find x if $y = 9$. Round your answer to two decimal places.

56. Cell Growth Cells are growing in a culture, and their number is increasing at the rate of 7% per hour. Initially, 1000 cells are present. After how many full hours will there be at least 3000 cells?

57. Bacteria Growth Refer to Example 1. How long will it take for 1000 bacteria to be present? Round your answer to the nearest tenth of a minute.

58. Demand Equation The demand equation for a new toy is

$$q = 10{,}000(0.95123)^p$$

(a) Evaluate q to the nearest integer when $p = 10$.

(b) Convert the demand equation to the form

$$q = 10{,}000e^{-xp}$$

(*Hint:* Find a number x such that $0.95123 \approx e^{-x}$.)

(c) Use the equation in part (b) to evaluate q to the nearest integer when $p = 10$. Your answers in parts (a) and (c) should be the same.

59. Investment If $2000 is invested in a savings account that earns interest at the rate of 9.9% compounded annually, after how many full years will the amount at least double?

Objective

To introduce logarithmic functions and their graphs. Properties of logarithms will be discussed in Section 4.3.

4.2 Logarithmic Functions

Since all exponential functions pass the horizontal line test, they are all one-to-one functions. It follows that each exponential function has an inverse. These functions, inverse to the exponential functions, are called the *logarithmic functions*.

More precisely, if $f(x) = b^x$, the exponential function base b (where $0 < b < 1$ or $1 < b$), then the inverse function $f^{-1}(x)$ is called the *logarithm function base b* and is denoted $\log_b x$. It follows from our general remarks about inverse functions in Section 2.4 that

TO REVIEW inverse functions, refer to Section 2.4.

$$y = \log_b x \quad \text{if and only if} \quad b^y = x$$

and we have the following fundamental equations:

$$\log_b b^x = x \tag{1}$$

and

$$b^{\log_b x} = x \tag{2}$$

where Equation (1) holds for all x in $(-\infty, \infty)$ and Equation (2) holds for all x in $(0, \infty)$. We recall that $(-\infty, \infty)$ is the domain of the exponential function base b and $(0, \infty)$ is the range of the exponential function base b. It follows that $(0, \infty)$ is the domain of the logarithm function base b and $(-\infty, \infty)$ is the range of the logarithm function base b.

Stated otherwise, given positive x, $\log_b x$ is the unique number with the property that $b^{\log_b x} = x$. The generalities about inverse functions also enable us to see immediately what the graph of a logarithmic function looks like.

In Figure 4.16 we have shown the graph of the particular exponential function $y = f(x) = 2^x$, whose general shape is typical of exponential functions $y = b^x$ for which the base b satisfies $1 < b$. We have added a (dashed) copy of the line $y = x$. The graph of $y = f^{-1}(x) = \log_2 x$ is obtained as the mirror image of $y = f(x) = 2^x$ in the line $y = x$.

In Table 4.3 we have tabulated the function values that appear as y-coordinates of the dots in Figure 4.16.

Table 4.3 Selected function values

x	2^x	x	$\log_2 x$
-2	$\frac{1}{4}$	$\frac{1}{4}$	-2
-1	$\frac{1}{2}$	$\frac{1}{2}$	-1
0	1	1	0
1	2	2	1
2	4	4	2
3	8	8	3

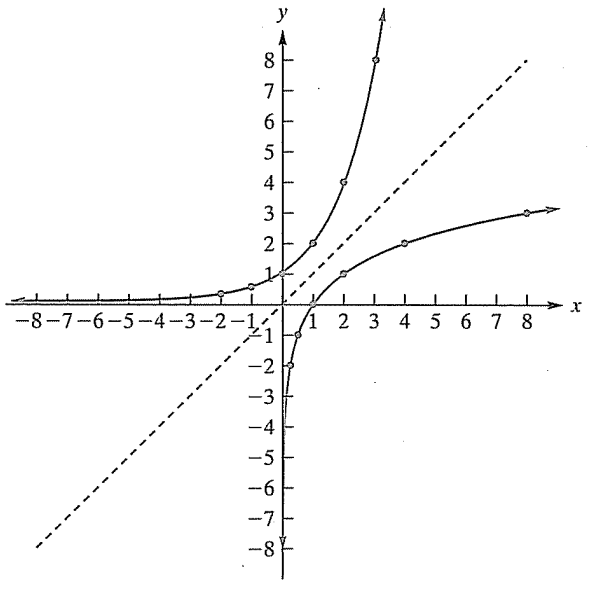


FIGURE 4.16 Graphs of $y = 2^x$ and $y = \log_2 x$.

It is clear that the exponential function base 2 and the logarithm function base 2 *undo* the effects of each other. Thus, for all x in the domain of 2^x [which is $(-\infty, \infty)$], we have

$$\log_2 2^x = x$$

and, for all x in the domain of $\log_2 x$ [which is the range of 2^x, which is $(0, \infty)$], we have

$$2^{\log_2 x} = x$$

It cannot be said too often that

$$y = \log_b x \quad \text{means} \quad b^y = x$$

and conversely

$$b^y = x \quad \text{means} \quad y = \log_b x$$

In this sense, *a logarithm of a number is an exponent:* $\log_b x$ is the power to which we must raise b to get x. For example,

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8$$

We say that $\log_2 8 = 3$ is the **logarithmic form** of the **exponential form** $2^3 = 8$. (See Figure 4.17.)

Logarithmic and exponential forms

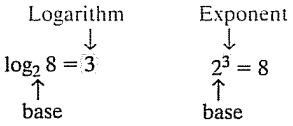


FIGURE 4.17 A logarithm can be considered an exponent.

APPLY IT ▶

8. If bacteria have been doubling every hour and the current amount is 16 times the amount first measured, then the situation can be represented by $16 = 2^t$. Represent this equation in logarithmic form. What does t represent?

APPLY IT ▶

9. An earthquake measuring 8.3 on the Richter scale can be represented by $8.3 = \log_{10}\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake and I_0 is the intensity of a zero-level earthquake. Represent this equation in exponential form.

APPLY IT ▶

10. Suppose a recycling plant has found that the amount of material being recycled has increased by 50% every year since the plant's first year of operation. Graph each year as a function of the multiplicative increase in recycling since the first year. Label the graph with the name of the function.

EXAMPLE 1 **Converting from Exponential to Logarithmic Form**

	Exponential Form		*Logarithmic Form*
a. Since	$5^2 = 25$	it follows that	$\log_5 25 = 2$
b. Since	$3^4 = 81$	it follows that	$\log_3 81 = 4$
c. Since	$10^0 = 1$	it follows that	$\log_{10} 1 = 0$

Now Work Problem 1 ◁

EXAMPLE 2 **Converting from Logarithmic to Exponential Form**

	Logarithmic Form		*Exponential Form*
a.	$\log_{10} 1000 = 3$	means	$10^3 = 1000$
b.	$\log_{64} 8 = \frac{1}{2}$	means	$64^{1/2} = 8$
c.	$\log_2 \frac{1}{16} = -4$	means	$2^{-4} = \frac{1}{16}$

Now Work Problem 3 ◁

EXAMPLE 3 **Graph of a Logarithmic Function with $b > 1$**

Examine again the graph of $y = \log_2 x$ in Figure 4.16. This graph is typical for a logarithmic function with $b > 1$.

Now Work Problem 9 ◁

APPLY IT ▶

11. Suppose a boat depreciates 20% every year. Graph the number of years the boat is owned as a function of the multiplicative decrease in its original value. Label the graph with the name of the function.

EXAMPLE 4 **Graph of a Logarithmic Function with $0 < b < 1$**

Graph $y = \log_{1/2} x$.

Solution: To plot points, we plot the inverse function $y = \left(\frac{1}{2}\right)^x$ and reflect the graph in the line $y = x$. (See Figure 4.18.)

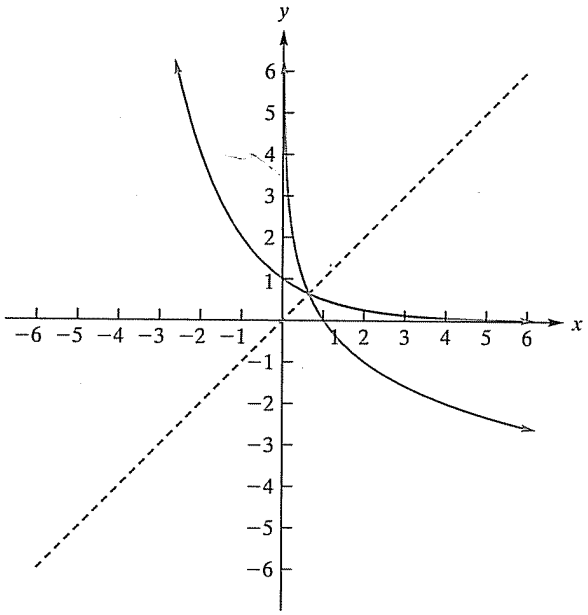


FIGURE 4.18 Graph of $y = \left(\frac{1}{2}\right)^x$ and $y = \log_{1/2} x$.

From the graph, we can see that the domain of $y = \log_{1/2} x$ is the set of all positive reals, for that is the range of $y = \left(\frac{1}{2}\right)^x$ and the range of $y = \log_{1/2} x$ is the set of all real numbers, which is the domain of $y = \left(\frac{1}{2}\right)^x$. The graph falls from left to right. Numbers between 0 and 1 have positive base $\frac{1}{2}$ logarithms, and the closer a number is to 0, the larger is its base $\frac{1}{2}$ logarithm. Numbers greater than 1 have negative base $\frac{1}{2}$ logarithms. The logarithm of 1 is 0, *regardless of the base b,* and corresponds to the x-intercept $(1, 0)$. This graph is typical for a logarithmic function with $0 < b < 1$.

Now Work Problem 11 ◁

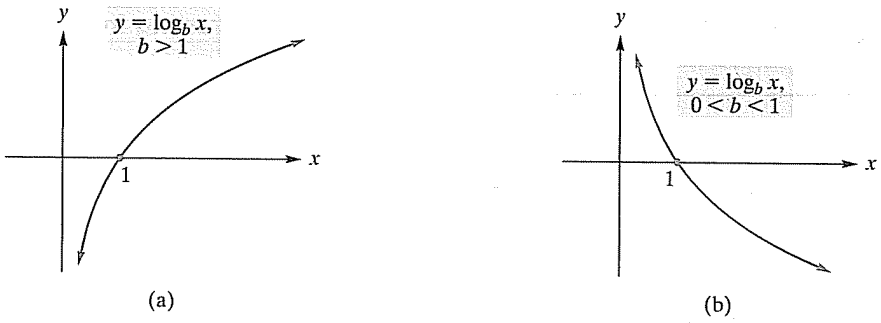


FIGURE 4.19 General shapes of $y = \log_b x$.

Summarizing the results of Examples 3 and 4, we can say that the graph of a logarithmic function has one of two general shapes, depending on whether $b > 1$ or $0 < b < 1$. (See Figure 4.19.) For $b > 1$, the graph rises from left to right; as x gets closer and closer to 0, the function values decrease without bound, and the graph gets closer and closer to the y-axis. For $0 < b < 1$, the graph falls from left to right;

as x gets closer and closer to 0, the function values increase without bound, and the graph gets closer and closer to the y-axis. In each case, note that

1. The domain of a logarithmic function is the interval $(0, \infty)$. Thus, the logarithm of a nonpositive number does not exist.
2. The range is the interval $(-\infty, \infty)$.
3. The logarithm of 1 is 0, which corresponds to the x-intercept $(1, 0)$.

The graph of the natural logarithmic function in Figure 4.20 is important too.

Logarithms to the base 10 are called **common logarithms.** They were frequently used for computational purposes before the calculator age. The subscript 10 is usually omitted from the notation:

$$\log x \quad \text{means} \quad \log_{10} x$$

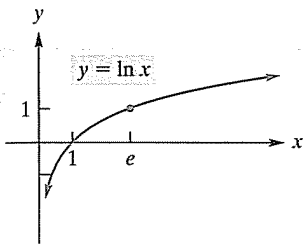


FIGURE 4.20 Graph of natural logarithmic function.

Important in calculus are logarithms to the base e, called **natural logarithms.** We use the notation "ln" for such logarithms:

$$\ln x \quad \text{means} \quad \log_e x$$

The symbol $\ln x$ can be read "natural log of x." Most calculators give approximate values for natural and common logarithms. For example, verify that $\ln 2 \approx 0.69315$. This means that $e^{0.69315} \approx 2$. Figure 4.20 shows the graph of $y = \ln x$. Because $e > 1$, the graph has the general shape of that of a logarithmic function with $b > 1$ [see Figure 4.19(a)] and rises from left to right. While the conventions about log, with no subscript, and ln are well established in elementary books, be careful when consulting an advanced book. In advanced texts, $\log x$ means $\log_e x$, ln is not used at all, and logarithms base 10 are written explicitly as $\log_{10} x$.

APPLY IT ▶

12. The number of years it takes for an amount invested at an annual rate of r and compounded continuously to quadruple is a function of the annual rate r given by $t(r) = \frac{\ln 4}{r}$. Use a calculator to find the rate needed to quadruple an investment in 10 years.

EXAMPLE 5 Finding Logarithms

a. Find $\log 100$.

Solution: Here the base is 10. Thus, $\log 100$ is the power to which we must raise 10 to get 100. Since $10^2 = 100$, $\log 100 = 2$.

b. Find $\ln 1$.

Solution: Here the base is e. Because $e^0 = 1$, $\ln 1 = 0$.

c. Find $\log 0.1$.

Solution: Since $0.1 = \frac{1}{10} = 10^{-1}$, $\log 0.1 = -1$.

d. Find $\ln e^{-1}$.

Solution: Since $\ln e^{-1}$ is the power to which e must be raised to obtain e^{-1}, clearly $\ln e^{-1} = -1$.

e. Find $\log_{36} 6$.

Solution: Because $36^{1/2}$ $(=\sqrt{36})$ is 6, $\log_{36} 6 = \frac{1}{2}$.

Now Work Problem 17 ◁

Remember the way in which a logarithm is an exponent.

Many equations involving logarithmic or exponential forms can be solved for an unknown quantity by first transforming from logarithmic form to exponential form or vice versa. Example 6 will illustrate.

APPLY IT ▶

13. The multiplicative increase m of an amount invested at an annual rate of r compounded continuously for a time t is given by $m = e^{rt}$. What annual percentage rate is needed to triple the investment in 12 years?

EXAMPLE 6 Solving Logarithmic and Exponential Equations

a. Solve $\log_2 x = 4$.

Solution: We can get an explicit expression for x by writing the equation in exponential form. This gives

$$2^4 = x$$

so $x = 16$.

b. Solve $\ln(x+1) = 7$.

Solution: The exponential form yields $e^7 = x + 1$. Thus, $x = e^7 - 1$.

c. Solve $\log_x 49 = 2$.

Solution: In exponential form, $x^2 = 49$, so $x = 7$. We reject $x = -7$ because a negative number cannot be a base of a logarithmic function.

d. Solve $e^{5x} = 4$.

Solution: We can get an explicit expression for x by writing the equation in logarithmic form. We have

$$\ln 4 = 5x$$
$$x = \frac{\ln 4}{5}$$

Now Work Problem 49 ◁

Radioactive Decay and Half-Life

From our discussion of the decay of a radioactive element in Section 4.1, we know that the amount of the element present at time t is given by

$$N = N_0 e^{-\lambda t} \tag{3}$$

where N_0 is the initial amount (the amount at time $t = 0$) and λ is the decay constant. Let us now determine the half-life T of the element. At time T, half of the initial amount is present. That is, when $t = T, N = N_0/2$. Thus, from Equation (3), we have

$$\frac{N_0}{2} = N_0 e^{-\lambda T}$$

Solving for T gives

$$\frac{1}{2} = e^{-\lambda T}$$
$$2 = e^{\lambda T} \qquad \text{taking reciprocals of both sides}$$

To get an explicit expression for T, we convert to logarithmic form. This results in

$$\lambda T = \ln 2$$
$$T = \frac{\ln 2}{\lambda}$$

Summarizing, we have the following:

If a radioactive element has decay constant λ, then the half-life of the element is given by

$$T = \frac{\ln 2}{\lambda} \tag{4}$$

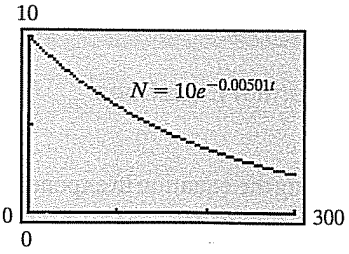


FIGURE 4.21 Radioactive decay function $N = 10e^{-0.00501t}$.

EXAMPLE 7 Finding Half-Life

A 10-milligram sample of radioactive polonium 210 (which is denoted ^{210}Po) decays according to the equation

$$N = 10e^{-0.00501t}$$

where N is the number of milligrams present after t days. (See Figure 4.21.) Determine the half-life of ^{210}Po.

Solution: Here the decay constant λ is 0.00501. By Equation (4), the half-life is given by

$$T = \frac{\ln 2}{\lambda} = \frac{\ln 2}{0.00501} \approx 138.4 \text{ days}$$

Now Work Problem 63 ◁

PROBLEMS 4.2

In Problems 1–8, express each logarithmic form exponentially and each exponential form logarithmically.

1. $10^4 = 10{,}000$
2. $2 = \log_{12} 144$
3. $\log_2 1024 = 10$
4. $8^{2/3} = 4$
5. $e^3 \approx 20.0855$
6. $e^{0.33647} \approx 1.4$
7. $\ln 3 \approx 1.09861$
8. $\log 7 \approx 0.84509$

In Problems 9–16, graph the functions.

9. $y = f(x) = \log_3 x$
10. $y = f(x) = \log_4 2x$
11. $y = f(x) = \log_{1/4} x$
12. $y = f(x) = \log_{1/5} x$
13. $y = f(x) = \log_2 (x + 4)$
14. $y = f(x) = \log_2 (-x)$
15. $y = f(x) = -2 \ln x$
16. $y = f(x) = \ln (x + 2)$

In Problems 17–28, evaluate the expression.

17. $\log_6 36$
18. $\log_2 512$
19. $\log_3 27$
20. $\log_{16} 4$
21. $\log_7 7$
22. $\log 10,000$
23. $\log 0.0001$
24. $\log_2 \sqrt[3]{2}$
25. $\log_5 1$
26. $\log_5 \frac{1}{25}$
27. $\log_2 \frac{1}{8}$
28. $\log_3 \sqrt[7]{3}$

In Problems 29–48, find x.

29. $\log_3 x = 4$
30. $\log_2 x = 8$
31. $\log_5 x = 3$
32. $\log_4 x = 0$
33. $\log x = -3$
34. $\ln x = 1$
35. $\ln x = -3$
36. $\log_x 25 = 2$
37. $\log_x 8 = 3$
38. $\log_x 4 = \frac{1}{3}$
39. $\log_x \frac{1}{6} = -1$
40. $\log_x y = 1$
41. $\log_3 x = -3$
42. $\log_x (2x - 3) = 1$
43. $\log_x (12 - x) = 2$
44. $\log_8 64 = x - 1$
45. $2 + \log_2 4 = 3x - 1$
46. $\log_3 (x + 2) = -2$
47. $\log_x (2x + 8) = 2$
48. $\log_x (16 - 4x - x^2) = 2$

In Problems 49–52, find x and express your answer in terms of natural logarithms.

49. $e^{3x} = 2$
50. $0.1e^{0.1x} = 0.5$
51. $e^{2x-5} + 1 = 4$
52. $6e^{2x} - 1 = \frac{1}{2}$

In Problems 53–56, use your calculator to find the approximate value of each expression. Round your answer to five decimal places.

53. $\ln 11$
54. $\ln 4.27$
55. $\ln 7.39$
56. $\ln 9.98$

57. Appreciation Suppose an antique gains 10% in value every year. Graph the number of years it is owned as a function of the multiplicative increase in its original value. Label the graph with the name of the function.

58. Cost Equation The cost for a firm producing q units of a product is given by the cost equation

$$c = (5q \ln q) + 15$$

Evaluate the cost when $q = 12$. (Round your answer to two decimal places.)

59. Supply Equation A manufacturer's supply equation is

$$p = \log\left(10 + \frac{q}{2}\right)$$

where q is the number of units supplied at a price p per unit. At what price will the manufacturer supply 1980 units?

60. Earthquake The magnitude, M, of an earthquake and its energy, E, are related by the equation[5]

$$1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$$

where M is given in terms of Richter's preferred scale of 1958 and E is in ergs. Solve the equation for E.

61. Biology For a certain population of cells, the number of cells at time t is given by $N = N_0(2^{t/k})$, where N_0 is the number of cells at $t = 0$ and k is a positive constant. (a) Find N when $t = k$. (b) What is the significance of k? (c) Show that the time it takes to have population N_1 can be written

$$t = k \log_2 \frac{N_1}{N_0}$$

62. Inferior Good In a discussion of an inferior good, Persky[6] solves an equation of the form

$$u_0 = A \ln (x_1) + \frac{x_2^2}{2}$$

for x_1, where x_1 and x_2 are quantities of two products, u_0 is a measure of utility, and A is a positive constant. Determine x_1.

63. Radioactive Decay A 1-gram sample of radioactive lead 211 (^{211}Pb) decays according to the equation $N = e^{-0.01920t}$, where N is the number of grams present after t minutes. How long will it take until only 0.25 grams remain? Express the answer to the nearest tenth of a minute.

---

[5] K. E. Bullen, *An Introduction to the Theory of Seismology* (Cambridge, U.K.: Cambridge at the University Press, 1963).

[6] A. L. Persky, "An Inferior Good and a Novel Indifference Map," *The American Economist*, XXIX, no. 1 (Spring 1985).

64. Radioactive Decay A 100-milligram sample of radioactive actinium 227 (^{227}Ac) decays according to the equation

$$N = 100e^{-0.03194t}$$

where N is the number of milligrams present after t years. Find the half-life of ^{227}Ac to the nearest tenth of a year.

65. If $\log_y x = 3$ and $\log_z x = 2$, find a formula for z as an explicit function of y only.

66. Solve for y as an explicit function of x if

$$x + 3e^{2y} - 8 = 0$$

67. Suppose $y = f(x) = x \ln x$. (a) For what values of x is $y < 0$? (*Hint:* Determine when the graph is below the x-axis.) (b) Determine the range of f.

68. Find the x-intercept of $y = x^3 \ln x$.

69. Use the graph of $y = e^x$ to estimate ln 3. Round your answer to two decimal places.

70. Use the graph of $y = \ln x$ to estimate e^2. Round your answer to two decimal places.

71. Determine the x-values of points of intersection of the graphs of $y = (x-2)^2$ and $y = \ln x$. Round your answers to two decimal places.

4.3 Properties of Logarithms

Objective

To study basic properties of logarithmic functions.

The logarithmic function has many important properties. For example,

1. $\log_b (mn) = \log_b m + \log_b n$

which says that the logarithm of a product of two numbers is the sum of the logarithms of the numbers. We can prove this property by deriving the exponential form of the equation:

$$b^{\log_b m + \log_b n} = mn$$

Using first a familiar rule for exponents, we have

$$b^{\log_b m + \log_b n} = b^{\log_b m} b^{\log_b n}$$
$$= mn$$

where the second equality uses two instances of the fundamental equation (2) of Section 4.2. We will not prove the next two properties, since their proofs are similar to that of Property 1.

2. $\log_b \frac{m}{n} = \log_b m - \log_b n$

That is, the logarithm of a quotient is the difference of the logarithm of the numerator and the logarithm of the denominator.

3. $\log_b m^r = r \log_b m$

That is, the logarithm of a power of a number is the exponent times the logarithm of the number.

Table 4.4 Common Logarithms

x	$\log x$	x	$\log x$
2	0.3010	7	0.8451
3	0.4771	8	0.9031
4	0.6021	9	0.9542
5	0.6990	10	1.0000
6	0.7782	e	0.4343

Table 4.4 gives the values of a few common logarithms. Most entries are approximate. For example, $\log 4 \approx 0.6021$, which means $10^{0.6021} \approx 4$. To illustrate the use of properties of logarithms, we will use this table in some of the examples that follow.

EXAMPLE 1 Finding Logarithms by Using Table 4.4

a. Find log 56.

Solution: Log 56 is not in the table. But we can write 56 as the product $8 \cdot 7$. Thus, by Property 1,

$$\log 56 = \log (8 \cdot 7) = \log 8 + \log 7 \approx 0.9031 + 0.8451 = 1.7482$$

Although the logarithms in Example 1 can be found with a calculator, we will make use of properties of logarithms.

b. Find $\log \frac{9}{2}$.

Solution: By Property 2,

$$\log \frac{9}{2} = \log 9 - \log 2 \approx 0.9542 - 0.3010 = 0.6532$$

c. Find $\log 64$.

Solution: Since $64 = 8^2$, by Property 3,

$$\log 64 = \log 8^2 = 2 \log 8 \approx 2(0.9031) = 1.8062$$

d. Find $\log \sqrt{5}$.

Solution: By Property 3, we have

$$\log \sqrt{5} = \log 5^{1/2} = \frac{1}{2} \log 5 \approx \frac{1}{2}(0.6990) = 0.3495$$

e. Find $\log \frac{16}{21}$.

Solution:

$$\begin{aligned} \log \frac{16}{21} &= \log 16 - \log 21 = \log (4^2) - \log (3 \cdot 7) \\ &= 2 \log 4 - [\log 3 + \log 7] \\ &\approx 2(0.6021) - [0.4771 + 0.8451] = -0.1180 \end{aligned}$$

Now Work Problem 3 ◁

EXAMPLE 2 Rewriting Logarithmic Expressions

a. Express $\log \frac{1}{x^2}$ in terms of $\log x$.

Solution:

$$\log \frac{1}{x^2} = \log x^{-2} = -2 \log x \qquad \text{Property 3}$$

Here we have assumed that $x > 0$. Although $\log (1/x^2)$ is defined for $x \neq 0$, the expression $-2 \log x$ is defined only if $x > 0$. Note that we do have

$$\log \frac{1}{x^2} = \log x^{-2} = -2 \log |x|$$

for all $x \neq 0$.

b. Express $\log \frac{1}{x}$ in terms of $\log x$, for $x > 0$.

Solution: By Property 3,

$$\log \frac{1}{x} = \log x^{-1} = -1 \log x = -\log x$$

Now Work Problem 21 ◁

From Example 2(b), we see that $\log (1/x) = -\log x$. Generalizing gives the following property:

4. $\log_b \frac{1}{m} = -\log_b m$

That is, the logarithm of the reciprocal of a number is the negative of the logarithm of the number.

For example, $\log \frac{2}{3} = -\log \frac{3}{2}$.

EXAMPLE 3 Writing Logarithms in Terms of Simpler Logarithms

Manipulations such as those in Example 3 are frequently used in calculus.

a. Write $\ln \dfrac{x}{zw}$ in terms of $\ln x$, $\ln z$, and $\ln w$.

Solution:

$$\begin{aligned} \ln \frac{x}{zw} &= \ln x - \ln (zw) && \text{Property 2} \\ &= \ln x - (\ln z + \ln w) && \text{Property 1} \\ &= \ln x - \ln z - \ln w \end{aligned}$$

b. Write $\ln \sqrt[3]{\dfrac{x^5(x-2)^8}{x-3}}$ in terms of $\ln x$, $\ln (x-2)$, and $\ln (x-3)$.

Solution:

$$\begin{aligned} \ln \sqrt[3]{\frac{x^5(x-2)^8}{x-3}} &= \ln \left[\frac{x^5(x-2)^8}{x-3}\right]^{1/3} = \frac{1}{3} \ln \frac{x^5(x-2)^8}{x-3} \\ &= \frac{1}{3}\{\ln [x^5(x-2)^8] - \ln (x-3)\} \\ &= \frac{1}{3}[\ln x^5 + \ln (x-2)^8 - \ln (x-3)] \\ &= \frac{1}{3}[5 \ln x + 8 \ln (x-2) - \ln (x-3)] \end{aligned}$$

Now Work Problem 29 ◁

APPLY IT ▷

14. The Richter scale measure of an earthquake is given by $R = \log \left(\dfrac{I}{I_0}\right)$, where I is the intensity of the earthquake and I_0 is the intensity of a zero-level earthquake. How much more on the Richter scale is an earthquake with intensity 900,000 times the intensity of a zero-level earthquake than an earthquake with intensity 9000 times the intensity of a zero-level earthquake? Write the answer as an expression involving logarithms. Simplify the expression by combining logarithms, and then evaluate the resulting expression.

EXAMPLE 4 Combining Logarithms

a. Write $\ln x - \ln (x+3)$ as a single logarithm.

Solution:

$$\ln x - \ln (x+3) = \ln \frac{x}{x+3} \qquad \text{Property 2}$$

b. Write $\ln 3 + \ln 7 - \ln 2 - 2 \ln 4$ as a single logarithm.

Solution:

$$\begin{aligned} &\ln 3 + \ln 7 - \ln 2 - 2 \ln 4 \\ &= \ln 3 + \ln 7 - \ln 2 - \ln (4^2) && \text{Property 3} \\ &= \ln 3 + \ln 7 - [\ln 2 + \ln (4^2)] \\ &= \ln (3 \cdot 7) - \ln (2 \cdot 4^2) && \text{Property 1} \\ &= \ln 21 - \ln 32 \\ &= \ln \frac{21}{32} && \text{Property 2} \end{aligned}$$

Now Work Problem 37 ◁

Since $b^0 = 1$ and $b^1 = b$, by converting to logarithmic forms we have the following properties:

5. $\log_b 1 = 0$
6. $\log_b b = 1$

APPLY IT ▷

15. If an earthquake is 10,000 times as intense as a zero-level earthquake, what is its measurement on the Richter scale? Write the answer as a logarithmic expression and simplify it. (See the preceding Apply it for the formula.)

EXAMPLE 5 **Simplifying Logarithmic Expressions**

a. Find $\ln e^{3x}$.

Solution: By the fundamental equation (1) of Section 4.2 with $b = e$, we have $\ln e^{3x} = 3x$.

b. Find $\log 1 + \log 1000$.

Solution: By Property 5, $\log 1 = 0$. Thus,

$$\begin{aligned} \log 1 + \log 1000 &= 0 + \log 10^3 \\ &= 0 + 3 \qquad \text{Fundamental equation (1) of} \\ &= 3 \qquad\qquad\quad \text{Section 4.2 with } b = 10 \end{aligned}$$

c. Find $\log_7 \sqrt[9]{7^8}$.

Solution:
$$\log_7 \sqrt[9]{7^8} = \log_7 7^{8/9} = \frac{8}{9}$$

d. Find $\log_3 \left(\frac{27}{81}\right)$.

Solution:
$$\log_3 \left(\frac{27}{81}\right) = \log_3 \left(\frac{3^3}{3^4}\right) = \log_3 (3^{-1}) = -1$$

e. Find $\ln e + \log \frac{1}{10}$.

Solution:
$$\begin{aligned} \ln e + \log \frac{1}{10} &= \ln e + \log 10^{-1} \\ &= 1 + (-1) = 0 \end{aligned}$$

Now Work Problem 15 ◁

Do not confuse $\ln x^2$ with $(\ln x)^2$. We have

$$\ln x^2 = \ln (x \cdot x)$$

but

$$(\ln x)^2 = (\ln x)(\ln x)$$

Sometimes $(\ln x)^2$ is written as $\ln^2 x$. This is not a new formula but merely a notation. More generally, some people write $f^2(x)$ for $(f(x))^2$. We recommend avoiding the notation $f^2(x)$.

EXAMPLE 6 **Using Equation (2) of Section 4.2**

a. Find $e^{\ln x^2}$.

Solution: By (2) with $b = e$, $e^{\ln x^2} = x^2$.

b. Solve $10^{\log x^2} = 25$ for x.

Solution:
$$\begin{aligned} 10^{\log x^2} &= 25 \\ x^2 &= 25 \qquad \text{By Equation (2) of Section 4.2} \\ x &= \pm 5 \end{aligned}$$

Now Work Problem 45 ◁

EXAMPLE 7 **Evaluating a Logarithm Base 5**

Use a calculator to find $\log_5 2$.

Solution: Calculators typically have keys for logarithms in base 10 and base e, but not for base 5. However, we can convert logarithms in one base to logarithms in another base. Let us convert from base 5 to base 10. First, let $x = \log_5 2$. Then $5^x = 2$. Taking the common logarithms of both sides of $5^x = 2$ gives

$$\log 5^x = \log 2$$
$$x \log 5 = \log 2$$
$$x = \frac{\log 2}{\log 5} \approx 0.4307$$

If we had taken natural logarithms of both sides, the result would be $x = (\ln 2)/(\ln 5) \approx 0.4307$, the same as before.

◁

Generalizing the method used in Example 7, we obtain the so-called *change-of-base* formula:

Change-of-Base Formula

7. $\log_b m = \dfrac{\log_a m}{\log_a b}$

Some students find the change-of-base formula more memorable when it is expressed in the form

$$(\log_a b)(\log_b m) = \log_a m$$

in which the two instances of b apparently cancel. Let us see how to prove this identity, for the ability to see the truth of such statements greatly enhances one's ability to use them in practical applications. Since $\log_a m = y$ precisely if $a^y = m$, our task is equivalently to show that

$$a^{(\log_a b)(\log_b m)} = m$$

and we have

$$a^{(\log_a b)(\log_b m)} = \left(a^{\log_a b}\right)^{\log_b m}$$
$$= b^{\log_b m}$$
$$= m$$

using a rule for exponents and fundamental equation (2) twice.

The change-of-base formula allows logarithms to be converted from base a to base b.

EXAMPLE 8 **Change-of-Base Formula**

Express $\log x$ in terms of natural logarithms.

Solution: We must transform from base 10 to base e. Thus, we use the change-of-base formula (Property 7) with $b = 10$, $m = x$, and $a = e$:

$$\log x = \log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}$$

Now Work Problem 49 ◁

TECHNOLOGY

Problem: Display the graph of $y = \log_2 x$.

Solution: To enter the function, we must first convert it to base e or base 10. We choose base e. By Property 7,

$$y = \log_2 x = \frac{\log_e x}{\log_e 2} = \frac{\ln x}{\ln 2}$$

Now we graph $y = (\ln x)/(\ln 2)$, which is shown in Figure 4.22.

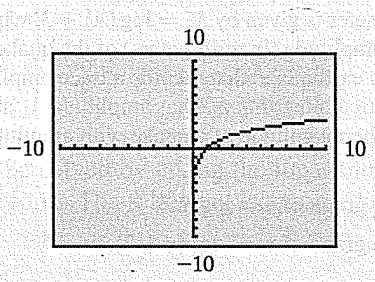


FIGURE 4.22 Graph of $y = \log_2 x$.

PROBLEMS 4.3

In Problems 1–10, let $\log 2 = a, \log 3 = b,$ *and* $\log 5 = c$. *Express the indicated logarithm in terms of a, b, and c.*

1. $\log 30$ **2.** $\log 1024$ **3.** $\log \frac{2}{3}$ **4.** $\log \frac{5}{2}$

5. $\log \frac{8}{3}$ **6.** $\log \frac{6}{25}$ **7.** $\log 100$ **8.** $\log 0.00003$

9. $\log_2 3$ **10.** $\log_3 5$

In Problems 11–20, determine the value of the expression without the use of a calculator.

11. $\log_7 7^{48}$ **12.** $\log_{11}(11\sqrt[3]{11})^7$ **13.** $\log 0.0000001$

14. $10^{\log 3.4}$ **15.** $\ln e^{5.01}$ **16.** $\ln e$ **17.** $\ln \frac{1}{\sqrt{e}}$

18. $\log_3 81$ **19.** $\log \frac{1}{10} + \ln e^3$ **20.** $e^{\ln \pi}$

In Problems 21–32, write the expression in terms of $\ln x$, $\ln(x+1)$, *and* $\ln(x+2)$.

21. $\ln(x(x+1)^2)$ **22.** $\ln \frac{\sqrt[5]{x}}{(x+1)^3}$

23. $\ln \frac{x^2}{(x+1)^3}$ **24.** $\ln(x(x+1))^3$

25. $\ln \left(\frac{x+1}{x+2}\right)^4$ **26.** $\ln \sqrt{x(x+1)(x+2)}$

27. $\ln \frac{x(x+1)}{x+2}$ **28.** $\ln \frac{x^2(x+1)}{x+2}$

29. $\ln \frac{\sqrt{x}}{(x+1)^2(x+2)^3}$ **30.** $\ln \frac{x}{(x+1)(x+2)}$

31. $\ln \left(\frac{1}{x+2}\sqrt[5]{\frac{x^2}{x+1}}\right)$ **32.** $\ln \sqrt[4]{\frac{x^2(x+2)^3}{(x+1)^5}}$

In Problems 33–40, express each of the given forms as a single logarithm.

33. $\log 6 + \log 4$ **34.** $\log_3 10 - \log_3 5$

35. $\log_2(2x) - \log_2(x+1)$ **36.** $2\log x - \frac{1}{2}\log(x-2)$

37. $7\log_3 5 + 4\log_3 17$ **38.** $5(2\log x + 3\log y - 2\log z)$

39. $2 + 10\log 1.05$

40. $\frac{1}{2}(\log 215 + 8\log 6 - 3\log 169)$

In Problems 41–44, determine the values of the expressions without using a calculator.

41. $e^{4\ln 3 - 3\ln 4}$

42. $\log_3(\ln(\sqrt{7+e^3} + \sqrt{7}) + \ln(\sqrt{7+e^3} - \sqrt{7}))$

43. $\log_6 54 - \log_6 9$

44. $\log_3 \sqrt{3} - \log_2 \sqrt[3]{2} - \log_5 \sqrt[4]{5}$

In Problems 45–48, find x.

45. $e^{\ln(2x)} = 5$ **46.** $4^{\log_4 x + \log_4 2} = 3$

47. $10^{\log(x^2+2x)} = 3$ **48.** $e^{3\ln x} = 8$

In Problems 49–53, write each expression in terms of natural logarithms.

49. $\log_2(2x+1)$ **50.** $\log_3(x^2+2x+2)$

51. $\log_3(x^2+1)$ **52.** $\log_7(x^2+1)$

53. If $e^{\ln z} = 7e^y$, solve for y in terms of z.

54. Statistics In statistics, the sample regression equation $y = ab^x$ is reduced to a linear form by taking logarithms of both sides. Express $\log y$ in terms of x, $\log a$, and $\log b$ and explain what is meant by saying that the resulting expression is linear.

55. Military Compensation In a study of military enlistments, Brown[7] considers total military compensation C as the sum of basic military compensation B (which includes the value of allowances, tax advantages, and base pay) and educational benefits E. Thus, $C = B + E$. Brown states that

$$\ln C = \ln B + \ln\left(1 + \frac{E}{B}\right)$$

Verify this.

[7] C. Brown, "Military Enlistments: What Can We Learn from Geographic Variation?" *The American Economic Review*, 75, no. 1 (1985), 228–34.

56. Earthquake According to Richter,[8] the magnitude M of an earthquake occurring 100 km from a certain type of seismometer is given by $M = \log(A) + 3$, where A is the recorded trace amplitude (in millimeters) of the quake. (a) Find the magnitude of an earthquake that records a trace amplitude of 10 mm. (b) If a particular earthquake has amplitude A_1 and magnitude M_1, determine the magnitude of a quake with amplitude $10A_1$ in terms of M_1.

57. Display the graph of $y = \log_4 x$.

58. Display the graph of $y = \log_4(x + 2)$.

59. Display the graphs of $y = \log x$ and $y = \dfrac{\ln x}{\ln 10}$ on the same screen. The graphs appear to be identical. Why?

60. On the same screen, display the graphs of $y = \ln x$ and $y = \ln(4x)$. It appears that the graph of $y = \ln(4x)$ is the graph of $y = \ln x$ shifted upward. Determine algebraically the value of this shift.

61. On the same screen, display the graphs of $y = \ln(2x)$ and $y = \ln(6x)$. It appears that the graph of $y = \ln(6x)$ is the graph of $y = \ln(2x)$ shifted upward. Determine algebraically the value of this shift.

Objective

To develop techniques for solving logarithmic and exponential equations.

4.4 Logarithmic and Exponential Equations

Here we solve *logarithmic* and *exponential equations*. A **logarithmic equation** is an equation that involves the logarithm of an expression containing an unknown. For example, $2\ln(x+4) = 5$ is a logarithmic equation. On the other hand, an **exponential equation** has the unknown appearing in an exponent, as in $2^{3x} = 7$.

To solve some logarithmic equations, it is convenient to use the fact that, for any base b, the function $y = \log_b x$ is one-to-one. This means, of course, that

$$\text{if} \quad \log_b m = \log_b n, \quad \text{then} \quad m = n$$

This follows from the fact that the function $y = \log_b x$ has an inverse and is visually apparent by inspecting the two possible shapes of $y = \log_b x$ given in Figure 4.19. In either event, the function passes the horizontal line test of Section 2.5. Also useful for solving logarithmic and exponential equations are the fundamental equations (1) and (2) in Section 4.2.

EXAMPLE 1 Oxygen Composition

An experiment was conducted with a particular type of small animal.[9] The logarithm of the amount of oxygen consumed per hour was determined for a number of the animals and was plotted against the logarithms of the weights of the animals. It was found that

$$\log y = \log 5.934 + 0.885 \log x$$

where y is the number of microliters of oxygen consumed per hour and x is the weight of the animal (in grams). Solve for y.

Solution: We first combine the terms on the right side into a single logarithm:

$$\begin{aligned} \log y &= \log 5.934 + 0.885 \log x \\ &= \log 5.934 + \log x^{0.885} && \text{Property 3 of Section 4.3} \\ \log y &= \log(5.934x^{0.885}) && \text{Property 1 of Section. 4.3} \end{aligned}$$

Since log is one-to-one, we have

$$y = 5.934x^{0.885}$$

Now Work Problem 1 ◁

APPLY IT ▷

16. Greg took a number and multiplied it by a power of 32. Jean started with the same number and got the same result when she multiplied it by 4 raised to a number that was nine less than three times the exponent that Greg used. What power of 32 did Greg use?

EXAMPLE 2 Solving an Exponential Equation

Find x if $(25)^{x+2} = 5^{3x-4}$.

Solution: Since $25 = 5^2$, we can express both sides of the equation as powers of 5:

$$\begin{aligned} (25)^{x+2} &= 5^{3x-4} \\ (5^2)^{x+2} &= 5^{3x-4} \\ 5^{2x+4} &= 5^{3x-4} \end{aligned}$$

[8] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, 1958).

[9] R. W. Poole, *An Introduction to Quantitative Ecology* (New York: McGraw-Hill, 1974).

Since 5^x is a one-to-one function,

$$2x + 4 = 3x - 4$$
$$x = 8$$

Now Work Problem 7 ◁

Some exponential equations can be solved by taking the logarithm of both sides after the equation is put in a desirable form. The following example illustrates.

APPLY IT ▷

17. The sales manager at a fast-food chain finds that breakfast sales begin to fall after the end of a promotional campaign. The sales in dollars as a function of the number of days d after the campaign's end are given by $S = 800\left(\frac{4}{3}\right)^{-0.1d}$. If the manager does not want sales to drop below 450 per day before starting a new campaign, when should he start such a campaign?

EXAMPLE 3 **Using Logarithms to Solve an Exponential Equation**

Solve $5 + (3)4^{x-1} = 12$.

Solution: We first isolate the exponential expression 4^{x-1} on one side of the equation:

$$5 + (3)4^{x-1} = 12$$
$$(3)4^{x-1} = 7$$
$$4^{x-1} = \frac{7}{3}$$

Now we take the natural logarithm of both sides:

$$\ln 4^{x-1} = \ln 7 - \ln 3$$

Simplifying gives

$$(x - 1)\ln 4 = \ln 7 - \ln 3$$
$$x - 1 = \frac{\ln 7 - \ln 3}{\ln 4}$$
$$x = \frac{\ln 7 - \ln 3}{\ln 4} + 1 \approx 1.61120$$

Now Work Problem 13 ◁

In Example 3, we used natural logarithms to solve the given equation. However, logarithms in any base can be used. If we use common logarithms, we would obtain

$$x = \frac{\log 7 - \log 3}{\log 4} + 1 \approx 1.61120$$

TECHNOLOGY

Figure 4.23 shows a graphical solution of the equation $5+(3)4^{x-1} = 12$ of Example 3. This solution occurs at the intersection of the graphs of $y = 5 + (3)4^{x-1}$ and $y = 12$.

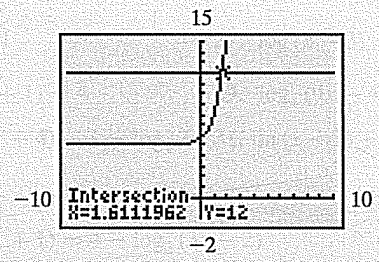


FIGURE 4.23 The solution of $5+(3)4^{x-1}=12$ is approximately 1.61120.

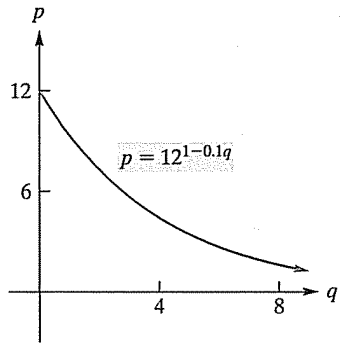


FIGURE 4.24 Graph of the demand equation $p = 12^{1-0.1q}$.

EXAMPLE 4 Demand Equation

The demand equation for a product is $p = 12^{1-0.1q}$. Use common logarithms to express q in terms of p.

Solution: Figure 4.24 shows the graph of this demand equation for $q \geq 0$. As is typical of a demand equation, the graph falls from left to right. We want to solve the equation for q. Taking the common logarithms of both sides of $p = 12^{1-0.1q}$ gives

$$\log p = \log(12^{1-0.1q})$$
$$\log p = (1 - 0.1q)\log 12$$
$$\frac{\log p}{\log 12} = 1 - 0.1q$$
$$0.1q = 1 - \frac{\log p}{\log 12}$$
$$q = 10\left(1 - \frac{\log p}{\log 12}\right)$$

Now Work Problem 43 ◁

To solve some exponential equations involving base e or base 10, such as $10^{2x} = 3$, the process of taking logarithms of both sides can be combined with the identity $\log_b b^r = r$ [Fundamental equation (1) from Section 4.2] to transform the equation into an equivalent logarithmic form. In this case, we have

$$10^{2x} = 3$$
$$2x = \log 3 \qquad \text{logarithmic form}$$
$$x = \frac{\log 3}{2} \approx 0.2386$$

EXAMPLE 5 Predator–Prey Relation

In an article concerning predators and prey, Holling[10] refers to an equation of the form

$$y = K(1 - e^{-ax})$$

where x is the prey density, y is the number of prey attacked, and K and a are constants. Verify his claim that

$$\ln \frac{K}{K - y} = ax$$

Solution: To find ax, we first solve the given equation for e^{-ax}:

$$y = K(1 - e^{-ax})$$
$$\frac{y}{K} = 1 - e^{-ax}$$
$$e^{-ax} = 1 - \frac{y}{K}$$
$$e^{-ax} = \frac{K - y}{K}$$

[10] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, 91, no. 7 (1959), 385–98.

Now we convert to logarithmic form:

$$\ln \frac{K-y}{K} = -ax$$
$$-\ln \frac{K-y}{K} = ax$$
$$\ln \frac{K}{K-y} = ax \qquad \text{Property 4 of Section 4.3}$$

as was to be shown.

Now Work Problem 9 ◁

Some logarithmic equations can be solved by rewriting them in exponential forms.

EXAMPLE 6 **Solving a Logarithmic Equation**

Solve $\log_2 x = 5 - \log_2 (x+4)$.

Solution: Here we must assume that both x and $x+4$ are positive, so that their logarithms are defined. Both conditions are satisfied if $x > 0$. To solve the equation, we first place all logarithms on one side so that we can combine them:

$$\log_2 x + \log_2 (x+4) = 5$$
$$\log_2 [x(x+4)] = 5$$

In exponential form, we have

$$x(x+4) = 2^5$$
$$x^2 + 4x = 32$$
$$x^2 + 4x - 32 = 0 \qquad \text{quadratic equation}$$
$$(x-4)(x+8) = 0$$
$$x = 4 \quad \text{or} \quad x = -8$$

Because we must have $x > 0$, the only solution is 4, as can be verified by substituting into the original equation. Indeed, replacing x by 4 in $\log_2 x$ gives $\log_2 4 = \log_2 2^2 = 2$ while replacing x by 4 in $5 - \log_2 (x+4)$ gives $5 - \log_2 (4+4) = 5 - \log_2 (8) = 5 - \log_2 2^3 = 5 - 3 = 2$. Since the results are the same, 4 is a solution of the equation.

In solving a logarithmic equation, it is a good idea to check for extraneous solutions.

Now Work Problem 5 ◁

APPLY IT ▶

18. The Richter scale measure of an earthquake is given by $R = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake, and I_0 is the intensity of a zero-level earthquake. An earthquake that is 675,000 times as intense as a zero-level earthquake has a magnitude on the Richter scale that is 4 more than another earthquake. What is the intensity of the other earthquake?

PROBLEMS 4.4

In Problems 1–36, find x. Round your answers to three decimal places.

1. $\log(5x+1) = \log(4x+2)$
2. $\log x - \log 5 = \log 7$
3. $\log 7 - \log(x-1) = \log 4$
4. $\log_2 x + 3\log_2 2 = \log_2 \frac{2}{x}$
5. $\ln(-x) = \ln(x^2 - 6)$
6. $\ln(x+3) + \ln 4 = 2\ln x$
7. $e^{2x} \cdot e^{5x} = e^{14}$
8. $(e^{3x-2})^3 = e^3$
9. $(81)^{4x} = 9$
10. $(27)^{2x+1} = \frac{1}{3}$
11. $e^{5x} = 7$
12. $e^{4x} = \frac{3}{4}$
13. $2e^{5x+2} = 17$
14. $5e^{2x-1} - 2 = 23$
15. $10^{4/x} = 6$
16. $\frac{2(10)^{0.3x}}{7} = 5$
17. $\frac{5}{10^{2x}} = 7$
18. $2(10)^x + (10)^{x+1} = 4$
19. $2^x = 5$
20. $7^{2x+3} = 9$
21. $5^{7x+5} = 2$
22. $4^{x/2} = 20$
23. $2^{-2x/3} = \frac{4}{5}$
24. $5(3^x - 6) = 10$
25. $(4)5^{3-x} - 7 = 2$
26. $\frac{5}{2^x} = 11$
27. $\log(x-3) = 3$
28. $\log_2(x+1) = 4$
29. $\log_4(9x-4) = 2$
30. $\log_4(2x+4) - 3 = \log_4 3$
31. $\ln(x-2) + \ln(2x+1) = 5$
32. $\log(x-3) + \log(x-5) = 1$
33. $\log_2(5x+1) = 4 - \log_2(3x-2)$
34. $\log(x+2)^2 = 2$, where $x > 0$
35. $\log_2\left(\frac{2}{x}\right) = 3 + \log_2 x$
36. $\log(x+5) = \log(3x+2) + 1$

37. Rooted Plants In a study of rooted plants in a certain geographic region,[11] it was determined that on plots of size A (in square meters), the average number of species that occurred was S. When $\log S$ was graphed as a function of $\log A$, the result was a straight line given by

$$\log S = \log 12.4 + 0.26 \log A$$

Solve for S.

38. Gross National Product In an article, Taagepera and Hayes refer to an equation of the form

$$\log T = 1.7 + 0.2068 \log P - 0.1334(\log P)^2$$

Here T is the percentage of a country's gross national product (GNP) that corresponds to foreign trade (exports plus imports), and P is the country's population (in units of 100,000).[12] Verify the claim that

$$T = 50P^{(0.2068 - 0.1334 \log P)}$$

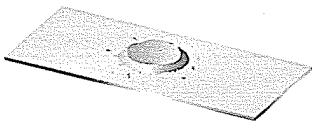
You may assume that $\log 50 = 1.7$. Also verify that, for any base b, $(\log_b x)^2 = \log_b (x^{\log_b x})$.

39. Radioactivity The number of milligrams of a radioactive substance present after t years is given by

$$Q = 100e^{-0.035t}$$

(a) How many milligrams are present after 0 years?
(b) After how many years will there be 20 milligrams present? Give your answer to the nearest year.

40. Blood Sample On the surface of a glass slide is a grid that divides the surface into 225 equal squares. Suppose a blood sample containing N red cells is spread on the slide and the cells are randomly distributed. Then the number of squares containing no cells is (approximately) given by $225e^{-N/225}$. If 100 of the squares contain no cells, estimate the number of cells the blood sample contained.



41. Population In Springfield the population P grows at the rate of 3% per year. The equation $P = 1{,}500{,}000(1.03)^t$ gives the population t years after 2009. Find the value of t for which the population will be 2,000,000. Give your answer to the nearest tenth.

42. Market Penetration In a discussion of market penetration by new products, Hurter and Rubenstein[13] refer to the function

$$F(t) = \frac{q - pe^{-(t+C)(p+q)}}{q[1 + e^{(t+C)(p+q)}]}$$

where p, q, and C are constants. They claim that if $F(0) = 0$, then

$$C = -\frac{1}{p+q} \ln \frac{q}{p}$$

Show that their claim is true.

43. Demand Equation The demand equation for a consumer product is $q = 80 - 2^p$. Solve for p and express your answer in terms of common logarithms, as in Example 4. Evaluate p to two decimal places when $q = 60$.

44. Investment The equation $A = P(1.105)^t$ gives the value A at the end of t years of an investment of P dollars compounded annually at an annual interest rate of 10.5%. How many years will it take for an investment to double? Give your answer to the nearest year.

45. Sales After t years the number of units of a product sold per year is given by $q = 1000\left(\frac{1}{2}\right)^{0.8^t}$. Such an equation is called a *Gompertz equation* and describes natural growth in many areas of study. Solve this equation for t in the same manner as in Example 4, and show that

$$t = \frac{\log\left(\dfrac{3 - \log q}{\log 2}\right)}{\log 0.8}$$

Also, for any A and suitable b and a, solve $y = Ab^{a^x}$ for x and explain how the previous solution is a special case.

46. Learning Equation Suppose that the daily output of units of a new product on the tth day of a production run is given by

$$q = 100(1 - e^{-0.1t})$$

Such an equation is called a *learning equation* and indicates that as time progresses, output per day will increase. This may be due to a gain in a worker's proficiency at his or her job. Determine, to the nearest complete unit, the output on (a) the first day and (b) the tenth day after the start of a production run. (c) After how many days will a daily production run of 80 units be reached? Give your answer to the nearest day.

47. Verify that 4 is the only solution to the logarithmic equation in Example 6 by graphing the function

$$y = 5 - \log_2 (x + 4) - \log_2 x$$

and observing when $y = 0$.

48. Solve $2^{3x+0.5} = 17$. Round your answer to two decimal places.

49. Solve $\ln (x + 2) = 5 - x$. Round your answer to two decimal places.

50. Graph the equation $(3)2^y - 4x = 5$. (*Hint:* Solve for y as a function of x.)

[11] R. W. Poole, *An Introduction to Quantitative Ecology* (New York: McGraw-Hill, 1974).

[12] R. Taagepera and J. P. Hayes, "How Trade/GNP Ratio Decreases with Country Size," *Social Science Research*, 6 (1977), 108–32.

[13] A. P. Hurter, Jr., A. H. Rubenstein et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

Chapter 4 Review

Important Terms and Symbols

Examples

Section 4.1 Exponential Functions

exponential function, b^x, for $b > 1$ and for $0 < b < 1$ — Ex. 2,3, p. 175,176
compound interest — principal — compound amount — Ex. 6, p. 178
interest period — periodic rate — nominal rate
e — natural exponential function, e^x — Ex. 8, p. 181
exponential law of decay — initial amount — decay constant — half-life — Ex. 11, p. 183

Section 4.2 Logarithmic Functions

logarithmic function, $\log_b x$ — common logarithm, $\log x$ — Ex. 5, p. 189
natural logarithm, $\ln x$ — Ex. 5, p. 189

Section 4.3 Properties of Logarithms

change-of-base formula — Ex. 8, p. 196

Section 4.4 Logarithmic and Exponential Equations

logarithmic equation — exponential equation — Ex. 1, p. 198

Summary

An exponential function has the form $f(x) = b^x$. The graph of $y = b^x$ has one of two general shapes, depending on the value of the base b. (See Figure 4.3.) The compound interest formula

$$S = P(1 + r)^n$$

expresses the compounded future amount S of a principal P at periodic rate r, as an exponential function of the number of interest periods n.

The irrational number $e \approx 2.71828$ provides the most important base for an exponential function. This base occurs in economic analysis and many situations involving growth of populations or decay of radioactive elements. Radioactive elements follow the exponential law of decay,

$$N = N_0 e^{-\lambda t}$$

where N is the amount of an element present at time t, N_0 is the initial amount, and λ is the decay constant. The time required for half of the amount of the element to decay is called the half-life.

The logarithmic function is the inverse function of the exponential function, and vice versa. The logarithmic function with base b is denoted $\log_b$, and $y = \log_b x$ if and only if $b^y = x$. The graph of $y = \log_b x$ has one of two general shapes, depending on the value of the base b. (See Figure 4.19.) Logarithms with base e are called natural logarithms and are denoted ln; those with base 10 are called common logarithms and are denoted log. The half-life T of a radioactive element can be given in terms of a natural logarithm and the decay constant: $T = (\ln 2)/\lambda$.

Some important properties of logarithms are the following:

$$\log_b (mn) = \log_b m + \log_b n$$
$$\log_b \frac{m}{n} = \log_b m - \log_b n$$
$$\log_b m^r = r \log_b m$$
$$\log_b \frac{1}{m} = -\log_b m$$
$$\log_b 1 = 0$$
$$\log_b b = 1$$
$$\log_b b^r = r$$
$$b^{\log_b m} = m$$
$$\log_b m = \frac{\log_a m}{\log_a b}$$

Moreover, if $\log_b m = \log_b n$, then $m = n$. Similarly, if $b^m = b^n$, then $m = n$. Many of these properties are used in solving logarithmic and exponential equations.

Review Problems

In Problems 1–6, write each exponential form logarithmically and each logarithmic form exponentially.

1. $3^5 = 243$ **2.** $\log_5 625 = 4$ **3.** $\log_{81} 3 = \frac{1}{4}$
4. $10^5 = 100{,}000$ **5.** $e^7 \approx 1096.63$ **6.** $\log_9 9 = 1$

In Problems 7–12, find the value of the expression without using a calculator.

7. $\log_5 125$ **8.** $\log_4 16$ **9.** $\log_3 \frac{1}{81}$
10. $\log_{1/5} \frac{1}{625}$ **11.** $\log_{1/3} 9$ **12.** $\log_4 2$

In Problems 13–18, find x without using a calculator.

13. $\log_5 625 = x$ **14.** $\log_x \frac{1}{81} = -4$ **15.** $\log_2 x = -10$
16. $\ln \frac{1}{e} = x$ **17.** $\ln (2x + 3) = 0$ **18.** $e^{\ln (x+4)} = 7$

10 Limits and Continuity

10.1 Limits

10.2 Limits (Continued)

10.3 Continuity

10.4 Continuity Applied to Inequalities

Chapter 10 Review

EXPLORE & EXTEND
National Debt

The philosopher Zeno of Elea was fond of paradoxes about motion. His most famous one goes something like this: The warrior Achilles agrees to run a race against a tortoise. Achilles can run 10 meters per second and the tortoise only 1 meter per second, so the tortoise gets a 10-meter head start. Since Achilles is so much faster, he should still win. But by the time he has covered his first 10 meters and reached the place where the tortoise started, the tortoise has advanced 1 meter and is still ahead. And after Achilles has covered that 1 meter, the tortoise has advanced another 0.1 meter and is still ahead. And after Achilles has covered that 0.1 meter, the tortoise has advanced another 0.01 meter and is still ahead. And so on. Therefore, Achilles gets closer and closer to the tortoise but can never catch up.

Zeno's audience knew that the argument was fishy. The position of Achilles at time t after the race has begun is $(10\text{ m/s})t$. The position of the tortoise at the same time t is $(1\text{ m/s})t + 10\text{ m}$. When these are equal, Achilles and the tortoise are side by side. To solve the resulting equation

$$(10\text{ m/s})t = (1\text{ m/s})t + 10\text{ m}$$

for t is to find the time at which Achilles pulls even with the tortoise.

The solution is $t = 1\frac{1}{9}$ seconds, at which time Achilles will have run $\left(1\frac{1}{9}\text{ s}\right)(10\text{ m/s}) = 11\frac{1}{9}$ meters.

What puzzled Zeno and his listeners is how it could be that

$$10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots = 11\frac{1}{9}$$

where the left side represents an *infinite sum* and the right side is a finite result. The solution to this problem is the concept of a limit, which is the key topic of this chapter. The left side of the equation is the sum of an infinite geometric sequence. Using limit notation, summation notation, and the formula from Section 1.6 for the sum of a finite geometric sequence, we write

$$\lim_{k\to\infty}\sum_{n=0}^{k} 10^{1-n} = \lim_{k\to\infty}\frac{10\left(1-\left(\frac{1}{10}\right)^{k+1}\right)}{1-\frac{1}{10}} = \frac{100}{9} = 11\frac{1}{9}$$

and find the sum of this particular infinite geometric sequence. (In Section 1.6 we showed that, for an infinite sequence with first term a and common ratio r, the sum of the infinite sequence exists and is given by $\frac{a}{1-r}$, *provided that* $|r| < 1$.)

Objective

To study limits and their basic properties.

10.1 Limits

Perhaps you have been in a parking-lot situation in which you must "inch up" to the car in front, but yet you do not want to bump or touch it. This notion of getting closer and closer to something, but yet not touching it, is very important in mathematics and is involved in the concept of *limit,* which lies at the foundation of calculus. We will let a variable "inch up" to a particular value and examine the effect this process has on the values of a function.

For example, consider the function

$$f(x) = \frac{x^3 - 1}{x - 1}$$

Although this function is not defined at $x = 1$, we may be curious about the behavior of the function values as x gets very close to 1. Table 10.1 gives some values of x that are slightly less than 1 and some that are slightly greater, as well as their corresponding function values. Notice that as x takes on values closer and closer to 1, regardless of whether x approaches it *from the left* ($x < 1$) or *from the right* ($x > 1$), the corresponding values of $f(x)$ get closer and closer to one and only one number, namely 3. This is also clear from the graph of f in Figure 10.1. Notice there that even though the function is not defined at $x = 1$ (as indicated by the hollow dot), the function values get closer and closer to 3 as x gets closer and closer to 1. To express this, we say that the **limit** of $f(x)$ as x approaches 1 is 3 and write

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3$$

We can make $f(x)$ as close as we like to 3, and keep it that close, by taking x sufficiently close to, but different from, 1. The limit exists at 1, even though 1 is not in the domain of f.

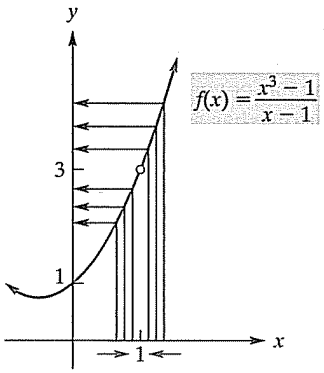


FIGURE 10.1 $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = 3.$

Table 10.1

$x < 1$		$x > 1$	
x	$f(x)$	x	$f(x)$
0.8	2.44	1.2	3.64
0.9	2.71	1.1	3.31
0.95	2.8525	1.05	3.1525
0.99	2.9701	1.01	3.0301
0.995	2.985025	1.005	3.015025
0.999	2.997001	1.001	3.003001

We can also consider the limit of a function as x approaches a number that is in the domain. Let us examine the limit of $f(x) = x + 3$ as x approaches 2:

$$\lim_{x \to 2} (x + 3)$$

Obviously, if x is close to 2 (but not equal to 2), then $x + 3$ is close to 5. This is also apparent from the table and graph in Figure 10.2. Thus,

$$\lim_{x \to 2} (x + 3) = 5$$

Given a function f and a number a, there *may* be two ways of associating a number to the pair (f, a). One such number is the *evaluation of f at a,* namely $f(a)$. It *exists* precisely when a is in the domain of f. For example, if $f(x) = \frac{x^3 - 1}{x - 1}$, our first example, then $f(1)$ does not *exist*. Another way of associating a number to the pair (f, a) is *the limit*

$x < 2$

x	$f(x)$
1.5	4.5
1.9	4.9
1.95	4.95
1.99	4.99
1.999	4.999

$x > 2$

x	$f(x)$
2.5	5.5
2.1	5.1
2.05	5.05
2.01	5.01
2.001	5.001

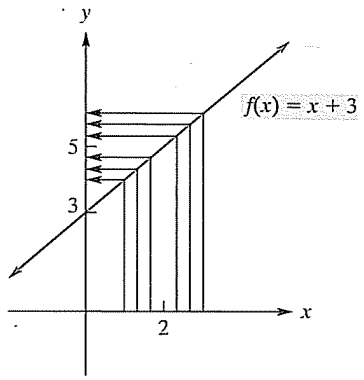


FIGURE 10.2 $\lim_{x \to 2}(x+3) = 5$.

of $f(x)$ *as* x *approaches* a, which is denoted $\lim_{x \to a} f(x)$. We have given two examples. Here is the general case.

Definition

The limit of $f(x)$ *as* x *approaches* a is the number L, written

$$\lim_{x \to a} f(x) = L$$

provided that we can make the values $f(x)$ as close as we like to L, and keep them that close, by taking x sufficiently close to, but different from, a. If there is no such number, we say that the limit of $f(x)$ as x approaches a *does not exist.*

We emphasize that, when finding a limit, we are concerned not with what happens to $f(x)$ when x equals a, but only with what happens to $f(x)$ when x is close to a. In fact, even if $f(a)$ *exists,* the preceding definition explicitly rules out consideration of it. In our second example, $f(x) = x + 3$, we have $f(2) = 5$ and also $\lim_{x \to 2}(x + 3) = 5$, but it is quite possible to have a function f and a number a for which both $f(a)$ and $\lim_{x \to a} f(x)$ exist and are different. Moreover, a limit must be independent of the way in which x *approaches* a, meaning the way in which x gets close to a. That is, the limit must be the same whether x approaches a from the left or from the right (for $x < a$ or $x > a$, respectively).

EXAMPLE 1 Estimating a Limit from a Graph

a. Estimate $\lim_{x \to 1} f(x)$, where the graph of f is given in Figure 10.3(a).

Solution: If we look at the graph for values of x near 1, we see that $f(x)$ is near 2. Moreover, as x gets closer and closer to 1, $f(x)$ appears to get closer and closer to 2.

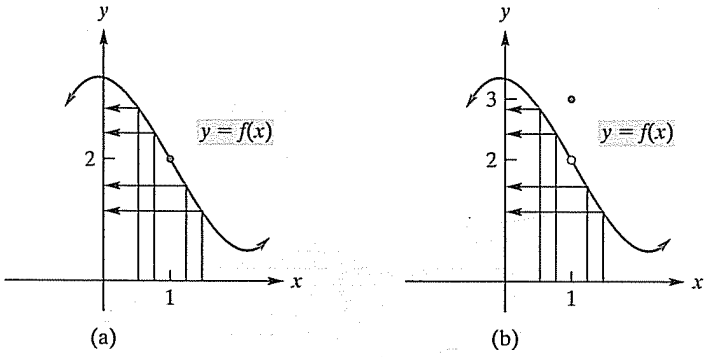


FIGURE 10.3 Investigation of $\lim_{x \to 1} f(x)$.

Thus, we estimate that

$$\lim_{x\to 1} f(x) = 2$$

b. Estimate $\lim_{x\to 1} f(x)$, where the graph of f is given in Figure 10.3(b).

Solution: Although $f(1) = 3$, this fact has no bearing whatsoever on the limit of $f(x)$ as x approaches 1. We see that as x gets closer and closer to 1, $f(x)$ appears to get closer and closer to 2. Thus, we estimate that

$$\lim_{x\to 1} f(x) = 2$$

Now Work Problem 1 ◁

Up to now, all of the limits that we have considered did indeed exist. Next we look at some situations in which a limit does not exist.

APPLY IT ▷

1. The greatest integer function, denoted, $f(x) = \lfloor x \rfloor$, is used every day by cashiers making change for customers. This function tells the amount of paper money for each amount of change owed. (For example, if a customer is owed \$1.25 in change, he or she would get \$1 in paper money; thus, $\lfloor 1.25 \rfloor = 1$.) Formally, $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to x. Graph f, sometimes called a step function, on a graphing calculator in the standard viewing rectangle. (It is in the numbers menu; it's called "integer part.") Explore this graph using TRACE. Determine whether $\lim_{x\to a} f(x)$ exists.

EXAMPLE 2 Limits That Do Not Exist

a. Estimate $\lim_{x\to -2} f(x)$ if it exists, where the graph of f is given in Figure 10.4.

Solution: As x approaches -2 from the left ($x < -2$), the values of $f(x)$ appear to get closer to 1. But as x approaches -2 from the right ($x > -2$), $f(x)$ appears to get closer to 3. Hence, as x approaches -2, the function values do not settle down to one and only one number. We conclude that

$$\lim_{x\to -2} f(x) \text{ does not exist}$$

Note that the limit does not exist even though the function is defined at $x = -2$.

b. Estimate $\lim_{x\to 0} \frac{1}{x^2}$ if it exists.

Solution: Let $f(x) = 1/x^2$. The table in Figure 10.5 gives values of $f(x)$ for some values of x near 0. As x gets closer and closer to 0, the values of $f(x)$ get larger and larger without bound. This is also clear from the graph. Since the values of $f(x)$ do not approach a *number* as x approaches 0,

$$\lim_{x\to 0} \frac{1}{x^2} \text{ does not exist}$$

Now Work Problem 3 ◁

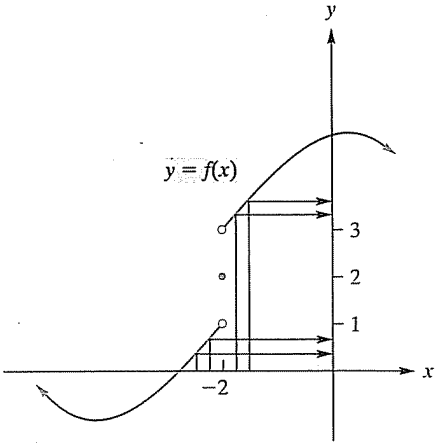


FIGURE 10.4 $\lim_{x\to -2} f(x)$ does not exist.

x	$f(x)$
±1	1
±0.5	4
±0.1	100
±0.01	10,000
±0.001	1,000,000

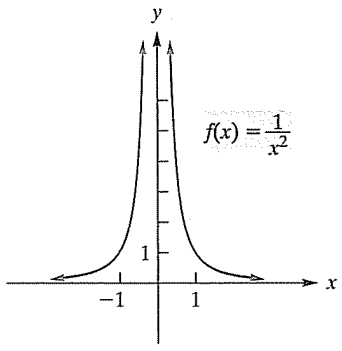


FIGURE 10.5 $\lim_{x\to 0} \frac{1}{x^2}$ does not exist.

TECHNOLOGY

Problem: Estimate $\lim_{x\to 2} f(x)$ if

$$f(x) = \frac{x^3 + 2.1x^2 - 10.2x + 4}{x^2 + 2.5x - 9}$$

Solution: One method of finding the limit is by constructing a table of function values $f(x)$ when x is close to 2. From Figure 10.6, we estimate the limit to be 1.57. Alternatively, we can estimate the limit from the graph of f. Figure 10.7 shows the graph of f in the standard window of $[-10, 10] \times [-10, 10]$. First we zoom in several times around $x = 2$ and obtain Figure 10.8. After tracing around $x = 2$, we estimate the limit to be 1.57.

X	Y1
1.9	1.4688
1.99	1.5592
1.999	1.5682
1.9999	1.5691
2.01	1.5793
2.001	1.5702
2.0001	1.5693

X=2.0001

FIGURE 10.6 $\lim_{x\to 2} f(x) \approx 1.57$.

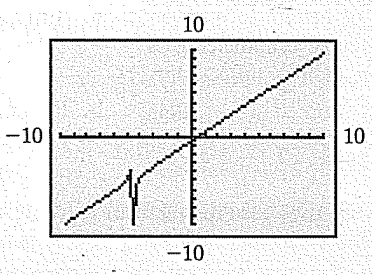


FIGURE 10.7 Graph of $f(x)$ in standard window.

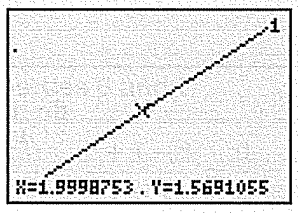


FIGURE 10.8 Zooming and tracing around $x = 2$ gives $\lim_{x\to 2} f(x) \approx 1.57$.

Properties of Limits

To determine limits, we do not always want to compute function values or sketch a graph. Alternatively, there are several properties of limits that we may be able to employ. The following properties may seem reasonable to you:

1. If $f(x) = c$ is a constant function, then
$$\lim_{x\to a} f(x) = \lim_{x\to a} c = c$$
2. $\lim_{x\to a} x^n = a^n$, for any positive integer n

EXAMPLE 3 Applying Limit Properties 1 and 2

a. $\lim_{x\to 2} 7 = 7$; $\lim_{x\to -5} 7 = 7$

b. $\lim_{x\to 6} x^2 = 6^2 = 36$

c. $\lim_{t\to -2} t^4 = (-2)^4 = 16$

Now Work Problem 9 ◁

Some other properties of limits are as follows:

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

3. $$\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

That is, the limit of a sum or difference is the sum or difference, respectively, of the limits.

4. $$\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

That is, the limit of a product is the product of the limits.

5. $$\lim_{x\to a} [cf(x)] = c \cdot \lim_{x\to a} f(x), \text{ where } c \text{ is a constant}$$

That is, the limit of a constant times a function is the constant times the limit of the function.

APPLY IT ▷

2. The volume of helium in a spherical balloon (in cubic centimeters), as a function of the radius r in centimeters, is given by $V(r) = \frac{4}{3}\pi r^3$. Find $\lim_{r\to 1} V(r)$.

EXAMPLE 4 **Applying Limit Properties**

a. $\lim_{x\to 2}(x^2 + x) = \lim_{x\to 2} x^2 + \lim_{x\to 2} x$ Property 3

$= 2^2 + 2 = 6$ Property 2

b. Property 3 can be extended to the limit of a finite number of sums and differences. For example,

$$\lim_{q\to -1}(q^3 - q + 1) = \lim_{q\to -1} q^3 - \lim_{q\to -1} q + \lim_{q\to -1} 1$$
$$= (-1)^3 - (-1) + 1 = 1$$

c. $\lim_{x\to 2}[(x+1)(x-3)] = \lim_{x\to 2}(x+1) \cdot \lim_{x\to 2}(x-3)$ Property 4

$= \left(\lim_{x\to 2} x + \lim_{x\to 2} 1\right) \cdot \left(\lim_{x\to 2} x - \lim_{x\to 2} 3\right)$

$= (2+1)\cdot(2-3) = 3(-1) = -3$

d. $\lim_{x\to -2} 3x^3 = 3 \cdot \lim_{x\to -2} x^3$ Property 5

$= 3(-2)^3 = -24$

Now Work Problem 11 ◁

APPLY IT ▷

3. The revenue function for a certain product is given by $R(x) = 500x - 6x^2$. Find $\lim_{x\to 8} R(x)$.

EXAMPLE 5 **Limit of a Polynomial Function**

Let $f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$ define a polynomial function. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0)$$
$$= c_n \cdot \lim_{x\to a} x^n + c_{n-1} \cdot \lim_{x\to a} x^{n-1} + \cdots + c_1 \cdot \lim_{x\to a} x + \lim_{x\to a} c_0$$
$$= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0 = f(a)$$

Thus, we have the following property:

If f is a polynomial function, then

$$\lim_{x\to a} f(x) = f(a)$$

In other words, if f is a polynomial and a is any number, then both ways of associating a number to the pair (f, a), namely formation of the limit and evaluation, exist and are equal.

Now Work Problem 13 ◁

The result of Example 5 allows us to find many limits simply by evaluation. For example, we can find

$$\lim_{x\to -3}(x^3 + 4x^2 - 7)$$

by substituting -3 for x because $x^3 + 4x^2 - 7$ is a polynomial function:

$$\lim_{x\to -3}(x^3 + 4x^2 - 7) = (-3)^3 + 4(-3)^2 - 7 = 2$$

Similarly,

$$\lim_{h\to 3}(2(h-1)) = 2(3-1) = 4$$

We want to stress that we do not find limits simply by evaluating unless there is a rule that covers the situation. We were able to find the previous two limits by evaluation because we have a rule that applies to limits of polynomial functions. However, indiscriminate use of evaluation can lead to errors. To illustrate, in Example 1(b) we

have $f(1) = 3$, which is not $\lim_{x\to 1} f(x)$; in Example 2(a), $f(-2) = 2$, which is not $\lim_{x\to -2} f(x)$.

The next two limit properties concern quotients and roots.

If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

6. $$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0$$

That is, the limit of a quotient is the quotient of limits, provided that the denominator does not have a limit of 0.

7. $$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$ See footnote 1

EXAMPLE 6 **Applying Limit Properties 6 and 7**

a. $$\lim_{x\to 1} \frac{2x^2 + x - 3}{x^3 + 4} = \frac{\lim_{x\to 1}(2x^2 + x - 3)}{\lim_{x\to 1}(x^3 + 4)} = \frac{2 + 1 - 3}{1 + 4} = \frac{0}{5} = 0$$

b. $$\lim_{t\to 4} \sqrt{t^2 + 1} = \sqrt{\lim_{t\to 4}(t^2 + 1)} = \sqrt{17}$$

c. $$\lim_{x\to 3} \sqrt[3]{x^2 + 7} = \sqrt[3]{\lim_{x\to 3}(x^2 + 7)} = \sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = 2\sqrt[3]{2}$$

Now Work Problem 15 ◁

CAUTION!

Note that in Example 6(a) the numerator and denominator of the function are polynomials. In general, we can determine a limit of a rational function by evaluation, provided that the denominator is not 0 at a.

Limits and Algebraic Manipulation

We now consider limits to which our limit properties do not apply and which cannot be determined by evaluation. A fundamental result is the following:

If f and g are two functions for which $f(x) = g(x)$, for all $x \neq a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

(meaning that if either limit exists, then the other exists and they are equal).

CAUTION!

The condition for equality of the limits does not preclude the possibility that $f(a) = g(a)$. The condition only concerns $x \neq a$.

The result follows directly from the definition of *limit* since the value of $\lim_{x\to a} f(x)$ depends only on those values $f(x)$ for x that are close to a. We repeat: The evaluation of f at a, $f(a)$, or lack of its existence, is irrelevant in the determination of $\lim_{x\to a} f(x)$ unless we have a specific rule that applies, such as in the case when f is a polynomial.

APPLY IT ▷

4. The rate of change of productivity p (in number of units produced per hour) increases with time on the job by the function

$$p(t) = \frac{50(t^2 + 4t)}{t^2 + 3t + 20}$$

Find $\lim_{t\to 2} p(t)$.

EXAMPLE 7 **Finding a Limit**

Find $\lim_{x\to -1} \frac{x^2 - 1}{x + 1}$.

Solution: As $x \to -1$, both numerator and denominator approach zero. Because the limit of the denominator is 0, we *cannot* use Property 6. However, since what happens to the quotient when x equals -1 is of no concern, we can assume that $x \neq -1$ and simplify the fraction:

$$\frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1} = x - 1 \quad \text{for } x \neq -1$$

[1] If n is even, we require that $\lim_{x\to a} f(x)$ be nonnegative.

This algebraic manipulation (factoring and cancellation) of the original function $\frac{x^2-1}{x+1}$ yields a new function $x-1$, which is the same as the original function for $x \neq -1$. Thus the fundamental result displayed in the box at the beginning of this subsection applies and we have

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} (x-1) = -1-1 = -2$$

Notice that, although the original function is not defined at -1, it *does* have a limit as $x \to -1$.

Now Work Problem 21 ◁

When both $f(x)$ and $g(x)$ approach 0 as $x \to a$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is said to have the *form* 0/0. Similarly, we speak of *form* $k/0$, for $k \neq 0$ if $f(x)$ approaches $k \neq 0$ as $x \to a$ but $g(x)$ approaches 0 as $x \to a$.

In Example 7, the method of finding a limit by evaluation does not work. Replacing x by -1 gives 0/0, which has no meaning. When the meaningless form 0/0 arises, algebraic manipulation (as in Example 7) may result in a function that agrees with the original function, except possibly at the limiting value. In Example 7 the new function, $x-1$, is a polynomial and its limit *can* be found by evaluation.

In the beginning of this section, we found

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

by examining a table of function values of $f(x) = (x^3-1)/(x-1)$ and also by considering the graph of f. This limit has the form 0/0. Now we will determine the limit by using the technique used in Example 7.

EXAMPLE 8 **Form 0/0**

Find $\lim_{x\to 1} \frac{x^3-1}{x-1}$.

Solution: As $x \to 1$, both the numerator and denominator approach 0. Thus, we will try to express the quotient in a different form for $x \neq 1$. By factoring, we have

$$\frac{x^3-1}{x-1} = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2+x+1 \qquad \text{for } x \neq 1$$

(Alternatively, long division would give the same result.) Therefore,

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} (x^2+x+1) = 1^2+1+1 = 3$$

as we showed before.

Now Work Problem 23 ◁

CAUTION!

There is frequently confusion about which principle is being used in this example and in Example 7. It is this:

If $f(x) = g(x)$ for $x \neq a$,

then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.

EXAMPLE 9 **Form 0/0**

If $f(x) = x^2+1$, find $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

Solution:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2+1]-(x^2+1)}{h}$$

Here we treat x as a constant because h, not x, is changing. As $h \to 0$, both the numerator and denominator approach 0. Therefore, we will try to express the quotient in a different form, for $h \neq 0$. We have

$$\lim_{h\to 0} \frac{[(x+h)^2+1]-(x^2+1)}{h} = \lim_{h\to 0} \frac{[x^2+2xh+h^2+1]-x^2-1}{h}$$

$$= \lim_{h\to 0} \frac{2xh+h^2}{h}$$

APPLY IT ▷

5. The length of a material increases as it is heated up according to the equation $l = 125 + 2x$. The rate at which the length is increasing is given by

$$\lim_{h\to 0} \frac{125+2(x+h)-(125+2x)}{h}$$

Calculate this limit.

The expression

$$\frac{f(x+h)-f(x)}{h}$$

is called a *difference quotient*. The limit of the difference quotient lies at the heart of differential calculus. We will encounter such limits in Chapter 11.

$$= \lim_{h\to 0} \frac{h(2x+h)}{h}$$
$$= \lim_{h\to 0} (2x+h)$$
$$= 2x$$

Note: It is the fourth equality above, $\lim_{h\to 0} \frac{h(2x+h)}{h} = \lim_{h\to 0} (2x+h)$, that uses the fundamental result. When $\frac{h(2x+h)}{h}$ and $2x+h$ are considered as *functions of* h, they are seen to be equal, for all $h \neq 0$. It follows that their limits as h approaches 0 are equal.

Now Work Problem 35 ◁

A Special Limit

We conclude this section with a note concerning a most important limit, namely,

$$\lim_{x\to 0} (1+x)^{1/x}$$

Figure 10.9 shows the graph of $f(x) = (1+x)^{1/x}$. Although $f(0)$ does not exist, as $x \to 0$ it is clear that the limit of $(1+x)^{1/x}$ exists. It is approximately 2.71828 and is denoted by the letter e. This, you may recall, is the base of the system of natural logarithms. The limit

This limit will be used in Chapter 12.

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

can actually be considered the definition of e. It can be shown that this agrees with the definition of e that we gave in Section 4.1.

x	$(1+x)^{1/x}$
0.5	2.2500
0.1	2.5937
0.01	2.7048
0.001	2.7169

x	$(1+x)^{1/x}$
−0.5	4.0000
−0.1	2.8680
−0.01	2.7320
−0.001	2.7196

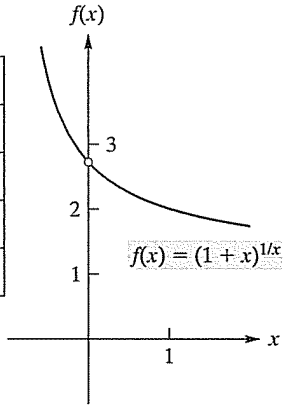


FIGURE 10.9 $\lim_{x\to 0} (1+x)^{1/x} = e$.

PROBLEMS 10.1

In Problems 1–4, use the graph of f to estimate each limit if it exists.

1. Graph of f appears in Figure 10.10.

(a) $\lim_{x\to 0} f(x)$ **(b)** $\lim_{x\to 1} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

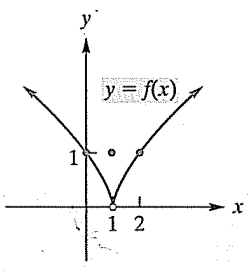


FIGURE 10.10

2. Graph of f appears in Figure 10.11.

(a) $\lim_{x\to -1} f(x)$ **(b)** $\lim_{x\to 0} f(x)$ **(c)** $\lim_{x\to 1} f(x)$

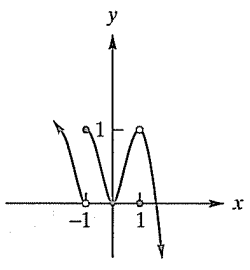


FIGURE 10.11

3. Graph of f appears in Figure 10.12.

(a) $\lim_{x\to -1} f(x)$ **(b)** $\lim_{x\to 1} f(x)$ **(c)** $\lim_{x\to 2} f(x)$

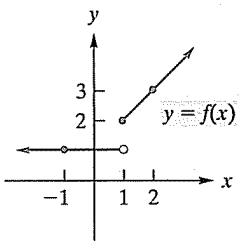


FIGURE 10.12

4. Graph of f appears in Figure 10.13.

(a) $\lim_{x\to -1} f(x)$ **(b)** $\lim_{x\to 0} f(x)$ **(c)** $\lim_{x\to 1} f(x)$

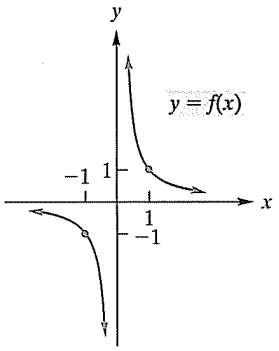


FIGURE 10.13

In Problems 5–8, use your calculator to complete the table, and use your results to estimate the given limit.

5. $\lim_{x\to -1} \dfrac{3x^2+2x-1}{x+1}$

x	−0.9	−0.99	−0.999	−1.001	−1.01	−1.1
$f(x)$						

6. $\lim_{x\to -3} \dfrac{x^2-9}{x+3}$

x	−3.1	−3.01	−3.001	−2.999	−2.99	−2.9
$f(x)$						

7. $\lim_{x\to 0} |x|^{|x|}$

x	−0.00001	0.00001	0.0001	0.001	0.01	0.1
$f(x)$						

8. $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$

h	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

In Problems 9–34, find the limits.

9. $\lim_{x\to 2} 16$
10. $\lim_{x\to 3} 2x$
11. $\lim_{t\to -5} (t^2-5)$
12. $\lim_{t\to 1/2} (3t-5)$
13. $\lim_{x\to -2} (3x^3-4x^2+2x-3)$
14. $\lim_{r\to 9} \dfrac{4r-3}{11}$
15. $\lim_{t\to -3} \dfrac{t-2}{t+5}$
16. $\lim_{x\to -6} \dfrac{x^2+6}{x-6}$
17. $\lim_{t\to 0} \dfrac{t}{t^3-4t+3}$
18. $\lim_{z\to 0} \dfrac{z^2-5z-4}{z^2+1}$
19. $\lim_{p\to 4} \sqrt{p^2+p+5}$
20. $\lim_{y\to 15} \sqrt{y+3}$
21. $\lim_{x\to -2} \dfrac{x^2+2x}{x+2}$
22. $\lim_{x\to -1} \dfrac{x^2-1}{x^2-1}$
23. $\lim_{x\to 2} \dfrac{x^2-x-2}{x-2}$
24. $\lim_{t\to 0} \dfrac{t^3+3t^2}{t^3-4t^2}$
25. $\lim_{x\to 3} \dfrac{x^2-x-6}{x-3}$
26. $\lim_{t\to 2} \dfrac{t^2-4}{t-2}$
27. $\lim_{x\to -4} \dfrac{x+4}{x^2-16}$
28. $\lim_{x\to 0} \dfrac{x^2-2x}{x}$
29. $\lim_{x\to 4} \dfrac{x^2-9x+20}{x^2-3x-4}$
30. $\lim_{x\to -3} \dfrac{x^4-81}{x^2+8x+15}$
31. $\lim_{x\to 2} \dfrac{3x^2-x-10}{x^2+5x-14}$
32. $\lim_{x\to 3} \dfrac{x^2-2x-3}{x^2+2x-15}$
33. $\lim_{h\to 0} \dfrac{(2+h)^2-2^2}{h}$
34. $\lim_{x\to 0} \dfrac{(x+2)^2-4}{x}$

35. Find $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$ by treating x as a constant.

36. Find $\lim_{h\to 0} \dfrac{3(x+h)^2+7(x+h)-3x^2-7x}{h}$ by treating x as a constant.

In Problems 37–42, find $\lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

37. $f(x)=5+2x$
38. $f(x)=2x+3$
39. $f(x)=x^2-3$
40. $f(x)=x^2+x+1$
41. $f(x)=x^3-4x^2$
42. $f(x)=2-5x+x^2$

43. Find $\lim_{x\to 6} \dfrac{\sqrt{x-2}-2}{x-6}$ (*Hint:* First rationalize the numerator by multiplying both the numerator and denominator by $\sqrt{x-2}+2$.)

44. Find the constant c so that $\lim_{x\to 3} \dfrac{x^2+x+c}{x^2-5x+6}$ exists. For that value of c, determine the limit. (*Hint:* Find the value of c for which $x-3$ is a factor of the numerator.)

45. **Power Plant** The maximum theoretical efficiency of a power plant is given by

$$E=\frac{T_h-T_c}{T_h}$$

where T_h and T_c are the absolute temperatures of the hotter and colder reservoirs, respectively. Find (a) $\lim_{T_c\to 0} E$ and (b) $\lim_{T_c\to T_h} E$.

46. **Satellite** When a 3200-lb satellite revolves about the earth in a circular orbit of radius r ft, the total mechanical energy E of the earth–satellite system is given by

$$E=-\frac{7.0\times 10^{17}}{r} \text{ ft-lb}$$

Find the limit of E as $r \to 7.5\times 10^7$ ft.

In Problems 47–50, use a graphing calculator to graph the functions, and then estimate the limits. Round your answers to two decimal places.

47. $\lim_{x\to 3} \dfrac{x^4-2x^3+2x^2-2x-3}{x^2-9}$
48. $\lim_{x\to 0} x^x$
49. $\lim_{x\to 9} \dfrac{x-10\sqrt{x}+21}{3-\sqrt{x}}$
50. $\lim_{x\to 1} \dfrac{x^3+x^2-5x+3}{x^3+2x^2-7x+4}$

51. Water Purification The cost of purifying water is given by $C = \dfrac{50{,}000}{p} - 6500$, where p is the percent of impurities remaining after purification. Graph this function on your graphing calculator, and determine $\lim_{p \to 0} C$. Discuss what this means.

52. Profit Function The profit function for a certain business is given by $P(x) = 225x - 3.2x^2 - 700$. Graph this function on your graphing calculator, and use the evaluation function to determine $\lim_{x \to 40.2} P(x)$, using the rule about the limit of a polynomial function.

Objective

To study one-sided limits, infinite limits, and limits at infinity.

10.2 Limits (Continued)

One-Sided Limits

Figure 10.14 shows the graph of a function f. Notice that $f(x)$ is not defined when $x = 0$. As x approaches 0 *from the right*, $f(x)$ approaches 1. We write this as

$$\lim_{x \to 0^+} f(x) = 1$$

On the other hand, as x approaches 0 *from the left*, $f(x)$ approaches -1, and we write

$$\lim_{x \to 0^-} f(x) = -1$$

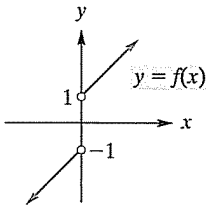


FIGURE 10.14 $\lim_{x \to 0} f(x)$ does not exist.

Limits like these are called **one-sided limits.** From the preceding section, we know that the limit of a function as $x \to a$ is independent of the way x approaches a. Thus, the limit will exist if and only if both one-sided limits exist and are equal. We therefore conclude that

$$\lim_{x \to 0} f(x) \text{ does not exist}$$

As another example of a one-sided limit, consider $f(x) = \sqrt{x - 3}$ as x approaches 3. Since f is defined only when $x \geq 3$, we can speak of the limit of $f(x)$ as x approaches 3 from the right. If x is slightly greater than 3, then $x - 3$ is a positive number that is close to 0, so $\sqrt{x - 3}$ is close to 0. We conclude that

$$\lim_{x \to 3^+} \sqrt{x - 3} = 0$$

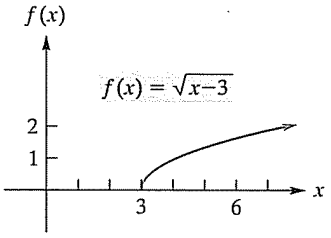


FIGURE 10.15 $\lim_{x \to 3^+} \sqrt{x - 3} = 0$.

This limit is also evident from Figure 10.15.

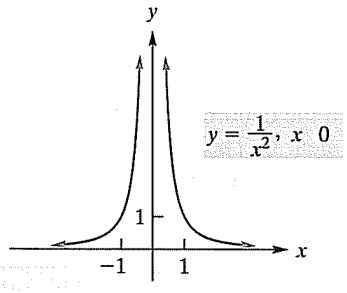
Infinite Limits

In the previous section, we considered limits of the form 0/0—that is, limits where both the numerator and denominator approach 0. Now we will examine limits where the denominator approaches 0, but the numerator approaches a number different from 0. For example, consider

$$\lim_{x \to 0} \frac{1}{x^2}$$

Here, as x approaches 0, the denominator approaches 0 and the numerator approaches 1. Let us investigate the behavior of $f(x) = 1/x^2$ when x is close to 0. The number x^2 is positive and also close to 0. Thus, dividing 1 by such a number results in a very large number. In fact, the closer x is to 0, the larger the value of $f(x)$. For example, see the table of values in Figure 10.16, which also shows the graph of f. Clearly, as $x \to 0$ both from the left and from the right, $f(x)$ increases without bound. Hence, no limit exists at 0. We say that as $x \to 0$, $f(x)$ becomes positively infinite, and symbolically we express this "infinite limit" by writing

$$\lim_{x \to 0} \frac{1}{x^2} = +\infty = \infty$$



x	$f(x)$
±1	1
±0.5	4
±0.1	100
±0.01	10,000
±0.001	1,000,000

FIGURE 10.16 $\lim_{x \to 0} \dfrac{1}{x^2} = \infty$.

If $\lim_{x \to a} f(x)$ does not exist, it may be for a reason other than that the values $f(x)$ become arbitrarily large as x gets close to a. For example, look again at the situation in Example 2(a) of Section 10.1. Here we have

$$\lim_{x \to -2} f(x) \text{ does not exist but } \lim_{x \to -2} f(x) \neq \infty$$

CAUTION!

The use of the "equality" sign in this situation does not mean that the limit exists. On the contrary, it is a way of saying specifically that there is no limit and *why* there is no limit.

Consider now the graph of $y = f(x) = 1/x$ for $x \neq 0$. (See Figure 10.17.) As x approaches 0 from the right, $1/x$ becomes positively infinite; as x approaches 0 from the left, $1/x$ becomes negatively infinite. Symbolically, these infinite limits are written

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

x	$f(x)$
0.01	100
0.001	1000
0.0001	10,000
−0.01	−100
−0.001	−1000
−0.0001	−10,000

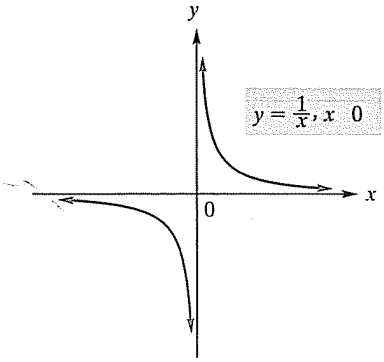


FIGURE 10.17 $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Either one of these facts implies that

$$\lim_{x\to 0} \frac{1}{x} \text{ does not exist}$$

EXAMPLE 1 **Infinite Limits**

Find the limit (if it exists).

a. $\lim_{x\to -1^+} \frac{2}{x+1}$

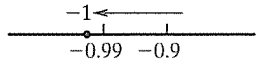


FIGURE 10.18 $x \to -1^+$.

Solution: As x approaches -1 from the right (think of values of x such as -0.9, -0.99, and so on, as shown in Figure 10.18), $x + 1$ approaches 0 but is always positive. Since we are dividing 2 by positive numbers approaching 0, the results, $2/(x+1)$, are positive numbers that are becoming arbitrarily large. Thus,

$$\lim_{x\to -1^+} \frac{2}{x+1} = \infty$$

and the limit does not exist. By a similar analysis, we can show that

$$\lim_{x\to -1^-} \frac{2}{x+1} = -\infty$$

b. $\lim_{x\to 2} \frac{x+2}{x^2-4}$

Solution: As $x \to 2$, the numerator approaches 4 and the denominator approaches 0. Hence, we are dividing numbers near 4 by numbers near 0. The results are numbers that become arbitrarily large in magnitude. At this stage, we can write

$$\lim_{x\to 2} \frac{x+2}{x^2-4} \text{ does not exist}$$

However, let us see if we can use the symbol ∞ or $-\infty$ to be more specific about "does not exist." Notice that

$$\lim_{x\to 2} \frac{x+2}{x^2-4} = \lim_{x\to 2} \frac{x+2}{(x+2)(x-2)} = \lim_{x\to 2} \frac{1}{x-2}$$

Since

$$\lim_{x\to 2^+} \frac{1}{x-2} = \infty \quad \text{and} \quad \lim_{x\to 2^-} \frac{1}{x-2} = -\infty$$

$\lim_{x\to 2} \frac{x+2}{x^2-4}$ is neither ∞ nor $-\infty$.

Now Work Problem 31 ◁

Example 1 considered limits of the form $k/0$, where $k \neq 0$. It is important to distinguish the form $k/0$ from the form $0/0$, which was discussed in Section 10.1. These two forms are handled quite differently.

EXAMPLE 2 Finding a Limit

Find $\lim_{t\to 2} \frac{t-2}{t^2-4}$.

Solution: As $t \to 2$, *both* numerator and denominator approach 0 (form $0/0$). Thus, we first simplify the fraction, for $t \neq 2$, as we did in Section 10.1, and then take the limit:

$$\lim_{t\to 2} \frac{t-2}{t^2-4} = \lim_{t\to 2} \frac{t-2}{(t+2)(t-2)} = \lim_{t\to 2} \frac{1}{t+2} = \frac{1}{4}$$

Now Work Problem 37 ◁

Limits at Infinity

Now let us examine the function

$$f(x) = \frac{1}{x}$$

as x becomes infinite, first in a positive sense and then in a negative sense. From Table 10.2, we can see that as x increases without bound through positive values, the values of $f(x)$ approach 0. Likewise, as x decreases without bound through negative values, the values of $f(x)$ also approach 0. These observations are also apparent from the graph in Figure 10.17. There, moving to the right along the curve through positive x-values, the corresponding y-values approach 0 through positive values. Similarly, moving to the left along the curve through negative x-values, the corresponding y-values approach 0 through negative values. Symbolically, we write

$$\lim_{x\to\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x} = 0$$

Both of these limits are called *limits at infinity*.

We can obtain

$$\lim_{x\to\infty} \frac{1}{x} \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x}$$

without the benefit of a graph or a table. Dividing 1 by a large positive number results in a small positive number, and as the divisors get arbitrarily large, the quotients get arbitrarily small. A similar argument can be made for the limit as $x \to -\infty$.

Table 10.2 Behavior of $f(x)$ as $x \to \pm\infty$

x	$f(x)$	x	$f(x)$
1000	0.001	−1000	−0.001
10,000	0.0001	−10,000	−0.0001
100,000	0.00001	−100,000	−0.00001
1,000,000	0.000001	−1,000,000	−0.000001

EXAMPLE 3 Limits at Infinity

Find the limit (if it exists).

a. $\lim_{x\to\infty} \frac{4}{(x-5)^3}$

Solution: As x becomes very large, so does $x-5$. Since the cube of a large number is also large, $(x-5)^3 \to \infty$. Dividing 4 by very large numbers results in numbers near 0. Thus,

$$\lim_{x\to\infty} \frac{4}{(x-5)^3} = 0$$

b. $\lim_{x\to-\infty} \sqrt{4-x}$

Solution: As x gets negatively infinite, $4-x$ becomes positively infinite. Because square roots of large numbers are large numbers, we conclude that

$$\lim_{x\to-\infty} \sqrt{4-x} = \infty$$

◁

APPLY IT ▷

6. The demand function for a certain product is given by $p(x) = \frac{10{,}000}{(x+1)^2}$, where p is the price in dollars and x is the quantity sold. Graph this function on your graphing calculator in the window $[0, 10] \times [0, 10{,}000]$. Use the TRACE function to find $\lim_{x\to\infty} p(x)$. Determine what is happening to the graph and what this means about the demand function.

In our next discussion we will need a certain limit, namely, $\lim_{x\to\infty} 1/x^p$, where $p > 0$. As x becomes very large, so does x^p. Dividing 1 by very large numbers results in numbers near 0. Thus, $\lim_{x\to\infty} 1/x^p = 0$. In general,

$$\lim_{x\to\infty} \frac{1}{x^p} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^p} = 0$$

for $p > 0$.[2] For example,

$$\lim_{x\to\infty} \frac{1}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1}{x^{1/3}} = 0$$

Let us now find the limit of the rational function

$$f(x) = \frac{4x^2 + 5}{2x^2 + 1}$$

as $x \to \infty$. (Recall from Section 2.2 that a rational function is a quotient of polynomials.) As x gets larger and larger, *both* the numerator and denominator of any rational function become infinite in absolute value. However, the form of the quotient can be changed, so that we can draw a conclusion as to whether or not it has a limit. To do this, we divide both the numerator and denominator by the greatest power of x that occurs in the denominator. Here it is x^2. This gives

$$\lim_{x\to\infty} \frac{4x^2+5}{2x^2+1} = \lim_{x\to\infty} \frac{\frac{4x^2+5}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x\to\infty} \frac{\frac{4x^2}{x^2} + \frac{5}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{4 + \frac{5}{x^2}}{2 + \frac{1}{x^2}} = \frac{\lim\limits_{x\to\infty} 4 + 5 \cdot \lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty} \frac{1}{x^2}}$$

Since $\lim_{x\to\infty} 1/x^p = 0$ for $p > 0$,

$$\lim_{x\to\infty} \frac{4x^2+5}{2x^2+1} = \frac{4+5(0)}{2+0} = \frac{4}{2} = 2$$

Similarly, the limit as $x \to -\infty$ is 2. These limits are clear from the graph of f in Figure 10.19.

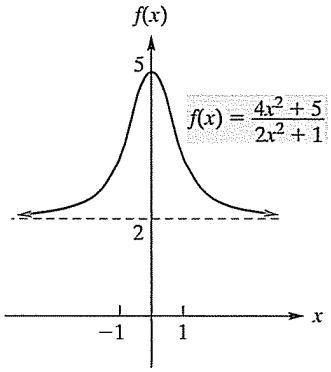


FIGURE 10.19 $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$.

For the preceding function, there is an easier way to find $\lim_{x\to\infty} f(x)$. For *large* values of x, in the numerator the term involving the greatest power of x, namely, $4x^2$, dominates the sum $4x^2 + 5$, and the dominant term in the denominator, $2x^2 + 1$, is $2x^2$. Thus, as $x \to \infty$, $f(x)$ can be approximated by $(4x^2)/(2x^2)$. As a result, to determine the limit of $f(x)$, it suffices to determine the limit of $(4x^2)/(2x^2)$. That is,

$$\lim_{x\to\infty} \frac{4x^2+5}{2x^2+1} = \lim_{x\to\infty} \frac{4x^2}{2x^2} = \lim_{x\to\infty} 2 = 2$$

as we saw before. In general, we have the following rule:

Limits at Infinity for Rational Functions

If $f(x)$ is a *rational function* and $a_n x^n$ and $b_m x^m$ are the terms in the numerator and denominator, respectively, with the greatest powers of x, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{a_n x^n}{b_m x^m}$$

[2] For $\lim_{x\to-\infty} 1/x^p$, we assume that p is such that $1/x^p$ is defined for $x < 0$.

Let us apply this rule to the situation where the degree of the numerator is greater than the degree of the denominator. For example,

$$\lim_{x\to-\infty}\frac{x^4-3x}{5-2x}=\lim_{x\to-\infty}\frac{x^4}{-2x}=\lim_{x\to-\infty}\left(-\frac{1}{2}x^3\right)=\infty$$

(Note that in the next-to-last step, as x becomes very negative, so does x^3; moreover, $-\frac{1}{2}$ times a very negative number is very positive.) Similarly,

$$\lim_{x\to\infty}\frac{x^4-3x}{5-2x}=\lim_{x\to\infty}\left(-\frac{1}{2}x^3\right)=-\infty$$

From this illustration, we make the following conclusion:

If the degree of the numerator of a *rational function* is greater than the degree of the denominator, then the function has no limit as $x\to\infty$ and no limit as $x\to-\infty$.

APPLY IT ▷

7. The yearly amount of sales y of a certain company (in thousands of dollars) is related to the amount the company spends on advertising, x (in thousands of dollars), according to the equation $y(x)=\dfrac{500x}{x+20}$. Graph this function on your graphing calculator in the window $[0,1000]\times[0,550]$. Use TRACE to explore $\lim_{x\to\infty}y(x)$, and determine what this means to the company.

EXAMPLE 4 **Limits at Infinity for Rational Functions**

Find the limit (if it exists).

a. $\lim_{x\to\infty}\dfrac{x^2-1}{7-2x+8x^2}$

Solution:
$$\lim_{x\to\infty}\frac{x^2-1}{7-2x+8x^2}=\lim_{x\to\infty}\frac{x^2}{8x^2}=\lim_{x\to\infty}\frac{1}{8}=\frac{1}{8}$$

b. $\lim_{x\to-\infty}\dfrac{x}{(3x-1)^2}$

Solution:
$$\lim_{x\to-\infty}\frac{x}{(3x-1)^2}=\lim_{x\to-\infty}\frac{x}{9x^2-6x+1}=\lim_{x\to-\infty}\frac{x}{9x^2}$$
$$=\lim_{x\to-\infty}\frac{1}{9x}=\frac{1}{9}\cdot\lim_{x\to-\infty}\frac{1}{x}=\frac{1}{9}(0)=0$$

c. $\lim_{x\to\infty}\dfrac{x^5-x^4}{x^4-x^3+2}$

Solution: Since the degree of the numerator is greater than that of the denominator, there is no limit. More precisely,

$$\lim_{x\to\infty}\frac{x^5-x^4}{x^4-x^3+2}=\lim_{x\to\infty}\frac{x^5}{x^4}=\lim_{x\to\infty}x=\infty$$

Now Work Problem 21 ◁

CAUTION!

The preceding technique applies only to limits of rational functions at *infinity*.

To find $\lim_{x\to0}\dfrac{x^2-1}{7-2x+8x^2}$, we cannot simply determine the limit of $\dfrac{x^2}{8x^2}$. That simplification applies only in case $x\to\infty$ or $x\to-\infty$. Instead, we have

$$\lim_{x\to0}\frac{x^2-1}{7-2x+8x^2}=\frac{\lim_{x\to0}x^2-1}{\lim_{x\to0}7-2x+8x^2}=\frac{0-1}{7-0+0}=-\frac{1}{7}$$

Let us now consider the limit of the polynomial function $f(x)=8x^2-2x$ as $x\to\infty$:

$$\lim_{x\to\infty}(8x^2-2x)$$

Because a polynomial is a rational function with denominator 1, we have

$$\lim_{x\to\infty}(8x^2-2x)=\lim_{x\to\infty}\frac{8x^2-2x}{1}=\lim_{x\to\infty}\frac{8x^2}{1}=\lim_{x\to\infty}8x^2$$

That is, the limit of $8x^2-2x$ as $x\to\infty$ is the same as the limit of the term involving the greatest power of x, namely, $8x^2$. As x becomes very large, so does $8x^2$. Thus,

$$\lim_{x\to\infty}(8x^2-2x)=\lim_{x\to\infty}8x^2=\infty$$

In general, we have the following:

As $x \to \infty$ (or $x \to -\infty$), the limit of a *polynomial function* is the same as the limit of its term that involves the greatest power of x.

APPLY IT ▶

8. The cost C of producing x units of a certain product is given by $C(x) = 50{,}000 + 200x + 0.3x^2$. Use your graphing calculator to explore $\lim_{x\to\infty} C(x)$ and determine what this means.

EXAMPLE 5 Limits at Infinity for Polynomial Functions

a. $\lim_{x\to-\infty}(x^3 - x^2 + x - 2) = \lim_{x\to-\infty} x^3$. As x becomes very negative, so does x^3. Thus,

$$\lim_{x\to-\infty}(x^3 - x^2 + x - 2) = \lim_{x\to-\infty} x^3 = -\infty$$

b. $\lim_{x\to-\infty}(-2x^3 + 9x) = \lim_{x\to-\infty} -2x^3 = \infty$, because -2 times a very negative number is very positive.

Now Work Problem 9 ◁

The technique of focusing on dominant terms to find limits as $x \to \infty$ or $x \to -\infty$ is valid for *rational functions*, but it is not necessarily valid for other types of functions. For example, consider

Do not use dominant terms when a function is not rational.

$$\lim_{x\to\infty}\left(\sqrt{x^2 + x} - x\right) \tag{1}$$

Notice that $\sqrt{x^2 + x} - x$ is not a rational function. It is *incorrect* to infer that because x^2 dominates in $x^2 + x$, the limit in (1) is the same as

$$\lim_{x\to\infty}\left(\sqrt{x^2} - x\right) = \lim_{x\to\infty}(x - x) = \lim_{x\to\infty} 0 = 0$$

It can be shown (see Problem 62) that the limit in (1) is not 0, but is $\frac{1}{2}$.

The ideas discussed in this section will now be applied to a case-defined function.

APPLY IT ▶

9. A plumber charges \$100 for the first hour of work at your house and \$75 for every hour (or fraction thereof) afterward. The function for what an x-hour visit will cost you is

$$f(x) = \begin{cases} \$100 & \text{if } 0 < x \le 1 \\ \$175 & \text{if } 1 < x \le 2 \\ \$250 & \text{if } 2 < x \le 3 \\ \$325 & \text{if } 3 < x \le 4 \end{cases}$$

Find $\lim_{x\to1} f(x)$ and $\lim_{x\to2.5} f(x)$.

EXAMPLE 6 Limits for a Case-Defined Function

If $f(x) = \begin{cases} x^2 + 1 & \text{if } x \ge 1 \\ 3 & \text{if } x < 1 \end{cases}$, find the limit (if it exists).

a. $\lim_{x\to1^+} f(x)$

Solution: Here x gets close to 1 from the right. For $x > 1$, we have $f(x) = x^2 + 1$. Thus,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x^2 + 1)$$

If x is greater than 1, but close to 1, then $x^2 + 1$ is close to 2. Therefore,

$$\lim_{x\to1^+} f(x) = \lim_{x\to1^+}(x^2 + 1) = 2$$

b. $\lim_{x\to1^-} f(x)$

Solution: Here x gets close to 1 from the left. For $x < 1$, $f(x) = 3$. Hence,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} 3 = 3$$

c. $\lim_{x\to1} f(x)$

Solution: We want the limit as x approaches 1. However, the rule of the function depends on whether $x \ge 1$ or $x < 1$. Thus, we must consider one-sided limits. The limit as x approaches 1 will exist if and only if both one-sided limits exist and are the same. From parts (a) and (b),

$$\lim_{x\to1^+} f(x) \ne \lim_{x\to1^-} f(x) \qquad \text{since } 2 \ne 3$$

Therefore,

$$\lim_{x\to1} f(x) \qquad \text{does not exist}$$

d. $\lim_{x\to\infty} f(x)$

Solution: For very large values of x, we have $x \geq 1$, so $f(x) = x^2 + 1$. Thus,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (x^2 + 1) = \lim_{x\to\infty} x^2 = \infty$$

e. $\lim_{x\to-\infty} f(x)$

Solution: For very negative values of x, we have $x < 1$, so $f(x) = 3$. Hence,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 3 = 3$$

All the limits in parts (a) through (c) should be obvious from the graph of f in Figure 10.20.

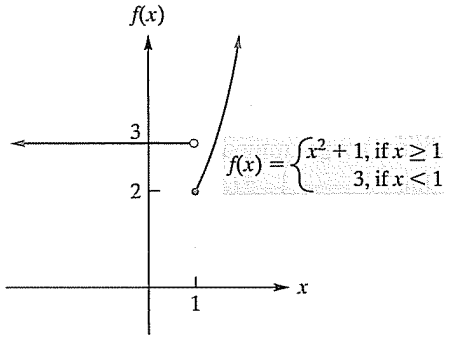


FIGURE 10.20 Graph of case-defined function.

Now Work Problem 57 ◁

PROBLEMS 10.2

1. For the function f given in Figure 10.21, find the following limits. If the limit does not exist, so state that, or use the symbol ∞ or $-\infty$ where appropriate.

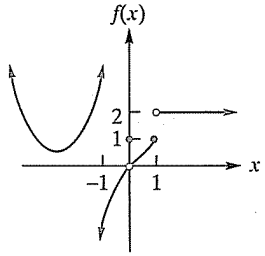


FIGURE 10.21

(a) $\lim_{x\to-\infty} f(x)$ **(b)** $\lim_{x\to-1^-} f(x)$ **(c)** $\lim_{x\to-1^+} f(x)$
(d) $\lim_{x\to-1} f(x)$ **(e)** $\lim_{x\to0^-} f(x)$ **(f)** $\lim_{x\to0^+} f(x)$
(g) $\lim_{x\to0} f(x)$ **(h)** $\lim_{x\to1^-} f(x)$ **(i)** $\lim_{x\to1^+} f(x)$
(j) $\lim_{x\to1} f(x)$ **(k)** $\lim_{x\to\infty} f(x)$

2. For the function f given in Figure 10.22, find the following limits. If the limit does not exist, so state that, or use the symbol ∞ or $-\infty$ where appropriate.

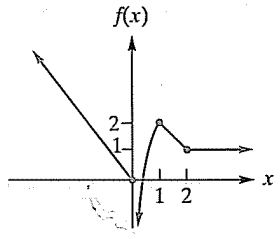


FIGURE 10.22

(a) $\lim_{x\to0^-} f(x)$ **(b)** $\lim_{x\to0^+} f(x)$ **(c)** $\lim_{x\to0} f(x)$
(d) $\lim_{x\to-\infty} f(x)$ **(e)** $\lim_{x\to1} f(x)$ **(f)** $\lim_{x\to\infty} f(x)$
(g) $\lim_{x\to2^+} f(x)$

In each of Problems 3–54, find the limit. If the limit does not exist, so state, or use the symbol ∞ or $-\infty$ where appropriate.

3. $\lim_{x\to3^+} (x-2)$ **4.** $\lim_{x\to-1^+} (1-x^2)$ **5.** $\lim_{x\to-\infty} 5x$

6. $\lim_{x\to-\infty} -6$ **7.** $\lim_{x\to0^-} \frac{6x}{x^4}$ **8.** $\lim_{x\to2} \frac{7}{x-1}$

9. $\lim_{x\to-\infty} x^2$ **10.** $\lim_{t\to\infty} (t-1)^3$ **11.** $\lim_{h\to1^+} \sqrt{h-1}$

12. $\lim_{h\to5^-} \sqrt{5-h}$ **13.** $\lim_{x\to-2^-} \frac{-3}{x+2}$ **14.** $\lim_{x\to0^-} 2^{1/2}$

15. $\lim_{x\to1^+} (4\sqrt{x-1})$ **16.** $\lim_{x\to2^-} (x\sqrt{4-x^2})$ **17.** $\lim_{x\to\infty} \sqrt{x+10}$

18. $\lim_{x\to-\infty} -\sqrt{1-10x}$ **19.** $\lim_{x\to\infty} \frac{3}{\sqrt{x}}$

20. $\lim_{x\to\infty} \frac{-6}{5x\sqrt[3]{x}}$ **21.** $\lim_{x\to\infty} \frac{x-5}{2x+1}$ **22.** $\lim_{x\to\infty} \frac{2x-4}{3-2x}$

23. $\lim_{x\to-\infty} \frac{x^2-1}{x^3+4x-3}$ **24.** $\lim_{r\to\infty} \frac{r^3}{r^2+1}$

25. $\lim_{t\to\infty} \frac{3t^3+2t^2+9t-1}{5t^2-5}$ **26.** $\lim_{x\to\infty} \frac{4x^2}{3x^3-x^2+2}$

27. $\lim_{x\to\infty} \frac{7}{2x+1}$ **28.** $\lim_{x\to-\infty} \frac{2}{(4x-1)^3}$

29. $\lim_{x\to\infty} \frac{3-4x-2x^3}{5x^3-8x+1}$ **30.** $\lim_{x\to-\infty} \frac{3-2x-2x^3}{7-5x^3+2x^2}$

31. $\lim_{x\to 3^+} \frac{x+3}{x^2-9}$ 32. $\lim_{x\to -3^-} \frac{3x}{9-x^2}$ 33. $\lim_{w\to\infty} \frac{2w^2-3w+4}{5w^2+7w-1}$

34. $\lim_{x\to\infty} \frac{4-3x^3}{x^3-1}$ 35. $\lim_{x\to\infty} \frac{6-4x^2+x^3}{4+5x-7x^2}$

36. $\lim_{x\to -\infty} \frac{2x-x^2}{x^2+19x-64}$ 37. $\lim_{x\to -3^-} \frac{5x^2+14x-3}{x^2+3x}$

38. $\lim_{t\to 3} \frac{t^2-4t+3}{t^2-2t-3}$ 39. $\lim_{x\to 1} \frac{x^2-3x+1}{x^2+1}$

40. $\lim_{x\to -1} \frac{3x^3-x^2}{2x+1}$ 41. $\lim_{x\to 2^-} \left(2-\frac{1}{x-2}\right)$

42. $\lim_{x\to -\infty} -\frac{x^5+2x^3-1}{x^5-4x^2}$ 43. $\lim_{x\to -7^-} \frac{x^2+1}{\sqrt{x^2-49}}$

44. $\lim_{x\to -2^+} \frac{x}{\sqrt{16-x^4}}$ 45. $\lim_{x\to 0^+} \frac{5}{x+x^2}$

46. $\lim_{x\to -\infty} \left(x^2+\frac{1}{x}\right)$ 47. $\lim_{x\to 1} x(x-1)^{-1}$ 48. $\lim_{x\to 1/2} \frac{1}{2x-1}$

49. $\lim_{x\to 1^+} \left(\frac{-5}{1-x}\right)$ 50. $\lim_{x\to 3} \left(-\frac{7}{x-3}\right)$ 51. $\lim_{x\to 1} |x-1|$

52. $\lim_{x\to 0} \left|\frac{1}{x}\right|$ 53. $\lim_{x\to -\infty} \frac{x+1}{x}$

54. $\lim_{x\to\infty} \left(\frac{3}{x}-\frac{2x^2}{x^2+1}\right)$

In Problems 55–58, find the indicated limits. If the limit does not exist, so state, or use the symbol ∞ or $-\infty$ where appropriate.

55. $f(x)=\begin{cases}2 & \text{if } x\le 2\\ 1 & \text{if } x>2\end{cases}$

(a) $\lim_{x\to 2^+} f(x)$ (b) $\lim_{x\to 2^-} f(x)$ (c) $\lim_{x\to 2} f(x)$
(d) $\lim_{x\to\infty} f(x)$ (e) $\lim_{x\to -\infty} f(x)$

56. $f(x)=\begin{cases}2-x & \text{if } x\le 3\\ -1+3x-x^2 & \text{if } x>3\end{cases}$

(a) $\lim_{x\to 3^+} f(x)$ (b) $\lim_{x\to 3^-} f(x)$ (c) $\lim_{x\to 3} f(x)$
(d) $\lim_{x\to\infty} f(x)$ (e) $\lim_{x\to -\infty} f(x)$

57. $g(x)=\begin{cases}x & \text{if } x<0\\ -x & \text{if } x>0\end{cases}$

(a) $\lim_{x\to 0^+} g(x)$ (b) $\lim_{x\to 0^-} g(x)$ (c) $\lim_{x\to 0} g(x)$
(d) $\lim_{x\to\infty} g(x)$ (e) $\lim_{x\to -\infty} g(x)$

58. $g(x)=\begin{cases}x^2 & \text{if } x<0\\ -x & \text{if } x>0\end{cases}$

(a) $\lim_{x\to 0^+} g(x)$ (b) $\lim_{x\to 0^-} g(x)$ (c) $\lim_{x\to 0} g(x)$
(d) $\lim_{x\to\infty} g(x)$ (e) $\lim_{x\to -\infty} g(x)$

59. Average Cost If c is the total cost in dollars to produce q units of a product, then the average cost per unit for an output of q units is given by $\bar{c}=c/q$. Thus, if the total cost equation is $c=5000+6q$, then

$$\bar{c}=\frac{5000}{q}+6$$

For example, the total cost of an output of 5 units is \$5030, and the average cost per unit at this level of production is \$1006. By finding $\lim_{q\to\infty} \bar{c}$, show that the average cost approaches a level of stability if the producer continually increases output. What is the limiting value of the average cost? Sketch the graph of the average-cost function.

60. Average Cost Repeat Problem 59, given that the fixed cost is \$12,000 and the variable cost is given by the function $c_v=7q$.

61. Population The population of a certain small city t years from now is predicted to be

$$N=40{,}000-\frac{5000}{t+3}$$

Find the population in the long run; that is, find $\lim_{t\to\infty} N$.

62. Show that

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\frac{1}{2}$$

(*Hint:* Rationalize the numerator by multiplying the expression $\sqrt{x^2+x}-x$ by

$$\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$$

Then express the denominator in a form such that x is a factor.)

63. Host–Parasite Relationship For a particular host–parasite relationship, it was determined that when the host density (number of hosts per unit of area) is x, the number of hosts parasitized over a period of time is

$$y=\frac{900x}{10+45x}$$

If the host density were to increase without bound, what value would y approach?

64. If $f(x)=\begin{cases}\sqrt{2-x} & \text{if } x<2\\ x^3+k(x+1) & \text{if } x\ge 2\end{cases}$, determine the value of the constant k for which $\lim_{x\to 2} f(x)$ exists.

In Problems 65 and 66, use a calculator to evaluate the given function when $x=1$, 0.5, 0.2, 0.1, 0.01, 0.001, and 0.0001. From your results, speculate about $\lim_{x\to 0^+} f(x)$.

65. $f(x)=x^{2x}$ **66.** $f(x)=e^{1/x}$

67. Graph $f(x)=\sqrt{4x^2-1}$. Use the graph to estimate $\lim_{x\to 1/2^+} f(x)$.

68. Graph $f(x)=\frac{\sqrt{x^2-9}}{x+3}$. Use the graph to estimate $\lim_{x\to -3^-} f(x)$ if it exists. Use the symbol ∞ or $-\infty$ if appropriate.

69. Graph $f(x)=\begin{cases}2x^2+3 & \text{if } x<2\\ 2x+5 & \text{if } x\ge 2\end{cases}$. Use the graph to estimate each of the following limits if it exists:

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$ (c) $\lim_{x\to 2} f(x)$

10.3 Continuity

Objective

To study continuity and to find points of discontinuity for a function.

Many functions have the property that there is no "break" in their graphs. For example, compare the functions

$$f(x)=x \quad \text{and} \quad g(x)=\begin{cases}x & \text{if } x\ne 1\\ 2 & \text{if } x=1\end{cases}$$

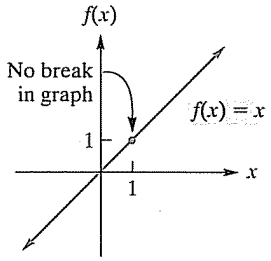


FIGURE 10.23 Continuous at 1.

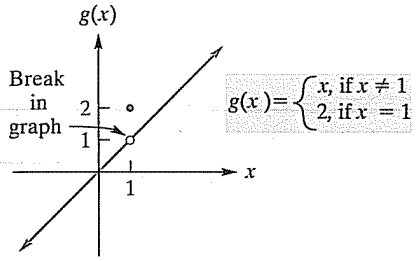


FIGURE 10.24 Discontinuous at 1.

whose graphs appear in Figures 10.23 and 10.24, respectively. The graph of f is unbroken, but the graph of g has a break at $x = 1$. Stated another way, if you were to trace both graphs with a pencil, you would have to lift the pencil off the graph of g when $x = 1$, but you would not have to lift it off the graph of f. These situations can be expressed by limits. As x approaches 1, compare the limit of each function with the value of the function at $x = 1$:

$$\lim_{x\to 1} f(x) = 1 = f(1)$$

whereas

$$\lim_{x\to 1} g(x) = 1 \neq 2 = g(1)$$

In Section 10.1 we stressed that given a function f and a number a, there are two important ways to associate a number to the pair (f, a). One is simple evaluation, $f(a)$, which *exists* precisely if a is in the domain of f. The other is $\lim_{x\to a} f(x)$, whose existence and determination can be more challenging. For the functions f and g above, the limit of f as $x \to 1$ is the same as $f(1)$, but the limit of g as $x \to 1$ is *not* the same as $g(1)$. For these reasons, we say that f is *continuous* at 1 and g is *discontinuous* at 1.

Definition

A function f is **continuous** at a if and only if the following three conditions are met:

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

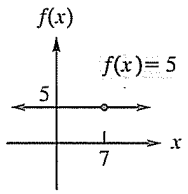


FIGURE 10.25 f is continuous at 7.

If f is not continuous at a, then f is said to be **discontinuous** at a, and a is called a **point of discontinuity** of f.

EXAMPLE 1 Applying the Definition of Continuity

a. Show that $f(x) = 5$ is continuous at 7.

Solution: We must verify that the preceding three conditions are met. First, $f(7) = 5$, so f is defined at $x = 7$. Second,

$$\lim_{x\to 7} f(x) = \lim_{x\to 7} 5 = 5$$

Thus, f has a limit as $x \to 7$. Third,

$$\lim_{x\to 7} f(x) = 5 = f(7)$$

Therefore, f is continuous at 7. (See Figure 10.25.)

b. Show that $g(x) = x^2 - 3$ is continuous at -4.

Solution: The function g is defined at $x = -4$: $g(-4) = 13$. Also,

$$\lim_{x\to -4} g(x) = \lim_{x\to -4} (x^2 - 3) = 13 = g(-4)$$

Therefore, g is continuous at -4. (See Figure 10.26.)

Now Work Problem 1 ◁

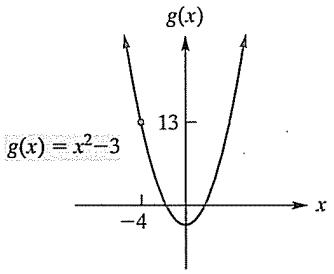


FIGURE 10.26 g is continuous at -4.

We say that a function is *continuous on an interval* if it is continuous at each point there. In this situation, the graph of the function is connected over the interval. For example, $f(x) = x^2$ is continuous on the interval [2, 5]. In fact, in Example 5 of Section 10.1, we showed that, for *any* polynomial function f, for any number a, $\lim_{x\to a} f(x) = f(a)$. This means that

A polynomial function is continuous at every point.

It follows that such a function is continuous on every interval. We say that a function is **continuous on its domain** if it is continuous at each point in its domain. If the domain of such a function is the set of all real numbers, we may simply say that the function is continuous.

EXAMPLE 2 **Continuity of Polynomial Functions**

The functions $f(x) = 7$ and $g(x) = x^2 - 9x + 3$ are polynomial functions. Therefore, they are continuous on their domains. For example, they are continuous at 3.

Now Work Problem 13 ◁

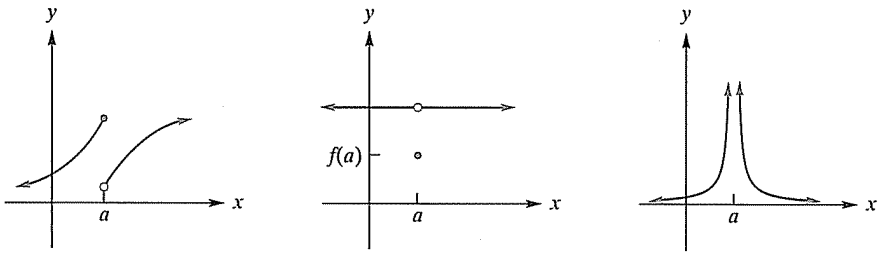
When is a function discontinuous? We can say that a function f defined on an open interval containing a is discontinuous at a if

1. f has no limit as $x \to a$

or

2. as $x \to a$, f has a limit that is different from $f(a)$

If f is not defined at a, we will say also, in that case, that f is discontinuous at a. In Figure 10.27, we can find points of discontinuity by inspection.



Defined at a but no limit as $x \to a$

Defined at a and limit as $x \to a$ exists, but limit is not $f(a)$

Not defined at a but defined for all nearby values of a

FIGURE 10.27 Discontinuities at a.

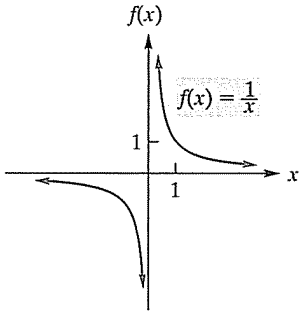


FIGURE 10.28 Infinite discontinuity at 0.

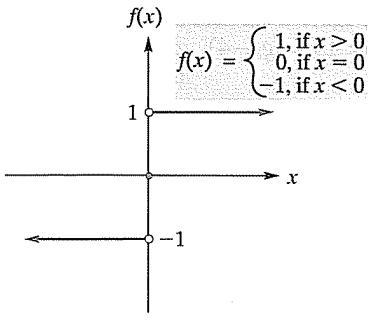


FIGURE 10.29 Discontinuous case-defined function.

EXAMPLE 3 **Discontinuities**

a. Let $f(x) = 1/x$. (See Figure 10.28.) Note that f is not defined at $x = 0$, but it is defined for all other x nearby. Thus, f is discontinuous at 0. Moreover, $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = -\infty$. A function is said to have an **infinite discontinuity** at a when at least one of the one-sided limits is either ∞ or $-\infty$ as $x \to a$. Hence, f has an *infinite discontinuity* at $x = 0$.

b. Let $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

(See Figure 10.29.) Although f is defined at $x = 0$, $\lim_{x \to 0} f(x)$ does not exist. Thus, f is discontinuous at 0.

Now Work Problem 29 ◁

The following property indicates where the discontinuities of a rational function occur:

Discontinuities of a Rational Function

A rational function is discontinuous at points where the denominator is 0 and is continuous otherwise. Thus, a rational function is continuous on its domain.

EXAMPLE 4 **Locating Discontinuities in Rational Functions**

The rational function $f(x) = \dfrac{x+1}{x+1}$ is continuous on its domain but it is not defined at -1. It is discontinuous at -1. The graph of f is a horizontal straight line with a "hole" in it at -1.

For each of the following functions, find all points of discontinuity.

a. $f(x) = \dfrac{x^2 - 3}{x^2 + 2x - 8}$

Solution: This rational function has denominator

$$x^2 + 2x - 8 = (x+4)(x-2)$$

which is 0 when $x = -4$ or $x = 2$. Thus, f is discontinuous only at -4 and 2.

b. $h(x) = \dfrac{x+4}{x^2+4}$

Solution: For this rational function, the denominator is never 0. (It is always positive.) Therefore, h has no discontinuity.

Now Work Problem 19 ◁

EXAMPLE 5 **Locating Discontinuities in Case-Defined Functions**

For each of the following functions, find all points of discontinuity.

a. $f(x) = \begin{cases} x+6 & \text{if } x \geq 3 \\ x^2 & \text{if } x < 3 \end{cases}$

Solution: The cases defining the function are given by polynomials, which are continuous, so the only possible place for a discontinuity is at $x = 3$, where the separation of cases occurs. We know that $f(3) = 3 + 6 = 9$. So because

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+6) = 9$$

and

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x^2 = 9$$

we can conclude that $\lim_{x \to 3} f(x) = 9 = f(3)$ and the function has no points of discontinuity. We can reach the same conclusion by inspecting the graph of f in Figure 10.30.

b. $f(x) = \begin{cases} x+2 & \text{if } x > 2 \\ x^2 & \text{if } x < 2 \end{cases}$

Solution: Since f is not defined at $x = 2$, it is discontinuous at 2. Note, however, that

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4 = \lim_{x \to 2^+} x + 2 = \lim_{x \to 2^+} f(x)$$

shows that $\lim_{x \to 2} f(x)$ exists. (See Figure 10.31.)

Now Work Problem 31 ◁

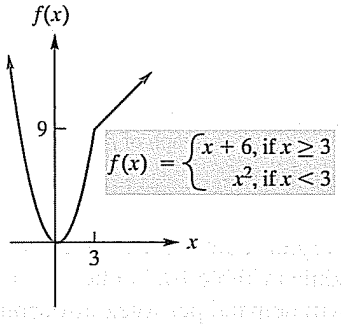


FIGURE 10.30 Continuous case-defined function.

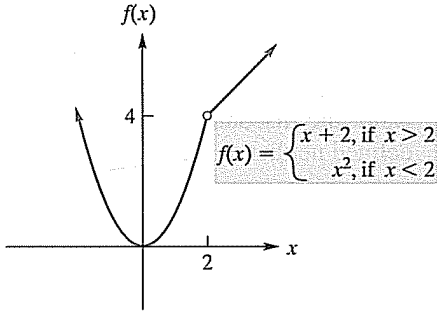


FIGURE 10.31 Discontinuous at 2.

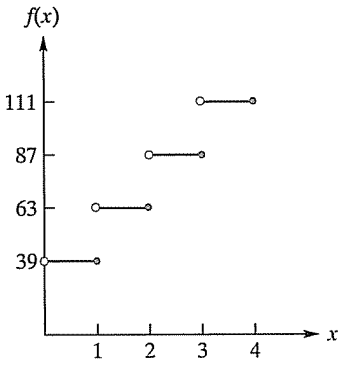


FIGURE 10.32 Post-office function.

EXAMPLE 6 Post-Office Function

The post-office function

$$c = f(x) = \begin{cases} 39 & \text{if } 0 < x \leq 1 \\ 63 & \text{if } 1 < x \leq 2 \\ 87 & \text{if } 2 < x \leq 3 \\ 111 & \text{if } 3 < x \leq 4 \end{cases}$$

gives the cost c (in cents) of mailing, first class, an item of weight x (ounces), for $0 < x \leq 4$, in July 2006. It is clear from its graph in Figure 10.32 that f has discontinuities at 1, 2, and 3 and is constant for values of x between successive discontinuities. Such a function is called a *step function* because of the appearance of its graph.

Now Work Problem 35 ◁

There is another way to express continuity besides that given in the definition. If we take the statement

$$\lim_{x \to a} f(x) = f(a)$$

and replace x by $a + h$, then as $x \to a$, we have $h \to 0$; and as $h \to 0$ we have $x \to a$. It follows that $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$, provided the limits exist (Figure 10.33). Thus, the statement

$$\lim_{h \to 0} f(a + h) = f(a)$$

assuming both sides exist, also defines continuity at a.

This method of expressing continuity at a is used frequently in mathematical proofs.

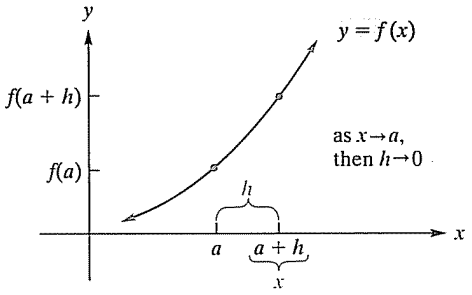


FIGURE 10.33 Diagram for continuity at a.

TECHNOLOGY

By observing the graph of a function, we may be able to determine where a discontinuity occurs. However, we can be fooled. For example, the function

$$f(x) = \frac{x - 1}{x^2 - 1}$$

is discontinuous at ± 1, but the discontinuity at 1 is not obvious from the graph of f in Figure 10.34. On the other hand, the discontinuity at -1 is obvious. Note that f is defined neither at -1 nor at 1.

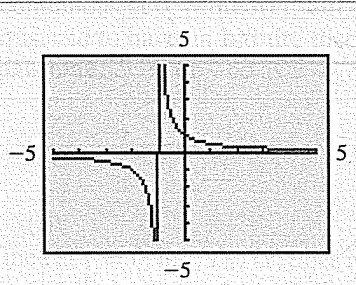


FIGURE 10.34 Discontinuity at 1 is not apparent from graph of $f(x) = \dfrac{x - 1}{x^2 - 1}$.

Often, it is helpful to describe a situation by a continuous function. For example, the demand schedule in Table 10.3 indicates the number of units of a particular product that consumers will demand per week at various prices. This information can be given graphically, as in Figure 10.35(a), by plotting each quantity–price pair as a point.

Table 10.3 Demand Schedule

Price/Unit, p	Quantity/Week, q
\$20	0
10	5
5	15
4	20
2	45
1	95

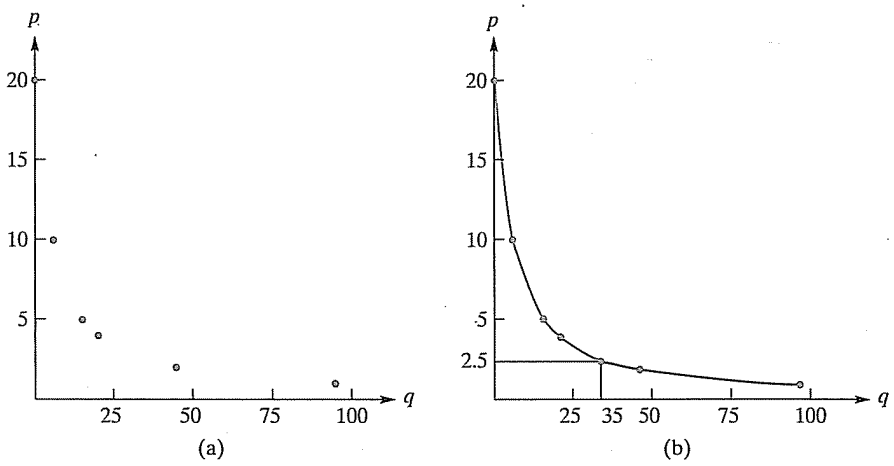


FIGURE 10.35 Viewing data via a continuous function.

Clearly, the graph does not represent a continuous function. Furthermore, it gives us no information as to the price at which, say, 35 units would be demanded. However, if we connect the points in Figure 10.35(a) by a smooth curve [see Figure 10.35(b)], we get a so-called demand curve. From it, we could guess that at about \$2.50 per unit, 35 units would be demanded.

Frequently, it is possible and useful to describe a graph, as in Figure 10.35(b), by means of an equation that defines a continuous function f. Such a function not only gives us a demand equation, $p = f(q)$, which allows us to anticipate corresponding prices and quantities demanded, but also permits a convenient mathematical analysis of the nature and basic properties of demand. Of course, some care must be used in working with equations such as $p = f(q)$. Mathematically, f may be defined when $q = \sqrt{37}$, but from a practical standpoint, a demand of $\sqrt{37}$ units could be meaningless to our particular situation. For example, if a unit is an egg, then a demand of $\sqrt{37}$ eggs make no sense.

We remark that functions of the form $f(x) = x^a$, for fixed a, are continuous on their domains. In particular, (square) root functions are continuous. Also, exponential functions and logarithmic functions are continuous on their domains. Thus, exponential functions have no discontinuities while a logarithmic function has only a discontinuity at 0 (which is an infinite discontinuity). Many more examples of continuous functions are provided by the observation that if f and g are continuous on their domains, then the composite function $f \circ g$, given by $f \circ g(x) = f(g(x))$ is continuous on its domain. For example, the function

$$f(x) = \sqrt{\ln\left(\frac{x^2+1}{x-1}\right)}$$

is continuous on its domain. Determining the domain of such a function may, of course, be fairly involved.

PROBLEMS 10.3

In Problems 1–6, use the definition of continuity to show that the given function is continuous at the indicated point.

1. $f(x) = x^3 - 5x; x = 2$
2. $f(x) = \dfrac{x-3}{5x}; x = -3$
3. $g(x) = \sqrt{2-3x}; x = 0$
4. $f(x) = \dfrac{x}{8}; x = 2$
5. $h(x) = \dfrac{x+3}{x-3}; x = -3$
6. $f(x) = \sqrt[3]{x}; x = -1$

In Problems 7–12, determine whether the function is continuous at the given points.

7. $f(x) = \dfrac{x+4}{x-2}; -2, 0$
8. $f(x) = \dfrac{x^2-4x+4}{6}; 2, -2$
9. $g(x) = \dfrac{x-3}{x^2-9}; 3, -3$
10. $h(x) = \dfrac{3}{x^2+9}; 3, -3$
11. $f(x) = \begin{cases} x+2 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}; 2, 0$
12. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}; 0, -1$

In Problems 13–16, give a reason why the function is continuous on its domain.

13. $f(x) = 2x^2 - 3$
14. $f(x) = \dfrac{2+3x-x^2}{5}$

15. $f(x) = \ln(\sqrt[3]{x})$ **16.** $f(x) = x(1-x)$

In Problems 17–34, find all points of discontinuity.

17. $f(x) = 3x^2 - 3$ **18.** $h(x) = x - 2$

19. $f(x) = \dfrac{3}{x+4}$ **20.** $f(x) = \dfrac{x^2+5x-2}{x^2-9}$

21. $g(x) = \dfrac{(2x^2-3)^3}{15}$ **22.** $f(x) = -1$

23. $f(x) = \dfrac{x^2+6x+9}{x^2+2x-15}$ **24.** $g(x) = \dfrac{x-3}{x^2+x}$

25. $h(x) = \dfrac{x-3}{x^3-9x}$ **26.** $f(x) = \dfrac{2x-3}{3-2x}$

27. $p(x) = \dfrac{x}{x^2+1}$ **28.** $f(x) = \dfrac{x^4}{x^4-1}$

29. $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$ **30.** $f(x) = \begin{cases} 3x+5 & \text{if } x \geq -2 \\ 2 & \text{if } x < -2 \end{cases}$

31. $f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$ **32.** $f(x) = \begin{cases} x-3 & \text{if } x > 2 \\ 3-2x & \text{if } x < 2 \end{cases}$

33. $f(x) = \begin{cases} x^2+1 & \text{if } x > 2 \\ 8x & \text{if } x < 2 \end{cases}$ **34.** $f(x) = \begin{cases} \dfrac{16}{x^2} & \text{if } x \geq 2 \\ 3x-2 & \text{if } x < 2 \end{cases}$

35. Telephone Rates Suppose the long-distance rate for a telephone call from Hazleton, Pennsylvania to Los Angeles, California, is \$0.08 for the first minute or fraction thereof and \$0.04 for each additional minute or fraction thereof. If $y = f(t)$ is a function that indicates the total charge y for a call of t minutes duration, sketch the graph of f for $0 < t \leq 3\frac{1}{2}$. Use your graph to determine the values of t, where $0 < t \leq 3\frac{1}{2}$, at which discontinuities occur.

36. The *greatest integer function*, $f(x) = \lfloor x \rfloor$, is defined to be the greatest integer less than or equal to x, where x is any real number. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 1.999 \rfloor = 1$, $\lfloor \frac{1}{4} \rfloor = 0$, and $\lfloor -4.5 \rfloor = -5$. Sketch the graph of this function for $-3.5 \leq x \leq 3.5$. Use your sketch to determine the values of x at which discontinuities occur.

37. Inventory Sketch the graph of

$$y = f(x) = \begin{cases} -100x + 600 & \text{if } 0 \leq x < 5 \\ -100x + 1100 & \text{if } 5 \leq x < 10 \\ -100x + 1600 & \text{if } 10 \leq x < 15 \end{cases}$$

A function such as this might describe the inventory y of a company at time x. Is f continuous at 2? At 5? At 10?

38. Graph $g(x) = e^{-1/x^2}$. Because g is not defined at $x = 0$, g is discontinuous at 0. Based on the graph of g, is

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

continuous at 0?

Objective

To develop techniques for solving nonlinear inequalities.

10.4 Continuity Applied to Inequalities

In Section 1.2, we solved linear inequalities. We now turn our attention to showing how the notion of continuity can be applied to solving a nonlinear inequality such as $x^2 + 3x - 4 < 0$. The ability to do this will be important in our study of calculus.

Recall (from Section 2.5) that the x-intercepts of the graph of a function g are precisely the roots of the equation $g(x) = 0$. Hence, from the graph of $y = g(x)$ in Figure 10.36, we conclude that r_1, r_2, and r_3 are roots of $g(x) = 0$ and any other roots will give rise to x-intercepts (beyond what is actually shown of the graph). Assume that in fact all the roots of $g(x) = 0$, and hence all the x-intercepts, are shown. Note further from Figure 10.36 that the three roots determine four open intervals on the x-axis:

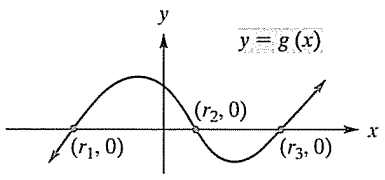


FIGURE 10.36 r_1, r_2, and r_3 are roots of $g(x) = 0$.

$$(-\infty, r_1) \quad (r_1, r_2) \quad (r_2, r_3) \quad (r_3, \infty)$$

To solve $x^2 + 3x - 4 > 0$, we let

$$f(x) = x^2 + 3x - 4 = (x+4)(x-1)$$

Because f is a polynomial function, it is continuous. The roots of $f(x) = 0$ are -4 and 1; hence, the graph of f has x-intercepts $(-4, 0)$ and $(1, 0)$. (See Figure 10.37.) The roots determine three intervals on the x-axis:

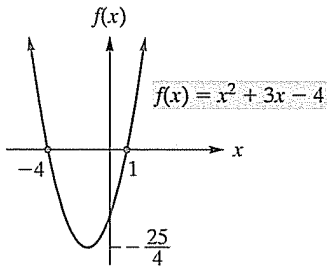


FIGURE 10.37 -4 and 1 are roots of $f(x) = 0$.

$$(-\infty, -4) \quad (-4, 1) \quad (1, \infty)$$

Consider the interval $(-\infty, -4)$. Since f is continuous on this interval, we claim that either $f(x) > 0$ or $f(x) < 0$ *throughout* the interval. If this were not the case, then $f(x)$ would indeed change sign on the interval. By the continuity of f, there would be a point where the graph intersects the x-axis—for example, at $(x_0, 0)$. (See Figure 10.38.) But then x_0 would be a root of $f(x) = 0$. However, this cannot be, because there is no root less than -4. Hence, $f(x)$ must be strictly positive or strictly negative on $(-\infty, -4)$. A similar argument can be made for each of the other intervals.

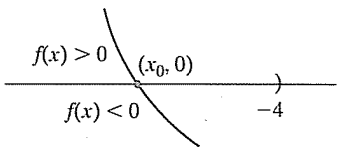


FIGURE 10.38 Change of sign for a continuous function.

To determine the sign of $f(x)$ on any one of the three intervals, it suffices to determine its sign at *any* point in the interval. For instance, -5 is in $(-\infty, -4)$ and

$$f(-5) = 6 > 0 \qquad \text{Thus, } f(x) > 0 \text{ on } (-\infty, -4)$$

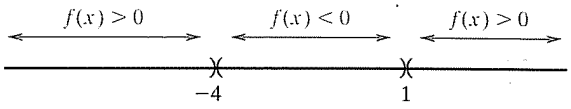


FIGURE 10.39 Simple sign chart for $x^2 + 3x - 4$.

Similarly, 0 is in $(-4, 1)$, and

$$f(0) = -4 < 0 \qquad \text{Thus, } f(x) < 0 \text{ on } (-4, 1)$$

Finally, 3 is in $(1, \infty)$, and

$$f(3) = 14 > 0 \qquad \text{Thus, } f(x) > 0 \text{ on } (1, \infty)$$

(See the "sign chart" in Figure 10.39.) Therefore,

$$x^2 + 3x - 4 > 0 \quad \text{on} \quad (-\infty, -4) \text{ and } (1, \infty)$$

so we have solved the inequality. These results are obvious from the graph in Figure 10.37. The graph lies above the x-axis, meaning that $f(x) > 0$, on $(-\infty, -4)$ and on $(1, \infty)$.

In more complicated examples it will be useful to exploit the multiplicative nature of signs. We noted that $f(x) = x^2 + 3x - 4 = (x + 4)(x - 1)$. Each of $x + 4$ and $x - 1$ has a sign chart that is simpler than that of $x^2 + 3x - 4$. Consider the "sign chart" in Figure 10.40. As before, we placed the roots of $f(x) = 0$ in ascending order, from left to right, so as to subdivide $(-\infty, \infty)$ into three open intervals. This forms the top line of the box. Directly below the top line we determined the signs of $x + 4$ on the three subintervals. We know that for the linear function $x + 4$ there is exactly one root of the equation $x + 4 = 0$, namely -4. We placed a 0 at -4 in the row labeled $x + 4$. By the argument illustrated in Figure 10.38, it follows that the sign of the function $x + 4$ is constant on $(-\infty, -4)$ and on $(-4, \infty)$ and two evaluations of $x + 4$ settle the distribution of signs for $x + 4$. From $(-5) + 4 = -1 < 0$, we have $x + 4$ *negative* on $(-\infty, -4)$, so we entered a $-$ sign in the $(-\infty, -4)$ space of the $x + 4$ row. From $(0) + 4 = 4 > 0$, we have $x + 4$ *positive* on $(-4, \infty)$. Since $(-4, \infty)$ has been further subdivided at 1, we entered a $+$ sign in each of the $(-4, 1)$ and $(1, \infty)$ spaces of the $x + 4$ row. In a similar way we constructed the row labeled $x - 1$.

	$-\infty$		-4		1		∞
$x + 4$	$-$		0	$+$		$+$	
$x - 1$	$-$			$-$	0	$+$	
$f(x)$	$+$		0	$-$	0	$+$	

FIGURE 10.40 Sign chart for $x^2 + 3x - 4$.

Now the bottom row is obtained by taking, for each component, the product of the entries above. Thus we have $(x + 4)(x - 1) = f(x)$, $(-)(-) = +$, 0(any number) $= 0$, $(+)(-) = -$, (any number)0 $= 0$, and $(+)(+) = +$. Sign charts of this kind are useful whenever a continuous function can be expressed as a product of several simpler, continuous functions, each of which has a simple sign chart. In Chapter 13 we will rely heavily on such sign charts.

EXAMPLE 1 Solving a Quadratic Inequality

Solve $x^2 - 3x - 10 > 0$.

Solution: If $f(x) = x^2 - 3x - 10$, then f is a polynomial (quadratic) function and thus is continuous everywhere. To find the real roots of $f(x) = 0$, we have

$$x^2 - 3x - 10 = 0$$
$$(x + 2)(x - 5) = 0$$
$$x = -2, 5$$

	$-\infty$		-2		5		∞
$x+2$	$-$	0	$+$		$+$		
$x-5$	$-$		$-$	0	$+$		
$f(x)$	$+$	0	$-$	0	$+$		

FIGURE 10.41 Sign chart for $x^2 - 3x - 10$.

The roots -2 and 5 determine three intervals:

$$(-\infty, -2) \quad (-2, 5) \quad (5, \infty)$$

In the manner of the last example, we construct the sign chart in Figure 10.41. We see that $x^2 - 3x - 10 > 0$ on $(-\infty, -2) \cup (5, \infty)$.

Now Work Problem 1 ◁

APPLY IT ▷

10. An open box is formed by cutting a square piece out of each corner of an 8-inch-by-10-inch piece of metal. If each side of the cut-out squares is x inches long, the volume of the box is given by $V(x) = x(8 - 2x)(10 - 2x)$. This problem makes sense only when this volume is positive. Find the values of x for which the volume is positive.

EXAMPLE 2 **Solving a Polynomial Inequality**

Solve $x(x - 1)(x + 4) \leq 0$.

Solution: If $f(x) = x(x - 1)(x + 4)$, then f is a polynomial function and hence continuous everywhere. The roots of $f(x) = 0$ are (in ascending order) -4, 0, and 1 and lead to the sign chart in Figure 10.42.

	$-\infty$		-4		0		1		∞
x	$-$		$-$	0	$+$		$+$		
$x-1$	$-$		$-$		$-$	0	$+$		
$x+1$	$-$	0	$+$		$+$		$+$		
$f(x)$	$-$	0	$+$	0	$-$	0	$+$		

FIGURE 10.42 Sign chart for $x(x - 1)(x + 4)$.

From the sign chart, noting the endpoints required, $x(x - 1)(x + 4) \leq 0$ on $(-\infty, -4] \cup [0, 1]$.

Now Work Problem 11 ◁

The sign charts we have described are certainly not limited to solving polynomial inequalities. The reader will have noticed that we used thicker vertical lines at the endpoints, $-\infty$ and ∞, of the chart. These symbols do not denote real numbers, let alone points in the domain of a function. We extend the thick vertical line convention to single out isolated real numbers that are not in the domain of the function in question. The next example will illustrate.

EXAMPLE 3 **Solving a Rational Function Inequality**

Solve $\dfrac{x^2 - 6x + 5}{x} \geq 0$.

Solution: Let

$$f(x) = \frac{x^2 - 6x + 5}{x} = \frac{(x - 1)(x - 5)}{x}$$

For a rational function $f = g/h$, we solve the inequality by considering the intervals determined by both the roots of $g(x) = 0$ and the roots of $h(x) = 0$. Observe that the roots of $g(x) = 0$ are the roots of $f(x) = 0$ because the only way for a fraction to be 0 is for its numerator to be 0. On the other hand, the roots of $h(x) = 0$ are precisely

	$-\infty$		0		1		5		∞
$x-1$		$-$		$-$	0	$+$		$+$	
$x-5$		$-$		$-$		$-$	0	$+$	
$1/x$		$-$	$\times$	$+$		$+$		$+$	
$f(x)$		$-$	$\times$	$+$	0	$-$	0	$+$	

FIGURE 10.43 Sign chart for $\dfrac{(x-1)(x-5)}{x}$.

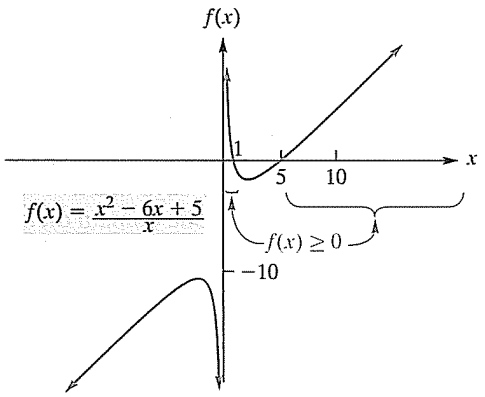


FIGURE 10.44 Graph of $f(x) = \dfrac{x^2-6x+5}{x}$.

the points at which f is not defined and these are also precisely the points at which f is discontinuous. The sign of f may change at a root and it may change at a discontinuity. Here the roots of the numerator are 1 and 5 and the root of the denominator is 0. In ascending order these give us 0, 1, and 5, which determine the open intervals

$$(-\infty, 0) \quad (0, 1) \quad (1, 5) \quad (5, \infty)$$

These, together with the observation that $1/x$ is a *factor* of f, lead to the sign chart in Figure 10.43.

Here the first two rows of the sign chart are constructed as before. In the third row we have placed a $\times$ sign at 0 to indicate that the factor $1/x$ is not defined at 0. The bottom row, as before, is constructed by taking the products of the entries above. Observe that a product is not defined at any point at which any of its factors is not defined. Hence we also have a $\times$ entry at 0 in the bottom row.

From the bottom row of the sign chart we can read that the solution of $\frac{(x-1)(x-5)}{x} \geq 0$ is $(0, 1] \cup [5, \infty]$. Observe that 1 and 5 are in the solution and 0 is not.

In Figure 10.44 we have graphed $f(x) = \frac{x^2-6x+5}{x}$, and we can confirm visually that the solution of the inequality $f(x) \geq 0$ is precisely the set of all real numbers at which the graph lies on or above the x-axis.

Now Work Problem 17 ◁

A sign chart is not always necessary, as the following example shows.

EXAMPLE 4 **Solving Nonlinear Inequalities**

a. Solve $x^2 + 1 > 0$.

Solution: The equation $x^2 + 1 = 0$ has no real roots. Thus, the continuous function $f(x) = x^2 + 1$ has no x-intercepts. It follows that either $f(x)$ is always positive or $f(x)$ is always negative. But x^2 is always positive or zero, so $x^2 + 1$ is always positive. Hence, the solution of $x^2 + 1 > 0$ is $(-\infty, \infty)$.

b. Solve $x^2 + 1 < 0$.

Solution: From part (a), $x^2 + 1$ is always positive, so $x^2 + 1 < 0$ has no solution, meaning that the set of solutions is $\emptyset$, the empty set.

Now Work Problem 7 ◁

We conclude with a nonrational example. The importance of the function introduced will become clear in later chapters.

EXAMPLE 5 **Solving a Nonrational Function Inequality**

Solve $x \ln x - x \geq 0$.

Solution: Let $f(x) = x \ln x - x = x(\ln x - 1)$, which, being a product of continuous functions, is continuous. From the *factored* form for f we see that the roots of $f(x) = 0$ are 0 and the roots of $\ln x - 1 = 0$. The latter is equivalent to $\ln x = 1$, which is equivalent to $e^{\ln x} = e^1$, since the exponential function is one-to-one. However, the last equality says that $x = e$. The domain of f is $(0, \infty)$ because $\ln x$ is only defined for $x > 0$. The domain dictates the top line of our sign chart in Figure 10.45.

	0		e		∞
x		+		+	
$\ln x - 1$		−	0	+	
$f(x)$		−	0	+	

FIGURE 10.45 Sign chart for $x \ln x - x$.

The first row of Figure 10.45 is straightforward. For the second row, we placed a 0 at e, the only root of $\ln x - 1 = 0$. By continuity of $\ln x - 1$, the sign of $\ln x - 1$ on $(0, e)$ and on (e, ∞) can be determined by suitable evaluations. For the first we evaluate at 1 in $(0, e)$ and get $\ln 1 - 1 = 0 - 1 = -1 < 0$. For the second we evaluate at e^2 in (e, ∞) and get $\ln e^2 - 1 = 2 - 1 = 1 > 0$. The bottom row is, as usual, determined by multiplying the others. From the bottom row of Figure 10.45 the solution of $x \ln x - x \geq 0$ is evidently $[e, \infty)$.

Now Work Problem 35 ◁

PROBLEMS 10.4

In Problems 1–26, solve the inequalities by the technique discussed in this section.

1. $x^2 - 3x - 4 > 0$
2. $x^2 - 8x + 15 > 0$
3. $x^2 - 3x - 10 \leq 0$
4. $15 - 2x - x^2 \geq 0$
5. $2x^2 + 11x + 14 < 0$
6. $x^2 - 4 < 0$
7. $x^2 + 4 < 0$
8. $2x^2 - x - 2 \leq 0$
9. $(x + 1)(x - 2)(x + 7) \leq 0$
10. $(x + 5)(x + 2)(x - 7) \leq 0$
11. $-x(x - 5)(x + 4) > 0$
12. $(x + 2)^2 > 0$
13. $x^3 + 4x \geq 0$
14. $(x + 3)^2(x^2 - 4) < 0$
15. $x^3 + 8x^2 + 15x \leq 0$
16. $x^3 + 6x^2 + 9x < 0$
17. $\dfrac{x}{x^2 - 9} < 0$
18. $\dfrac{x^2 - 1}{x} < 0$
19. $\dfrac{3}{x + 1} \geq 0$
20. $\dfrac{3}{x^2 - 5x + 6} > 0$
21. $\dfrac{x^2 - x - 6}{x^2 + 4x - 5} \geq 0$
22. $\dfrac{x^2 + 4x - 5}{x^2 + 3x + 2} \leq 0$
23. $\dfrac{3}{x^2 + 6x + 5} \leq 0$
24. $\dfrac{3x + 2}{(x - 1)^2} \leq 0$
25. $x^2 + 2x \geq 2$
26. $x^4 - 16 \geq 0$

27. Revenue Suppose that consumers will purchase q units of a product when the price of *each* unit is $28 - 0.2q$ dollars. How many units must be sold for the sales revenue to be at least \$750?

28. Forest Management A lumber company owns a forest that is of rectangular shape, 1 mi × 2 mi. The company wants to cut a uniform strip of trees along the outer edges of the forest. At most, how wide can the strip be if the company wants at least $1\frac{5}{16}$ mi^2 of forest to remain?

29. Container Design A container manufacturer wishes to make an open box by cutting a 3-in.-by-3-in. square from each corner of a square sheet of aluminum and then turning up the sides. The box is to contain at least 192 cubic inches. Find the dimensions of the smallest square sheet of aluminum that can be used.

30. Workshop Participation Imperial Education Services (I.E.S.) is offering a workshop in data processing to key personnel at Zeta Corporation. The price per person is \$50, and Zeta Corporation guarantees that at least 50 people will attend. Suppose I.E.S. offers to reduce the charge for *everybody* by \$0.50 for each person over the 50 who attends. How should I.E.S. limit the size of the group so that the total revenue it receives will never be less than that received for 50 persons?

31. Graph $f(x) = x^3 + 7x^2 - 5x + 4$. Use the graph to determine the solution of

$$x^3 + 7x^2 - 5x + 4 \leq 0$$

32. Graph $f(x) = \dfrac{3x^2 - 0.5x + 2}{6.2 - 4.1x}$. Use the graph to determine the solution of

$$\frac{3x^2 - 0.5x + 2}{6.2 - 4.1x} > 0$$

A novel way of solving a nonlinear inequality like $f(x) > 0$ is by examining the graph of $g(x) = f(x)/|f(x)|$, whose range consists only of 1 and -1:

$$g(x) = \frac{f(x)}{|f(x)|} = \begin{cases} 1 & \text{if } f(x) > 0 \\ -1 & \text{if } f(x) < 0 \end{cases}$$

The solution of $f(x) > 0$ consists of all intervals for which $g(x) = 1$. Using this technique, solve the inequalities in Problems 33 and 34.

33. $6x^2 - x - 2 > 0$

34. $\dfrac{x^2 + x - 1}{x^2 + x - 6} < 0$

35. Graph $x \ln x - x$. Does the function appear to be continuous? Does the graph support the conclusions of Example 5? At what value does the function appear to have a minimum value?

36. Graph e^{-x^2}. Does the function appear to be continuous? Can the conclusion be confirmed by invoking facts about continuous functions? At what value does the function appear to have a maximum value?

Chapter 10 Review

Important Terms and Symbols

Examples

Section 10.1 Limits

$\lim_{x \to a} f(x) = L$ — Ex. 8, p. 466

Section 10.2 Limits (Continued)

$\lim_{x \to a^-} f(x) = L$ $\quad \lim_{x \to a^+} f(x) = L$ $\quad \lim_{x \to a} f(x) = \infty$ — Ex. 1, p. 470

$\lim_{x \to \infty} f(x) = L$ $\quad \lim_{x \to -\infty} f(x) = L$ — Ex. 3, p. 471

Section 10.3 Continuity

continuous at a $\quad$ discontinuous at a — Ex. 3, p. 478

continuous on an interval $\quad$ continuous on its domain — Ex. 4, p. 479

Section 10.4 Continuity Applied to Inequalities

sign chart — Ex. 1, p. 483

Summary

The notion of limit is fundamental for calculus. To say that $\lim_{x \to a} f(x) = L$ means that the values of $f(x)$ can be made as close to the number L as we like by taking x sufficiently close to, but different from, a. If $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and c is a constant, then

1. $\lim_{x \to a} c = c$
2. $\lim_{x \to a} x^n = a^n$
3. $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} [cf(x)] = c \cdot \lim_{x \to a} f(x)$
6. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$,
7. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
8. If f is a polynomial function, then $\lim_{x \to a} f(x) = f(a)$.

Property 8 implies that the limit of a polynomial function as $x \to a$ can be found by simply evaluating the polynomial at a. However, with other functions, f, evaluation at a may lead to the meaningless form $0/0$. In such cases, algebraic manipulation such as factoring and canceling may yield a function g that agrees with f, for $x \neq a$, and for which the limit can be determined.

If $f(x)$ approaches L as x approaches a from the right, then we write $\lim_{x \to a^+} f(x) = L$. If $f(x)$ approaches L as x approaches a from the left, we write $\lim_{x \to a^-} f(x) = L$. These limits are called one-sided limits.

The infinity symbol ∞, which does not represent a number, is used in describing limits. The statement

$$\lim_{x \to \infty} f(x) = L$$

means that as x increases without bound, the values of $f(x)$ approach the number L. A similar statement applies for the situation when $x \to -\infty$, which means that x is decreasing without bound. In general, if $p > 0$, then

$$\lim_{x \to \infty} \frac{1}{x^p} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^p} = 0$$

If $f(x)$ increases without bound as $x \to a$, then we write $\lim_{x \to a} f(x) = \infty$. Similarly, if $f(x)$ decreases without bound, we have $\lim_{x \to a} f(x) = -\infty$. To say that the limit of a function is ∞ (or $-\infty$) does not mean that the limit exists. Rather, it is a way of saying that the limit does not exist and tells *why* there is no limit.

There is a rule for evaluating the limit of a rational function (quotient of polynomials) as $x \to \infty$ or $-\infty$. If $f(x)$ is a rational function and $a_n x^n$ and $b_m x^m$ are the terms in the numerator and denominator, respectively, with the greatest powers of x, then

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{a_n x^n}{b_m x^m}$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{a_n x^n}{b_m x^m}$$

In particular, as $x \to \infty$ or $-\infty$, the limit of a polynomial is the same as the limit of the term that involves the greatest power of x. This means that, for a nonconstant polynomial, the limit as $x \to \infty$ or $-\infty$ is either ∞ or $-\infty$.

A function f is continuous at a if and only if

1. $f(a)$ exists
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Geometrically this means that the graph of f has no break at $x = a$. If a function is not continuous at a, then the function is said to be discontinuous at a. Polynomial functions and rational functions are continuous on their domains. Thus polynomial functions have no discontinuities and a rational function is discontinuous only at points where its denominator is zero.

To solve the inequality $f(x) > 0$ (or $f(x) < 0$), we first find the real roots of $f(x) = 0$ and the values of x for which f is discontinuous. These values determine intervals, and on each interval, $f(x)$ is either always positive or always negative. To find the sign on any one of these intervals, it suffices to find the sign of $f(x)$ at any point there. After the signs are determined for all intervals and assembled on a sign chart, it is easy to give the solution of $f(x) > 0$ (or $f(x) < 0$).

Review Problems

In Problems 1–28, find the limits if they exist. If the limit does not exist, so state, or use the symbol ∞ or $-\infty$ where appropriate.

1. $\lim_{x\to -1} (2x^2 + 6x - 1)$
2. $\lim_{x\to 0} \dfrac{2x^2 - 3x + 1}{2x^2 - 2}$
3. $\lim_{x\to 4} \dfrac{x^2 - 16}{x^2 - 4x}$
4. $\lim_{x\to -4} \dfrac{2x + 3}{x^2 - 4}$
5. $\lim_{h\to 0} (x + h)$
6. $\lim_{x\to 2} \dfrac{x^2 - 4}{x^2 - 3x + 2}$
7. $\lim_{x\to -4} \dfrac{x^3 + 4x^2}{x^2 + 2x - 8}$
8. $\lim_{x\to 2} \dfrac{x^2 - 7x + 10}{x^2 + x - 6}$
9. $\lim_{x\to \infty} \dfrac{2}{x + 1}$
10. $\lim_{x\to \infty} \dfrac{x^2 + 1}{2x^2}$
11. $\lim_{x\to \infty} \dfrac{2x + 5}{7x - 4}$
12. $\lim_{x\to -\infty} \dfrac{1}{x^4}$
13. $\lim_{t\to 4} \dfrac{3t - 4}{t - 4}$
14. $\lim_{x\to -\infty} \dfrac{x^6}{x^5}$
15. $\lim_{x\to -\infty} \dfrac{x + 3}{1 - x}$
16. $\lim_{x\to 4} \sqrt[3]{64}$
17. $\lim_{x\to \infty} \dfrac{x^2 - 1}{(3x + 2)^2}$
18. $\lim_{x\to 5} \dfrac{x^2 - 2x - 15}{x - 5}$
19. $\lim_{x\to 3^-} \dfrac{x + 3}{x^2 - 9}$
20. $\lim_{x\to 2} \dfrac{2 - x}{x - 2}$
21. $\lim_{x\to \infty} \sqrt{3x}$
22. $\lim_{y\to 5^+} \sqrt{y - 5}$
23. $\lim_{x\to \infty} \dfrac{x^{100} + (1/x^4)}{e - x^{96}}$
24. $\lim_{x\to -\infty} \dfrac{ex^2 - x^4}{31x - 2x^3}$
25. $\lim_{x\to 1} f(x)$ if $f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ x & \text{if } x > 1 \end{cases}$
26. $\lim_{x\to 3} f(x)$ if $f(x) = \begin{cases} x + 5 & \text{if } x < 3 \\ 6 & \text{if } x \ge 3 \end{cases}$
27. $\lim_{x\to 4^+} \dfrac{\sqrt{x^2 - 16}}{4 - x}$ (*Hint:* For $x > 4$, $\sqrt{x^2 - 16} = \sqrt{x - 4}\sqrt{x + 4}$.)
28. $\lim_{x\to 3^+} \dfrac{x^2 + x - 12}{\sqrt{x - 3}}$ (*Hint:* For $x > 3$, $\dfrac{x - 3}{\sqrt{x - 3}} = \sqrt{x - 3}$.)
29. If $f(x) = 8x - 2$, find $\lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h}$.
30. If $f(x) = 2x^2 - 3$, find $\lim_{h\to 0} \dfrac{f(x + h) - f(x)}{h}$.
31. **Host–Parasite Relationship** For a particular host–parasite relationship, it was determined that when the host density (number of hosts per unit of area) is x, then the number of hosts parasitized over a certain period of time is
$$y = 23\left(1 - \frac{1}{1 + 2x}\right)$$
If the host density were to increase without bound, what value would y approach?
32. **Predator–Prey Relationship** For a particular predator–prey relationship, it was determined that the number y of prey consumed by an individual predator over a period of time was a function of the prey density x (the number of prey per unit of area). Suppose
$$y = f(x) = \frac{10x}{1 + 0.1x}$$
If the prey density were to increase without bound, what value would y approach?
33. Using the definition of *continuity*, show that the function $f(x) = x + 3$ is continuous at $x = 2$.
34. Using the definition of *continuity*, show that the function $f(x) = \dfrac{x - 5}{x^2 + 2}$ is continuous at $x = 5$.
35. State whether $f(x) = x^2/5$ is continuous at each real number. Give a reason for your answer.
36. State whether $f(x) = x^2 - 2$ is continuous everywhere. Give a reason for your answer.

In Problems 37–44, find the points of discontinuity (if any) for each function.

37. $f(x) = \dfrac{x^2}{x + 3}$
38. $f(x) = \dfrac{0}{x^2}$
39. $f(x) = \dfrac{x - 1}{2x^2 + 3}$
40. $f(x) = (2 - 3x)^3$
41. $f(x) = \dfrac{4 - x^2}{x^2 + 3x - 4}$
42. $f(x) = \dfrac{2x + 6}{x^3 + x}$
43. $f(x) = \begin{cases} 2x + 3 & \text{if } x > 2 \\ 3x + 5 & \text{if } x \le 2 \end{cases}$
44. $f(x) = \begin{cases} 1/x & \text{if } x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$

11 Differentiation

11.1 The Derivative

11.2 Rules for Differentiation

11.3 The Derivative as a Rate of Change

11.4 The Product Rule and the Quotient Rule

11.5 The Chain Rule

Chapter 11 Review

EXPLORE & EXTEND
Marginal Propensity to Consume

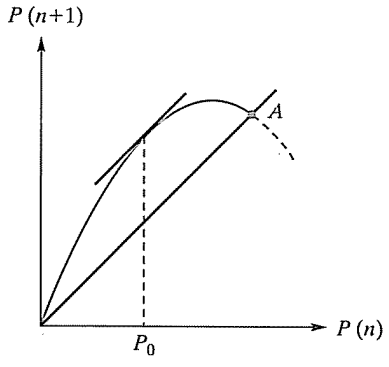
Government regulations generally limit the number of fish taken from a given fishing ground by commercial fishing boats in a season. This prevents overfishing, which depletes the fish population and leaves, in the long run, fewer fish to catch.

From a strictly commercial perspective, the ideal regulations would maximize the number of fish available for the year-to-year fish harvest. The key to finding those ideal regulations is a mathematical function called the reproduction curve. For a given fish habitat, this function estimates the fish population a year from now, $P(n+1)$, based on the population now, $P(n)$, assuming no external interventions such as fishing or influx of predators.

The figure to the bottom left shows a typical reproduction curve. Also graphed is the line $P(n+1) = P(n)$, the line along which the populations $P(n+1)$ and $P(n)$ would be equal. Notice the intersection of the curve and the straight line at point A. This is where, because of habitat crowding, the population has reached its maximum sustainable size. A population that is this size one year will be the same size the next year.

For any point on the horizontal axis, the distance between the reproduction curve and the line $P(n+1) = P(n)$ represents the sustainable harvest: the number of fish that could be caught, after the spawn have grown to maturity, so that in the end the population is back at the same size it was a year ago.

Commercially speaking, the optimal population size is the one where the distance between the reproduction curve and the line $P(n+1) = P(n)$ is the greatest. This condition is met where the slopes of the reproduction curve and the line $P(n+1) = P(n)$ are equal. [The slope of $P(n+1) = P(n)$ is, of course, 1.] Thus, for a maximum fish harvest year after year, regulations should aim to keep the fish population fairly close to P_0.



A central idea here is that of the slope of a curve at a given point. That idea is the cornerstone concept of this chapter.

Now we begin our study of calculus. The ideas involved in calculus are completely different from those of algebra and geometry. The power and importance of these ideas and their applications will become clear later in the book. In this chapter we introduce the *derivative* of a function and the important rules for finding derivatives. We also show how the derivative is used to analyze the rate of change of a quantity, such as the rate at which the position of a body is changing.

Objective

To develop the idea of a tangent line to a curve, to define the slope of a curve, and to define a derivative and give it a geometric interpretation. To compute derivatives by using the limit definition.

11.1 The Derivative

The main problem of differential calculus deals with finding the slope of the *tangent line* at a point on a curve. In high school geometry a tangent line, or *tangent*, to a circle is often defined as a line that meets the circle at exactly one point (Figure 11.1). However, this idea of a tangent is not very useful for other kinds of curves. For example, in Figure 11.2(a), the lines L_1 and L_2 intersect the curve at exactly one point P. Although we would not think of L_2 as the tangent at this point, it seems natural that L_1 is. In Figure 11.2(b) we intuitively would consider L_3 to be the tangent at point P, even though L_3 intersects the curve at other points.

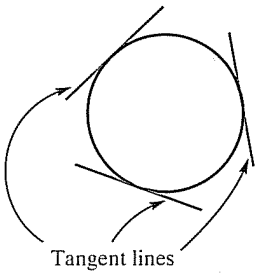
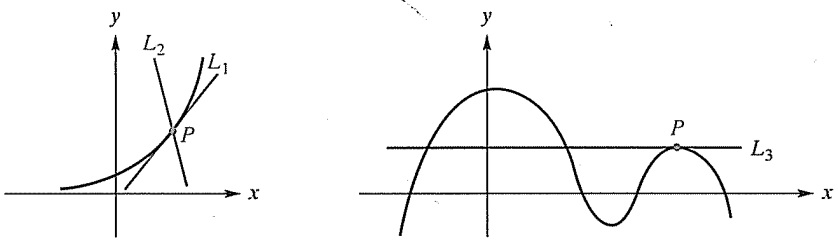


FIGURE 11.1 Tangent lines to a circle.



L_1 is a tangent line at P, but L_2 is not.

(a)

L_3 is a tangent line at P.

(b)

FIGURE 11.2 Tangent line at a point.

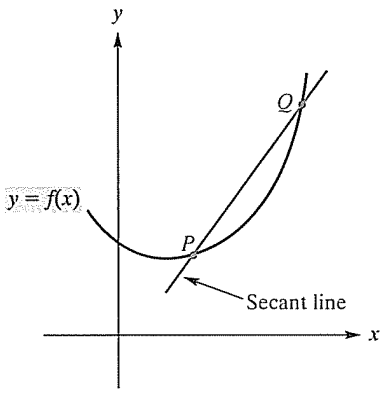


FIGURE 11.3 Secant line PQ.

From these examples, we see that the idea of a tangent as simply a line that intersects a curve at only one point is inadequate. To obtain a suitable definition of tangent line, we use the limit concept and the geometric notion of a *secant line*. A **secant line** is a line that intersects a curve at two or more points.

Look at the graph of the function $y = f(x)$ in Figure 11.3. We wish to define the tangent line at point P. If Q is a different point on the curve, the line PQ is a secant line. If Q moves along the curve and approaches P from the right (see Figure 11.4), typical secant lines are PQ', PQ'', and so on. As Q approaches P from the left, typical secant lines are PQ_1, PQ_2, and so on. *In both cases, the secant lines approach the same limiting position.* This common limiting position of the secant lines is defined to be the **tangent line** to the curve at P. This definition seems reasonable and applies to curves in general, not just circles.

A curve does not necessarily have a tangent line at each of its points. For example, the curve $y = |x|$ does not have a tangent at (0,0). As can be seen in Figure 11.5, a secant line through (0,0) and a nearby point to its right on the curve must always be the line

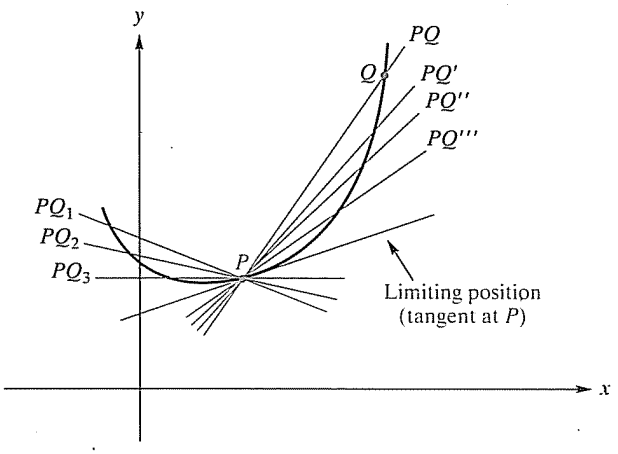


FIGURE 11.4 The tangent line is a limiting position of secant lines.

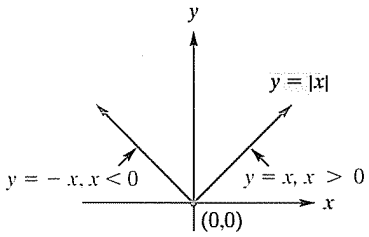


FIGURE 11.5 No tangent line to graph of $y = |x|$ at (0,0).

$y = x$. Thus the limiting position of such secant lines is also the line $y = x$. However, a secant line through (0,0) and a nearby point to its left on the curve must always be the line $y = -x$. Hence, the limiting position of such secant lines is also the line $y = -x$. Since there is no common limiting position, there is no tangent line at (0,0).

Now that we have a suitable definition of a tangent to a curve at a point, we can define the *slope of a curve* at a point.

Definition

The **slope of a curve** at a point P is the slope, if it exists, of the tangent line at P.

Since the tangent at P is a limiting position of secant lines PQ, we consider the slope of the tangent to be the limiting value of the slopes of the secant lines as Q approaches P. For example, let us consider the curve $f(x) = x^2$ and the slopes of some secant lines PQ, where $P = (1, 1)$. For the point $Q = (2.5, 6.25)$, the slope of PQ (see Figure 11.6) is

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{6.25 - 1}{2.5 - 1} = 3.5$$

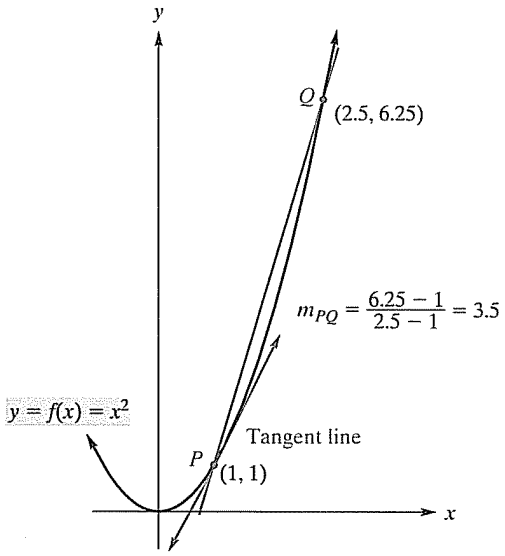


FIGURE 11.6 Secant line to $f(x) = x^2$ through (1, 1) and (2.5, 6.25).

Table 11.1 includes other points Q on the curve, as well as the corresponding slopes of PQ. Notice that as Q approaches P, the slopes of the secant lines seem to approach 2. Thus, we expect the slope of the indicated tangent line at (1, 1) to be 2. This will be confirmed later, in Example 1. But first, we wish to generalize our procedure.

Table 11.1 Slopes of Secant Lines to the Curve $f(x) = x^2$ at $P = (1, 1)$

Q	Slope of PQ
(2.5, 6.25)	$(6.25 - 1)/(2.5 - 1) = 3.5$
(2, 4)	$(4 - 1)/(2 - 1) = 3$
(1.5, 2.25)	$(2.25 - 1)/(1.5 - 1) = 2.5$
(1.25, 1.5625)	$(1.5625 - 1)/(1.25 - 1) = 2.25$
(1.1, 1.21)	$(1.21 - 1)/(1.1 - 1) = 2.1$
(1.01, 1.0201)	$(1.0201 - 1)/(1.01 - 1) = 2.01$

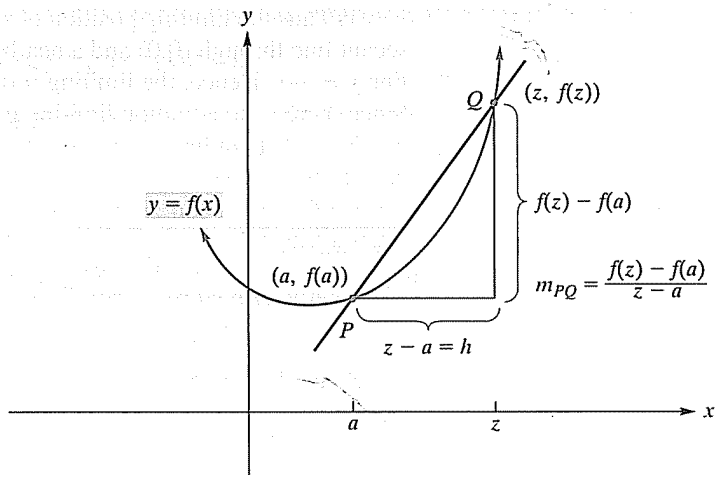


FIGURE 11.7 Secant line through P and Q.

For the curve $y = f(x)$ in Figure 11.7, we will find an expression for the slope at the point $P = (a, f(a))$. If $Q = (z, f(z))$, the slope of the secant line PQ is

$$m_{PQ} = \frac{f(z) - f(a)}{z - a}$$

If the difference $z - a$ is called h, then we can write z as $a + h$. Here we must have $h \neq 0$, for if $h = 0$, then $z = a$, and no secant line exists. Accordingly,

$$m_{PQ} = \frac{f(z) - f(a)}{z - a} = \frac{f(a + h) - f(a)}{h}$$

Which of these two forms for m_{PQ} is most convenient depends on the nature of the function f. As Q moves along the curve toward P, z approaches a. This means that h approaches zero. The limiting value of the slopes of the secant lines—which is the slope of the tangent line at $(a, f(a))$—is

$$m_{\tan} = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{1}$$

Again, which of these two forms is most convenient—which limit is easiest to determine—depends on the nature of the function f. In Example 1, we will use this limit to confirm our previous expectation that the slope of the tangent line to the curve $f(x) = x^2$ at (1, 1) is 2.

EXAMPLE 1 **Finding the Slope of a Tangent Line**

Find the slope of the tangent line to the curve $y = f(x) = x^2$ at the point (1, 1).

Solution: The slope is the limit in Equation (1) with $f(x) = x^2$ and $a = 1$:

$$\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} \frac{(1 + h)^2 - (1)^2}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 - 1}{h} = \lim_{h \to 0} \frac{2h + h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(2 + h)}{h} = \lim_{h \to 0} (2 + h) = 2$$

Therefore, the tangent line to $y = x^2$ at (1, 1) has slope 2. (Refer to Figure 11.6.)

Now Work Problem 1 ◁

We can generalize Equation (1) so that it applies to any point $(x,f(x))$ on a curve. Replacing a by x gives a function, called the *derivative* of f, whose input is x and whose output is the slope of the tangent line to the curve at $(x,f(x))$, provided that the tangent line *exists* and *has* a slope. (If the tangent line exists but is *vertical*, then it has no slope.) We thus have the following definition, which forms the basis of differential calculus:

Definition

The **derivative** of a function f is the function denoted f' (read "f prime") and defined by

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

provided that this limit exists. If $f'(a)$ can be found [while perhaps not all $f'(x)$ can be found] f is said to be ***differentiable*** at a, and $f'(a)$ is called the derivative of f at a or the derivative of f with respect to x at a. The process of finding the derivative is called ***differentiation.***

In the definition of the derivative, the expression

$$\frac{f(z) - f(x)}{z - x} = \frac{f(x+h) - f(x)}{h}$$

where $z = x + h$, is called a **difference quotient.** Thus $f'(x)$ is the limit of a difference quotient.

EXAMPLE 2 Using the Definition to Find the Derivative

If $f(x) = x^2$, find the derivative of f.

Solution: Applying the definition of a derivative gives

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x + h) = 2x \end{aligned}$$

Calculating a derivative via the definition requires precision. Typically, the difference quotient requires considerable manipulation before the limit step is taken. This requires that each written step be preceded by "$\lim_{h \to 0}$" to acknowledge that the limit step is still pending. Observe that after the limit step is taken, h will no longer be present.

Observe that, in taking the limit, we treated x as a constant, because it was h, not x, that was changing. Also, note that $f'(x) = 2x$ defines a function of x, which we can interpret as giving the slope of the tangent line to the graph of f at $(x,f(x))$. For example, if $x = 1$, then the slope is $f'(1) = 2 \cdot 1 = 2$, which confirms the result in Example 1.

Now Work Problem 3 ◁

Besides the notation $f'(x)$, other common ways to denote the derivative of $y = f(x)$ at x are

$\frac{dy}{dx}$	pronounced "dee y, dee x" or "dee y by dee x"
$\frac{d}{dx}(f(x))$	"dee $f(x)$, dee x" or "dee by dee x of $f(x)$"
y'	"y prime"
$D_x y$	"dee x of y"
$D_x(f(x))$	"dee x of $f(x)$"

CAUTION! The notation $\frac{dy}{dx}$, which is called *Leibniz notation*, should **not** be thought of as a fraction, although it looks like one. It is a single symbol for a derivative. We have not yet attached any meaning to individual symbols such as dy and dx.

Because the derivative gives the slope of the tangent line, $f'(a)$ is the slope of the line tangent to the graph of $y = f(x)$ at $(a,f(a))$.

Two other notations for the derivative of f at a are

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{and} \quad y'(a)$$

EXAMPLE 3 **Finding an Equation of a Tangent Line**

If $f(x) = 2x^2 + 2x + 3$, find an equation of the tangent line to the graph of f at $(1, 7)$.

Solution:

Strategy We will first determine the slope of the tangent line by computing the derivative and evaluating it at $x = 1$. Using this result and the point $(1, 7)$ in a point-slope form gives an equation of the tangent line.

We have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(2(x+h)^2 + 2(x+h) + 3) - (2x^2 + 2x + 3)}{h} \\ &= \lim_{h\to 0} \frac{2x^2 + 4xh + 2h^2 + 2x + 2h + 3 - 2x^2 - 2x - 3}{h} \\ &= \lim_{h\to 0} \frac{4xh + 2h^2 + 2h}{h} = \lim_{h\to 0} (4x + 2h + 2) \end{aligned}$$

In Example 3 it is *not* correct to say that, since the derivative is $4x + 2$, the tangent line at $(1, 7)$ is $y - 7 = (4x + 2)(x - 1)$. (This is not even the equation of a line.) The derivative must be **evaluated** at the point of tangency to determine the slope of the tangent line.

So

$$f'(x) = 4x + 2$$

and

$$f'(1) = 4(1) + 2 = 6$$

Thus, the tangent line to the graph at (1, 7) has slope 6. A point-slope form of this tangent is

$$y - 7 = 6(x - 1)$$

which in slope-intercept form is

$$y = 6x + 1$$

Now Work Problem 25 ◁

EXAMPLE 4 **Finding the Slope of a Curve at a Point**

Find the slope of the curve $y = 2x + 3$ at the point where $x = 6$.

Solution: The slope of the curve is the slope of the tangent line. Letting $y = f(x) = 2x + 3$, we have

$$\begin{aligned} \frac{dy}{dx} &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(2(x+h) + 3) - (2x + 3)}{h} \\ &= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2 \end{aligned}$$

Since $dy/dx = 2$, the slope when $x = 6$, or in fact at any point, is 2. Note that the curve is a straight line and thus has the same slope at each point.

Now Work Problem 19 ◁

EXAMPLE 5 **A Function with a Vertical Tangent Line**

Find $\dfrac{d}{dx}(\sqrt{x})$.

Solution: Letting $f(x) = \sqrt{x}$, we have

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalizing numerators or denominators of fractions is often helpful in calculating limits.

As $h \to 0$, both the numerator and denominator approach zero. This can be avoided by rationalizing the *numerator*:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$
$$= \frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})}$$

Therefore,

$$\frac{d}{dx}(\sqrt{x}) = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

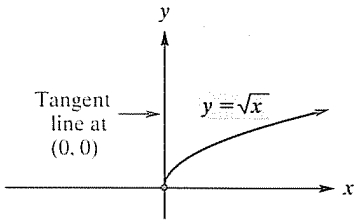


FIGURE 11.8 Vertical tangent line at (0, 0).

Note that the original function, $\sqrt{x}$, is defined for $x \geq 0$, but its derivative, $1/(2\sqrt{x})$, is defined only when $x > 0$. The reason for this is clear from the graph of $y = \sqrt{x}$ in Figure 11.8. When $x = 0$, the tangent is a vertical line, so its slope is not defined.

Now Work Problem 17 ◁

In Example 5 we saw that the function $y = \sqrt{x}$ is not differentiable when $x = 0$, because the tangent line is vertical at that point. It is worthwhile to mention that $y = |x|$ also is not differentiable when $x = 0$, but for a different reason: There is *no* tangent line at all at that point. (Refer to Figure 11.5.) Both examples show that the domain of f' may be strictly contained in the domain of f.

To indicate a derivative, Leibniz notation is often useful because it makes it convenient to emphasize the independent and dependent variables involved. For example, if the variable p is a function of the variable q, we speak of the derivative of p with respect to q, written dp/dq.

Variables other than x and y are often more natural in applied problems. Time denoted by t, quantity by q, and price by p are obvious examples. Example 6 illustrates.

APPLY IT ▷

1. If a ball is thrown upward at a speed of 40 ft/s from a height of 6 feet, its height H in feet after t seconds is given by $H = 6 + 40t - 16t^2$. Find $\frac{dH}{dt}$.

EXAMPLE 6 **Finding the Derivative of p with Respect to q**

If $p = f(q) = \frac{1}{2q}$, find $\frac{dp}{dq}$.

Solution: We will do this problem first using the $h \to 0$ limit (the only one we have used so far) and then using $r \to q$ to illustrate the other variant of the limit.

$$\frac{dp}{dq} = \frac{d}{dq}\left(\frac{1}{2q}\right) = \lim_{h\to 0} \frac{f(q+h)-f(q)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{2(q+h)} - \frac{1}{2q}}{h} = \lim_{h\to 0} \frac{\frac{q-(q+h)}{2q(q+h)}}{h}$$

$$= \lim_{h\to 0} \frac{q-(q+h)}{h(2q(q+h))} = \lim_{h\to 0} \frac{-h}{h(2q(q+h))}$$

$$= \lim_{h\to 0} \frac{-1}{2q(q+h)} = -\frac{1}{2q^2}$$

We also have

$$\frac{dp}{dq} = \lim_{r\to q} \frac{f(r)-f(q)}{r-q}$$

$$= \lim_{r\to q} \frac{\frac{1}{2r} - \frac{1}{2q}}{r-q} = \lim_{r\to q} \frac{\frac{q-r}{2rq}}{r-q}$$

$$= \lim_{r\to q} \frac{-1}{2rq} = \frac{-1}{2q^2}$$

We leave it you to decide which form leads to the simpler limit calculation in this case.

Note that when $q = 0$ the function is not defined, so the derivative is also not even defined when $q = 0$.

Now Work Problem 15 ◁

Keep in mind that the derivative of $y = f(x)$ at x is nothing more than a limit, namely

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

equivalently

$$\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

whose use we have just illustrated. Although we can interpret the derivative as a function that gives the slope of the tangent line to the curve $y = f(x)$ at the point $(x, f(x))$, this interpretation is simply a geometric convenience that assists our understanding. The preceding limit may exist, aside from any geometric considerations at all. As we will see later, there are other useful interpretations of the derivative.

In Section 11.4, we will make technical use of the following relationship between differentiability and continuity. However, it is of fundamental importance and needs to be understood from the outset.

If f is differentiable at a, then f is continuous at a.

To establish this result, we will assume that f is differentiable at a. Then $f'(a)$ exists, and

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a)$$

Consider the numerator $f(a+h)-f(a)$ as $h \to 0$. We have

$$\begin{aligned}\lim_{h\to 0}(f(a+h)-f(a)) &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right)\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot\lim_{h\to 0}h\\ &= f'(a)\cdot 0 = 0\end{aligned}$$

Thus, $\lim_{h\to 0}(f(a+h)-f(a)) = 0$. This means that $f(a+h)-f(a)$ approaches 0 as $h \to 0$. Consequently,

$$\lim_{h\to 0} f(a+h) = f(a)$$

As stated in Section 10.3, this condition means that f is continuous at a. The foregoing, then, proves that f is continuous at a when f is differentiable there. More simply, we say that **differentiability at a point implies continuity at that point.**

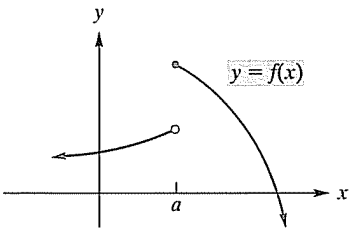


FIGURE 11.9 f is not continuous at a, so f is not differentiable at a.

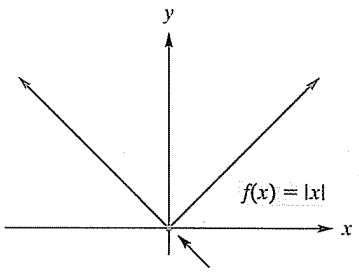
If a function is not continuous at a point, then it cannot have a derivative there. For example, the function in Figure 11.9 is discontinuous at a. The curve has no tangent at that point, so the function is not differentiable there.

EXAMPLE 7 Continuity and Differentiability

a. Let $f(x) = x^2$. The derivative, $2x$, is defined for all values of x, so $f(x) = x^2$ must be continuous for all values of x.

b. The function $f(p) = \dfrac{1}{2p}$ is not continuous at $p = 0$ because f is not defined there. Thus, the derivative does not exist at $p = 0$.

◁



Continuous at $x = 0$, but not differentiable at $x = 0$

FIGURE 11.10 Continuity does not imply differentiability.

The converse of the statement that differentiability implies continuity is *false*. That is, continuity does not imply differentiability. In Example 8, we give a function that is continuous at a point, but not differentiable there.

EXAMPLE 8 **Continuity Does Not Imply Differentiability**

The function $y = f(x) = |x|$ is continuous at $x = 0$. (See Figure 11.10.) As we mentioned earlier, there is no tangent line at $x = 0$. Thus, the derivative does not exist there. This shows that continuity does *not* imply differentiability.

◁

Finally, we remark that while differentiability of f at a implies continuity of f at a, the derivative function, f', is not necessarily continuous at a. Unfortunately, the classic example is constructed from a function not considered in this book.

PROBLEMS 11.1

In Problems 1 and 2, a function f and a point P on its graph are given.

(a) *Find the slope of the secant line PQ for each point $Q = (x, f(x))$ whose x-value is given in the table. Round your answers to four decimal places.*

(b) *Use your results from part (a) to estimate the slope of the tangent line at P.*

1. $f(x) = x^3 + 3, P = (-2, -5)$

x-value of Q	-3	-2.5	-2.2	-2.1	-2.01	-2.001
m_{PQ}						

2. $f(x) = e^x, P = (0, 1)$

x-value of Q	1	0.5	0.2	0.1	0.01	0.001
m_{PQ}						

In Problems 3–18, use the definition of the derivative to find each of the following.

3. $f'(x)$ if $f(x) = x$

4. $f'(x)$ if $f(x) = 4x - 1$

5. $\dfrac{dy}{dx}$ if $y = 3x + 5$

6. $\dfrac{dy}{dx}$ if $y = -5x$

7. $\dfrac{d}{dx}(3 - 2x)$

8. $\dfrac{d}{dx}\left(1 - \dfrac{x}{2}\right)$

9. $f'(x)$ if $f(x) = 3$

10. $f'(x)$ if $f(x) = 7.01$

11. $\dfrac{d}{dx}(x^2 + 4x - 8)$

12. y' if $y = x^2 + 3x + 2$

13. $\dfrac{dp}{dq}$ if $p = 3q^2 + 2q + 1$

14. $\dfrac{d}{dx}(x^2 - x - 3)$

15. y' if $y = \dfrac{6}{x}$

16. $\dfrac{dC}{dq}$ if $C = 7 + 2q - 3q^2$

17. $f'(x)$ if $f(x) = \sqrt{2x}$

18. $H'(x)$ if $H(x) = \dfrac{3}{x - 2}$

19. Find the slope of the curve $y = x^2 + 4$ at the point $(-2, 8)$.

20. Find the slope of the curve $y = 1 - x^2$ at the point $(1, 0)$.

21. Find the slope of the curve $y = 4x^2 - 5$ when $x = 0$.

22. Find the slope of the curve $y = \sqrt{2x}$ when $x = 18$.

In Problems 23–28, find an equation of the tangent line to the curve at the given point.

23. $y = x + 4; (3, 7)$

24. $y = 3x^2 - 4; (1, -1)$

25. $y = x^2 + 2x + 3; (1, 6)$

26. $y = (x - 7)^2; (6, 1)$

27. $y = \dfrac{4}{x + 1}; (3, 1)$

28. $y = \dfrac{5}{1 - 3x}; (2, -1)$

29. Banking Equations may involve derivatives of functions. In an article on interest rate deregulation, Christofi and Agapos[1] solve the equation

$$r = \left(\frac{\eta}{1 + \eta}\right)\left(r_L - \frac{dC}{dD}\right)$$

for η (the Greek letter "eta"). Here r is the deposit rate paid by commercial banks, r_L is the rate earned by commercial banks, C is the administrative cost of transforming deposits into return-earning assets, D is the savings deposits level, and η is the deposit elasticity with respect to the deposit rate. Find η.

In Problems 30 and 31, use the numerical derivative feature of your graphing calculator to estimate the derivatives of the functions at the indicated values. Round your answers to three decimal places.

30. $f(x) = \sqrt{2x^2 + 3x}; x = 1, x = 2$

31. $f(x) = e^x(4x - 7); x = 0, x = 1.5$

In Problems 32 and 33, use the "limit of a difference quotient" approach to estimate $f'(x)$ at the indicated values of x. Round your answers to three decimal places.

32. $f(x) = x \ln x - x; x = 1, x = 10$

33. $f(x) = \dfrac{x^2 + 4x + 2}{x^3 - 3}; x = 2, x = -4$

[1] A. Christofi and A. Agapos, "Interest Rate Deregulation: An Empirical Justification," *Review of Business and Economic Research*, XX, no. 1 (1984), 39–49.

34. Find an equation of the tangent line to the curve $f(x) = x^2 + x$ at the point $(-2, 2)$. Graph both the curve and the tangent line. Notice that the tangent line is a good approximation to the curve near the point of tangency.

35. The derivative of $f(x) = x^3 - x + 2$ is $f'(x) = 3x^2 - 1$. Graph both the function f and its derivative f'. Observe that there are two points on the graph of f where the tangent line is horizontal. For the x-values of these points, what are the corresponding values of $f'(x)$? Why are these results expected? Observe the intervals where $f'(x)$ is positive. Notice that tangent lines to the graph of f have positive slopes over these intervals. Observe the interval where $f'(x)$ is negative. Notice that tangent lines to the graph of f have negative slopes over this interval.

In Problems 36 and 37, verify the identity $(z - x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$ *for the indicated values of n and calculate the derivative using the* $z \to x$ *form of the definition of the derivative in Equation (2).*

36. $n = 4, n = 3, n = 2$; $f'(x)$ if $f(x) = 2x^4 + x^3 - 3x^2$

37. $n = 5, n = 3$; $f'(x)$ if $f(x) = 4x^5 - 3x^3$

11.2 Rules for Differentiation

Objective

To develop the basic rules for differentiating constant functions and power functions and the combining rules for differentiating a constant multiple of a function and a sum of two functions.

Differentiating a function by direct use of the definition of derivative can be tedious. However, if a function is constructed from simpler functions, then the derivative of the more complicated function can be constructed from the derivatives of the simpler functions. Ultimately, we need to know only the derivatives of a few basic functions and ways to assemble derivatives of constructed functions from the derivatives of their components. For example, if functions f and g have derivatives f' and g', respectively, then $f + g$ has a derivative given by $(f + g)' = f' + g'$. However, some *rules* are less intuitive. For example, if $f \cdot g$ denotes the function whose value at x is given by $(f \cdot g)(x) = f(x) \cdot g(x)$, then $(f \cdot g)' = f' \cdot g + f \cdot g'$. In this chapter we study most such combining rules and some basic rules for calculating derivatives of certain basic functions.

We begin by showing that the derivative of a constant function is zero. Recall that the graph of the constant function $f(x) = c$ is a horizontal line (see Figure 11.11), which has a slope of zero at each point. This means that $f'(x) = 0$ regardless of x. As a formal proof of this result, we apply the definition of the derivative to $f(x) = c$:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Thus, we have our first rule:

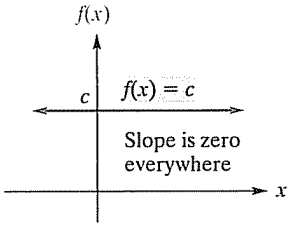


FIGURE 11.11 The slope of a constant function is 0.

BASIC RULE 0 Derivative of a Constant

If c is a constant, then

$$\frac{d}{dx}(c) = 0$$

That is, the derivative of a constant function is zero.

EXAMPLE 1 Derivatives of Constant Functions

a. $\frac{d}{dx}(3) = 0$ because 3 is a constant function.

b. If $g(x) = \sqrt{5}$, then $g'(x) = 0$ because g is a constant function. For example, the derivative of g when $x = 4$ is $g'(4) = 0$.

c. If $s(t) = (1{,}938{,}623)^{807.4}$, then $ds/dt = 0$.

Now Work Problem 1 ◁

The next rule gives a formula for the derivative of "x raised to a constant power"—that is, the derivative of $f(x) = x^a$, where a is an arbitrary real number. A function of this form is called a **power function.** For example, $f(x) = x^2$ is a power function. While the rule we record is valid for all real a, we will establish it only in the case where

a is a positive integer, n. The rule is so central to differential calculus that it warrants a detailed calculation—if only in the case where a is a positive integer, n. Whether we use the $h \to 0$ form of the definition of derivative or the $z \to x$ form, the calculation of $\dfrac{dx^n}{dx}$ is instructive and provides good practice with summation notation, whose use is more essential in later chapters. We provide a calculation for each possibility. We must either expand $(x+h)^n$, to use the $h \to 0$ form of Equation (2) from Section 11.1, or factor $z^n - x^n$, to use the $z \to x$ form.

For the first of these we recall the *binomial theorem* of Section 9.2:

$$(x+h)^n = \sum_{i=0}^{n} {}_nC_i x^{n-i} h^i$$

where the ${}_nC_i$ are the binomial coefficients, whose precise descriptions, except for ${}_nC_0 = 1$ and ${}_nC_1 = n$, are not necessary here (but are given in Section 8.2). For the second we have

$$(z-x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) = z^n - x^n$$

which is easily verified by carrying out the multiplication using the rules for manipulating summations given in Section 1.5. In fact, we have

$$\begin{aligned}(z-x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i}\right) &= z\sum_{i=0}^{n-1} x^i z^{n-1-i} - x\sum_{i=0}^{n-1} x^i z^{n-1-i} \\ &= \sum_{i=0}^{n-1} x^i z^{n-i} - \sum_{i=0}^{n-1} x^{i+1} z^{n-1-i} \\ &= \left(z^n + \sum_{i=1}^{n-1} x^i z^{n-i}\right) - \left(\sum_{i=0}^{n-2} x^{i+1} z^{n-1-i} + x^n\right) \\ &= z^n - x^n\end{aligned}$$

where the reader should check that the two summations in the second to last line really do cancel as shown.

BASIC RULE 1 Derivative of x^a

If a is any real number, then

$$\frac{d}{dx}(x^a) = ax^{a-1}$$

That is, the derivative of a constant power of x is the exponent times x raised to a power one less than the given power.

CAUTION!

There is a lot more to calculus than this rule.

For n a positive integer, if $f(x) = x^n$, the definition of the derivative gives

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

By our previous discussion on expanding $(x+h)^n$,

$$f'(x) = \lim_{h\to 0} \frac{\sum_{i=0}^{n} {}_nC_i x^{n-i} h^i - x^n}{h}$$

$$\overset{(1)}{=} \lim_{h\to 0} \frac{\sum_{i=1}^{n} {}_nC_i x^{n-i} h^i}{h}$$

$$\begin{aligned}
&\overset{(2)}{=} \lim_{h\to 0} \frac{h\sum_{i=1}^{n} {}_nC_i x^{n-i}h^{i-1}}{h} \\
&\overset{(3)}{=} \lim_{h\to 0} \sum_{i=1}^{n} {}_nC_i x^{n-i}h^{i-1} \\
&\overset{(4)}{=} \lim_{h\to 0} \left(nx^{n-1} + \sum_{i=2}^{n} {}_nC_i x^{n-i}h^{i-1} \right) \\
&\overset{(5)}{=} nx^{n-1}
\end{aligned}$$

where we justify the further steps as follows:

(1) The $i = 0$ term in the summation is ${}_nC_0 x^n h^0 = x^n$ so it cancels with the separate, last, term: $-x^n$.

(2) We are able to extract a common factor of h from each term in the sum.

(3) This is the crucial step. The expressions separated by the equal sign are limits as $h \to 0$ of functions of h that are equal for $h \neq 0$.

(4) The $i = 1$ term in the summation is ${}_nC_1 x^{n-1}h^0 = nx^{n-1}$. It is the only one that does not contain a factor of h, and we separated it from the other terms.

(5) Finally, in determining the limit we made use of the fact that the isolated term is independent of h; while all the others contain h as a factor and so have limit 0 as $h \to 0$.

Now, using the $z \to x$ limit for the definition of the derivative and $f(x) = x^n$, we have

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x} = \lim_{h\to 0} \frac{z^n - x^n}{z - x}$$

By our previous discussion on factoring $z^n - x^n$, we have

$$\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{(z - x)\left(\sum_{i=0}^{n-1} x^i z^{n-1-i} \right)}{z - x} \\
&\overset{(1)}{=} \lim_{z\to x} \sum_{i=0}^{n-1} x^i z^{n-1-i} \\
&\overset{(2)}{=} \sum_{i=0}^{n-1} x^i x^{n-1-i} \\
&\overset{(3)}{=} \sum_{i=0}^{n-1} x^{n-1} \\
&\overset{(4)}{=} nx^{n-1}
\end{aligned}$$

where this time we justify the further steps as follows:

(1) Here the crucial step comes first. The expressions separated by the equal sign are limits as $z \to x$ of functions of z that are equal for $z \neq x$.

(2) The limit is given by evaluation because the expression is a polynomial in the variable z.

(3) An obvious rule for exponents is used.

(4) Each term in the sum is x^{n-1}, independent of i, and there are n such terms.

EXAMPLE 2 Derivatives of Powers of x

a. By Basic Rule 1, $\dfrac{d}{dx}(x^2) = 2x^{2-1} = 2x$.

b. If $F(x) = x = x^1$, then $F'(x) = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$. Thus, the derivative of x with respect to x is 1.

c. If $f(x) = x^{-10}$, then $f'(x) = -10x^{-10-1} = -10x^{-11}$.

Now Work Problem 3 ◁

When we apply a differentiation rule to a function, sometimes the function must first be rewritten so that it has the proper form for that rule. For example, to differentiate $f(x) = \dfrac{1}{x^{10}}$ we would first rewrite f as $f(x) = x^{-10}$ and then proceed as in Example 2(c).

EXAMPLE 3 Rewriting Functions in the Form x^a

a. To differentiate $y = \sqrt{x}$, we rewrite $\sqrt{x}$ as $x^{1/2}$ so that it has the form x^n. Thus,

$$\frac{dy}{dx} = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

which agrees with our limit calculation in Example 5 of Section 11.1.

b. Let $h(x) = \dfrac{1}{x\sqrt{x}}$. To apply Basic Rule 1, we must rewrite $h(x)$ as $h(x) = x^{-3/2}$ so that it has the form x^n. We have

$$h'(x) = \frac{d}{dx}(x^{-3/2}) = -\frac{3}{2}x^{(-3/2)-1} = -\frac{3}{2}x^{-5/2}$$

Now Work Problem 39 ◁

CAUTION!

In Example 3(b), do not rewrite $\dfrac{1}{x\sqrt{x}}$ as $\dfrac{1}{x^{3/2}}$ and then merely differentiate the denominator.

Now that we can say immediately that the derivative of x^3 is $3x^2$, the question arises as to what we could say about the derivative of a *multiple* of x^3, such as $5x^3$. Our next rule will handle this situation of differentiating a constant times a function.

COMBINING RULE 1 Constant Factor Rule

If f is a differentiable function and c is a constant, then $cf(x)$ is differentiable, and

$$\frac{d}{dx}(cf(x)) = cf'(x)$$

That is, the derivative of a constant times a function is the constant times the derivative of the function.

Proof. If $g(x) = cf(x)$, applying the definition of the derivative of g gives

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= \lim_{h\to 0}\left(c\cdot\frac{f(x+h)-f(x)}{h}\right) = c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ is $f'(x)$; so $g'(x) = cf'(x)$.

EXAMPLE 4 Differentiating a Constant Times a Function

Differentiate the following functions.

a. $g(x) = 5x^3$

Solution: Here g is a constant (5) times a function (x^3). So

$$\frac{d}{dx}(5x^3) = 5\frac{d}{dx}(x^3) \qquad \text{Combining Rule 1}$$

$$= 5(3x^{3-1}) = 15x^2 \qquad \text{Basic Rule 1}$$

b. $f(q) = \dfrac{13q}{5}$

Solution:

Strategy We first rewrite f as a constant times a function and then apply Basic Rule 1.

Because $\dfrac{13q}{5} = \dfrac{13}{5}q$, f is the constant $\dfrac{13}{5}$ times the function q. Thus,

$$f'(q) = \frac{13}{5}\frac{d}{dq}(q) \qquad \text{Combining Rule 1}$$

$$= \frac{13}{5} \cdot 1 = \frac{13}{5} \qquad \text{Basic Rule 1}$$

c. $y = \dfrac{0.25}{\sqrt[5]{x^2}}$

Solution: We can express y as a constant times a function:

$$y = 0.25 \cdot \frac{1}{\sqrt[5]{x^2}} = 0.25x^{-2/5}$$

Hence,

$$y' = 0.25\frac{d}{dx}(x^{-2/5}) \qquad \text{Combining Rule 1}$$

$$= 0.25\left(-\frac{2}{5}x^{-7/5}\right) = -0.1x^{-7/5} \qquad \text{Basic Rule 1}$$

Now Work Problem 7 ◁

CAUTION!

In differentiating $f(x) = (4x)^3$, Basic Rule 1 cannot be applied directly. It applies to a power of the variable x, *not* to a power of an expression involving x, such as $4x$. To apply our rules, write $f(x) = (4x)^3 = 4^3x^3 = 64x^3$. Thus,

$$f'(x) = 64\frac{d}{dx}(x^3) = 64(3x^2) = 192x^2.$$

The next rule involves derivatives of sums and differences of functions.

COMBINING RULE 2 Sum or Difference Rule

If f and g are differentiable functions, then $f + g$ and $f - g$ are differentiable, and

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

and

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

That is, the derivative of the sum (difference) of two functions is the sum (difference) of their derivatives.

Proof. For the case of a sum, if $F(x) = f(x) + g(x)$, applying the definition of the derivative of F gives

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h\to 0}\frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0} \frac{(f(x+h)-f(x))+(g(x+h)-g(x))}{h} \qquad \text{regrouping}$$

$$= \lim_{h\to 0} \left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h} \right)$$

Because the limit of a sum is the sum of the limits,

$$F'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

But these two limits are $f'(x)$ and $g'(x)$. Thus,

$$F'(x) = f'(x) + g'(x)$$

The proof for the derivative of a difference of two functions is similar.

Combining Rule 2 can be extended to the derivative of any number of sums and differences of functions. For example,

$$\frac{d}{dx}[f(x) - g(x) + h(x) + k(x)] = f'(x) - g'(x) + h'(x) + k'(x)$$

APPLY IT ▷

2. If the revenue function for a certain product is $r(q) = 50q - 0.3q^2$, find the derivative of this function, also known as the marginal revenue.

EXAMPLE 5 **Differentiating Sums and Differences of Functions**

Differentiate the following functions.

a. $F(x) = 3x^5 + \sqrt{x}$

Solution: Here F is the sum of two functions, $3x^5$ and $\sqrt{x}$. Therefore,

$$F'(x) = \frac{d}{dx}(3x^5) + \frac{d}{dx}(x^{1/2}) \qquad \text{Combining Rule 2}$$

$$= 3\frac{d}{dx}(x^5) + \frac{d}{dx}(x^{1/2}) \qquad \text{Combining Rule 1}$$

$$= 3(5x^4) + \frac{1}{2}x^{-1/2} = 15x^4 + \frac{1}{2\sqrt{x}} \qquad \text{Basic Rule 1}$$

b. $f(z) = \frac{z^4}{4} - \frac{5}{z^{1/3}}$

Solution: To apply our rules, we will rewrite $f(z)$ in the form $f(z) = \frac{1}{4}z^4 - 5z^{-1/3}$. Since f is the difference of two functions,

$$f'(z) = \frac{d}{dz}\left(\frac{1}{4}z^4\right) - \frac{d}{dz}(5z^{-1/3}) \qquad \text{Combining Rule 2}$$

$$= \frac{1}{4}\frac{d}{dz}(z^4) - 5\frac{d}{dz}(z^{-1/3}) \qquad \text{Combining Rule 1}$$

$$= \frac{1}{4}(4z^3) - 5\left(-\frac{1}{3}z^{-4/3}\right) \qquad \text{Basic Rule 1}$$

$$= z^3 + \frac{5}{3}z^{-4/3}$$

c. $y = 6x^3 - 2x^2 + 7x - 8$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3) - \frac{d}{dx}(2x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(8)$$

$$= 6\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(8)$$

$$= 6(3x^2) - 2(2x) + 7(1) - 0$$

$$= 18x^2 - 4x + 7$$

Now Work Problem 47 ◁

In Examples 6 and 7, we need to rewrite the given function in a form to which our rules apply.

EXAMPLE 6 Finding a Derivative

Find the derivative of $f(x) = 2x(x^2 - 5x + 2)$ when $x = 2$.

Solution: We multiply and then differentiate each term:

$$f(x) = 2x^3 - 10x^2 + 4x$$
$$f'(x) = 2(3x^2) - 10(2x) + 4(1)$$
$$= 6x^2 - 20x + 4$$
$$f'(2) = 6(2)^2 - 20(2) + 4 = -12$$

Now Work Problem 75 ◁

EXAMPLE 7 Finding an Equation of a Tangent Line

Find an equation of the tangent line to the curve

$$y = \frac{3x^2 - 2}{x}$$

when $x = 1$.

Solution:

Strategy First we find $\frac{dy}{dx}$, which gives the slope of the tangent line at any point. Evaluating $\frac{dy}{dx}$ when $x = 1$ gives the slope of the required tangent line. We then determine the y-coordinate of the point on the curve when $x = 1$. Finally, we substitute the slope and both of the coordinates of the point in point-slope form to obtain an equation of the tangent line.

Rewriting y as a difference of two functions, we have

$$y = \frac{3x^2}{x} - \frac{2}{x} = 3x - 2x^{-1}$$

Thus,

$$\frac{dy}{dx} = 3(1) - 2((-1)x^{-2}) = 3 + \frac{2}{x^2}$$

The slope of the tangent line to the curve when $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 + \frac{2}{1^2} = 5$$

To find the y-coordinate of the point on the curve where $x = 1$, we evaluate $y = \frac{3x^2 - 2}{x}$ at $x = 1$. This gives

$$y = \frac{3(1)^2 - 2}{1} = 1$$

CAUTION!

To obtain the y-value of the point on the curve when $x = 1$, evaluate the *original* function at $x = 1$.

Hence, the point $(1, 1)$ lies on both the curve and the tangent line. Therefore, an equation of the tangent line is

$$y - 1 = 5(x - 1)$$

In slope-intercept form, we have

$$y = 5x - 4$$

Now Work Problem 81 ◁

PROBLEMS 11.2

In Problems 1–74, differentiate the functions.

1. $f(x) = \pi$
2. $f(x) = \left(\frac{6}{7}\right)^{2/3}$
3. $y = x^6$
4. $f(x) = x^{21}$
5. $y = x^{80}$
6. $y = x^{2.1}$
7. $f(x) = 9x^2$
8. $y = 4x^3$
9. $g(w) = 8w^7$
10. $v(x) = x^e$
11. $y = \frac{3}{5}x^6$
12. $f(p) = \sqrt{3}p^4$
13. $f(t) = \dfrac{t^7}{25}$
14. $y = \dfrac{x^7}{7}$
15. $f(x) = x + 3$
16. $f(x) = 5x - e$
17. $f(x) = 4x^2 - 2x + 3$
18. $F(x) = 5x^2 - 9x$
19. $g(p) = p^4 - 3p^3 - 1$
20. $f(t) = -13t^2 + 14t + 1$
21. $y = x^4 - \sqrt[3]{x}$
22. $y = -8x^4 + \ln 2$
23. $y = -13x^3 + 14x^2 - 2x + 3$
24. $V(r) = r^8 - 7r^6 + 3r^2 + 1$
25. $f(x) = 2(13 - x^4)$
26. $\psi(t) = e(t^7 - 5^3)$
27. $g(x) = \dfrac{13 - x^4}{3}$
28. $f(x) = \dfrac{5(x^4 - 6)}{2}$
29. $h(x) = 4x^4 + x^3 - \dfrac{9x^2}{2} + 8x$
30. $k(x) = -2x^2 + \dfrac{5}{3}x + 11$
31. $f(x) = \dfrac{5}{7}x^9 + \dfrac{3}{5}x^7$
32. $p(x) = \dfrac{x^7}{7} + \dfrac{2x}{3}$
33. $f(x) = x^{3/5}$
34. $f(x) = 2x^{-14/5}$
35. $y = x^{3/4} + 2x^{5/3}$
36. $y = 4x^2 - x^{-3/5}$
37. $y = 11\sqrt{x}$
38. $y = \sqrt{x^7}$
39. $f(r) = 6\sqrt[3]{r}$
40. $y = 4\sqrt[8]{x^2}$
41. $f(x) = x^{-6}$
42. $f(s) = 2s^{-3}$
43. $f(x) = x^{-3} + x^{-5} - 2x^{-6}$
44. $f(x) = 100x^{-3} + 10x^{1/2}$
45. $y = \dfrac{1}{x}$
46. $f(x) = \dfrac{3}{x^4}$
47. $y = \dfrac{8}{x^5}$
48. $y = \dfrac{1}{4x^5}$
49. $g(x) = \dfrac{4}{3x^3}$
50. $y = \dfrac{1}{x^2}$
51. $f(t) = \dfrac{3}{5t^3}$
52. $g(x) = \dfrac{7}{9x}$
53. $f(x) = \dfrac{x}{7} + \dfrac{7}{x}$
54. $\Phi(x) = \dfrac{x^3}{3} - \dfrac{3}{x^3}$
55. $f(x) = -9x^{1/3} + 5x^{-2/5}$
56. $f(z) = 5z^{3/4} - 6^2 - 8z^{1/4}$
57. $q(x) = \dfrac{1}{\sqrt[3]{8x^2}}$
58. $f(x) = \dfrac{3}{\sqrt[4]{x^3}}$
59. $y = \dfrac{2}{\sqrt{x}}$
60. $y = \dfrac{1}{2\sqrt{x}}$
61. $y = x^3\sqrt[3]{x}$
62. $f(x) = (2x^3)(4x^2)$
63. $f(x) = x(3x^2 - 10x + 7)$
64. $f(x) = x^3(3x^6 - 5x^2 + 4)$
65. $f(x) = x^3(3x)^2$
66. $s(x) = \sqrt{x}(\sqrt[5]{x} + 7x + 2)$
67. $v(x) = x^{-2/3}(x + 5)$
68. $f(x) = x^{3/5}(x^2 + 7x + 11)$
69. $f(q) = \dfrac{3q^2 + 4q - 2}{q}$
70. $f(w) = \dfrac{w - 5}{w^5}$
71. $f(x) = (x - 1)(x + 2)$
72. $f(x) = x^2(x - 2)(x + 4)$
73. $w(x) = \dfrac{x^2 + x^3}{x^2}$
74. $f(x) = \dfrac{7x^3 + x}{6\sqrt{x}}$

For each curve in Problems 75–78, find the slopes at the indicated points.

75. $y = 3x^2 + 4x - 8$; $(0, -8), (2, 12), (-3, 7)$
76. $y = 3 + 5x - 3x^3$; $(0, 3), (\frac{1}{2}, \frac{41}{8}), (2, -11)$
77. $y = 4$; when $x = -4, x = 7, x = 22$
78. $y = 3x - 4\sqrt{x}$; when $x = 4, x = 9, x = 25$

In Problems 79–82, find an equation of the tangent line to the curve at the indicated point.

79. $y = 4x^2 + 5x + 6$; $(1, 15)$
80. $y = \dfrac{1 - x^2}{5}$; $(4, -3)$
81. $y = \dfrac{1}{x^2}$; $(2, \frac{1}{4})$
82. $y = -\sqrt[3]{x}$; $(8, -2)$
83. Find an equation of the tangent line to the curve

$$y = 3 + x - 5x^2 + x^4$$

when $x = 0$.

84. Repeat Problem 83 for the curve

$$y = \frac{\sqrt{x}(2 - x^2)}{x}$$

when $x = 4$.

85. Find all points on the curve

$$y = \frac{5}{2}x^2 - x^3$$

where the tangent line is horizontal.

86. Repeat Problem 85 for the curve

$$y = \frac{x^6}{6} - \frac{x^2}{2} + 1$$

87. Find all points on the curve

$$y = x^2 - 5x + 3$$

where the slope is 1.

88. Repeat Problem 87 for the curve

$$y = x^4 - 31x + 11$$

89. If $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, evaluate the expression

$$\frac{x - 1}{2x\sqrt{x}} - f'(x)$$

90. Economics Eswaran and Kotwal[2] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks such as weeding, harvesting, and threshing. The difference z in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1 + b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a constant, and w_p is a function of w_c. Eswaran and Kotwal claim that

$$\frac{dz}{dw_c} = (1 + b)\left[\frac{dw_p}{dw_c} - \frac{b}{1 + b}\right]$$

Verify this.

91. Find an equation of the tangent line to the graph of $y = x^3 - 2x + 1$ at the point $(1, 0)$. Graph both the function and the tangent line on the same screen.

92. Find an equation of the tangent line to the graph of $y = \sqrt[3]{x}$, at the point $(-8, -2)$. Graph both the function and the tangent line on the same screen. Notice that the line passes through $(-8, -2)$ and the line appears to be tangent to the curve.

Objective

To motivate the instantaneous rate of change of a function by means of velocity and to interpret the derivative as an instantaneous rate of change. To develop the "marginal" concept, which is frequently used in business and economics.

11.3 The Derivative as a Rate of Change

We have given a geometric interpretation of the derivative as being the slope of the tangent line to a curve at a point. Historically, an important application of the derivative involves the motion of an object traveling in a straight line. This gives us a convenient way to interpret the derivative as a *rate of change*.

To denote the change in a variable such as x, the symbol Δx (read "delta x") is commonly used. For example, if x changes from 1 to 3, then the change in x is $\Delta x = 3 - 1 = 2$. The new value of $x(=3)$ is the old value plus the change, which is $1 + \Delta x$. Similarly, if t increases by Δt, the new value is $t + \Delta t$. We will use Δ-notation in the discussion that follows.

Suppose an object moves along the number line in Figure 11.12 according to the equation

$$s = f(t) = t^2$$

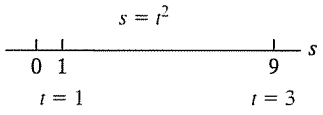


FIGURE 11.12 Motion along a number line.

where s is the position of the object at time t. This equation is called an **equation of motion,** and f is called a ***position function.*** Assume that t is in seconds and s is in meters. At $t = 1$ the position is $s = f(1) = 1^2 = 1$, and at $t = 3$ the position is $s = f(3) = 3^2 = 9$. Over this two-second time interval, the object has a change in position, or a *displacement,* of $9 - 1 = 8$ m, and the *average velocity* of the object is defined as

$$v_{\text{ave}} = \frac{\text{displacement}}{\text{length of time interval}} \tag{1}$$

$$= \frac{8}{2} = 4 \text{ m/s}$$

To say that the average velocity is 4 m/s from $t = 1$ to $t = 3$ means that, *on the average,* the position of the object changed by 4 m to the right each second during that time interval. Let us denote the changes in s-values and t-values by Δs and Δt, respectively. Then the average velocity is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = 4 \text{ m/s} \quad \text{(for the interval } t = 1 \text{ to } t = 3)$$

The ratio $\Delta s/\Delta t$ is also called the **average rate of change of s with respect to t** over the interval from $t = 1$ to $t = 3$.

Now, let the time interval be only 1 second long (that is, $\Delta t = 1$). Then, for the *shorter* interval from $t = 1$ to $t = 1 + \Delta t = 2$, we have $f(2) = 2^2 = 4$, so

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(2) - f(1)}{\Delta t} = \frac{4 - 1}{1} = 3 \text{ m/s}$$

[2] M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economies," *The American Economic Review,* 75, no. 1 (1985), 162–77.

Table 11.2

Length of Time Interval Δt	Time Interval $t = 1$ to $t = 1 + \Delta t$	Average Velocity $\frac{\Delta s}{\Delta t} = \frac{f(1+\Delta t) - f(1)}{\Delta t}$
0.1	$t = 1$ to $t = 1.1$	2.1 m/s
0.07	$t = 1$ to $t = 1.07$	2.07 m/s
0.05	$t = 1$ to $t = 1.05$	2.05 m/s
0.03	$t = 1$ to $t = 1.03$	2.03 m/s
0.01	$t = 1$ to $t = 1.01$	2.01 m/s
0.001	$t = 1$ to $t = 1.001$	2.001 m/s

More generally, over the time interval from $t = 1$ to $t = 1 + \Delta t$, the object moves from position $f(1)$ to position $f(1 + \Delta t)$. Thus, its displacement is

$$\Delta s = f(1 + \Delta t) - f(1)$$

Since the time interval has length Δt, the object's average velocity is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(1 + \Delta t) - f(1)}{\Delta t}$$

If Δt were to become smaller and smaller, the average velocity over the interval from $t = 1$ to $t = 1 + \Delta t$ would be close to what we might call the *instantaneous velocity* at time $t = 1$; that is, the velocity at a *point* in time ($t = 1$) as opposed to the velocity over an *interval* of time. For some typical values of Δt between 0.1 and 0.001, we get the average velocities in Table 11.2, which the reader can verify.

The table suggests that as the length of the time interval approaches zero, the average velocity approaches the value 2 m/s. In other words, as Δt approaches 0, $\Delta s/\Delta t$ approaches 2 m/s. We define the limit of the average velocity as $\Delta t \to 0$ to be the **instantaneous velocity** (or simply the **velocity**), v, at time $t = 1$. This limit is also called the **instantaneous rate of change** of s with respect to t at $t = 1$:

$$v = \lim_{\Delta t \to 0} v_{\text{ave}} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(1 + \Delta t) - f(1)}{\Delta t}$$

If we think of Δt as h, then the limit on the right is simply the derivative of s with respect to t at $t = 1$. Thus, the instantaneous velocity of the object at $t = 1$ is just ds/dt at $t = 1$. Because $s = t^2$ and

$$\frac{ds}{dt} = 2t$$

the velocity at $t = 1$ is

$$v = \left.\frac{ds}{dt}\right|_{t=1} = 2(1) = 2 \text{ m/s}$$

which confirms our previous conclusion.

In summary, if $s = f(t)$ is the position function of an object moving in a straight line, then the average velocity of the object over the time interval $[t, t + \Delta t]$ is given by

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

and the velocity at time t is given by

$$v = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \frac{ds}{dt}$$

Selectively combining equations for v, we have

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

which provides motivation for the otherwise bizarre Leibniz notation. (After all, Δ is the [uppercase] Greek letter corresponding to d.)

EXAMPLE 1 Finding Average Velocity and Velocity

Suppose the position function of an object moving along a number line is given by $s = f(t) = 3t^2 + 5$, where t is in seconds and s is in meters.

a. Find the average velocity over the interval [10, 10.1].

b. Find the velocity when $t = 10$.

Solution:

a. Here $t = 10$ and $\Delta t = 10.1 - 10 = 0.1$. So we have

$$\begin{aligned} v_{\text{ave}} &= \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} \\ &= \frac{f(10+0.1) - f(10)}{0.1} \\ &= \frac{f(10.1) - f(10)}{0.1} \\ &= \frac{311.03 - 305}{0.1} = \frac{6.03}{0.1} = 60.3 \text{ m/s} \end{aligned}$$

b. The velocity at time t is given by

$$v = \frac{ds}{dt} = 6t$$

When $t = 10$, the velocity is

$$\left.\frac{ds}{dt}\right|_{t=10} = 6(10) = 60 \text{ m/s}$$

Notice that the average velocity over the interval [10, 10.1] is close to the velocity at $t = 10$. This is to be expected because the length of the interval is small.

Now Work Problem 1 ◁

Our discussion of the rate of change of s with respect to t applies equally well to *any* function $y = f(x)$. This means that we have the following:

If $y = f(x)$, then

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \begin{cases} \text{average rate of change} \\ \text{of } y \text{ with respect to } x \\ \text{over the interval from} \\ x \text{ to } x + \Delta x \end{cases}$$

and

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \begin{cases} \text{instantaneous rate of change} \\ \text{of } y \text{ with respect to } x \end{cases} \tag{2}$$

Because the instantaneous rate of change of $y = f(x)$ at a point is a derivative, it is also the *slope of the tangent line* to the graph of $y = f(x)$ at that point. For convenience, we usually refer to the instantaneous rate of change simply as the **rate of change.** The interpretation of a derivative as a rate of change is extremely important.

Let us now consider the significance of the rate of change of y with respect to x. From Equation (2), if Δx (a change in x) is close to 0, then $\Delta y/\Delta x$ is close to dy/dx. That is,

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx}$$

Therefore,

$$\Delta y \approx \frac{dy}{dx}\Delta x \tag{3}$$

That is, if x changes by Δx, then the change in y, Δy, is approximately dy/dx times the change in x. In particular,

if x changes by 1, an estimate of the change in y is $\dfrac{dy}{dx}$

APPLY IT ▶

3. Suppose that the profit P made by selling a certain product at a price of p per unit is given by $P = f(p)$ and the rate of change of that profit with respect to change in price is $\dfrac{dP}{dp} = 5$ at $p = 25$. Estimate the change in the profit P if the price changes from 25 to 25.5.

EXAMPLE 2 Estimating Δy by Using dy/dx

Suppose that $y = f(x)$ and $\dfrac{dy}{dx} = 8$ when $x = 3$. Estimate the change in y if x changes from 3 to 3.5.

Solution: We have $dy/dx = 8$ and $\Delta x = 3.5 - 3 = 0.5$. The change in y is given by Δy, and, from Equation (3),

$$\Delta y \approx \frac{dy}{dx}\Delta x = 8(0.5) = 4$$

We remark that, since $\Delta y = f(3.5) - f(3)$, we have $f(3.5) = f(3) + \Delta y$. For example, if $f(3) = 5$, then $f(3.5)$ can be estimated by $5 + 4 = 9$.

◁

APPLY IT ▶

4. The position of an object thrown upward at a speed of 16 feet/s from a height of 0 feet is given by $y(t) = 16t - 16t^2$. Find the rate of change of y with respect to t, and evaluate it when $t = 0.5$. Use your graphing calculator to graph $y(t)$. Use the graph to interpret the behavior of the object when $t = 0.5$.

EXAMPLE 3 Finding a Rate of Change

Find the rate of change of $y = x^4$ with respect to x, and evaluate it when $x = 2$ and when $x = -1$. Interpret your results.

Solution: The rate of change is

$$\frac{dy}{dx} = 4x^3$$

When $x = 2, dy/dx = 4(2)^3 = 32$. This means that if x increases, from 2, by a small amount, then y increases approximately 32 times as much. More simply, we say that, when $x = 2$, y is increasing 32 times as fast as x does. When $x = -1$, $dy/dx = 4(-1)^3 = -4$. The significance of the minus sign on -4 is that, when $x = -1$, y is *decreasing* 4 times as fast as x increases.

Now Work Problem 11 ◁

EXAMPLE 4 Rate of Change of Price with Respect to Quantity

Let $p = 100 - q^2$ be the demand function for a manufacturer's product. Find the rate of change of price p per unit with respect to quantity q. How fast is the price changing with respect to q when $q = 5$? Assume that p is in dollars.

Solution: The rate of change of p with respect to q is

$$\frac{dp}{dq} = \frac{d}{dq}(100 - q^2) = -2q$$

Thus,

$$\left.\frac{dp}{dq}\right|_{q=5} = -2(5) = -10$$

This means that when five units are demanded, an *increase* of one extra unit demanded corresponds to a decrease of approximately \$10 in the price per unit that consumers are willing to pay.

◁

EXAMPLE 5 **Rate of Change of Volume**

A spherical balloon is being filled with air. Find the rate of change of the volume of air in the balloon with respect to its radius. Evaluate this rate of change when the radius is 2 ft.

Solution: The formula for the volume V of a ball of radius r is $V = \frac{4}{3}\pi r^3$. The rate of change of V with respect to r is

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

When $r = 2$ ft, the rate of change is

$$\left.\frac{dV}{dr}\right|_{r=2} = 4\pi(2)^2 = 16\pi \frac{\text{ft}^3}{\text{ft}}$$

This means that when the radius is 2 ft, changing the radius by 1 ft will change the volume by approximately 16π ft^3.

◁

EXAMPLE 6 **Rate of Change of Enrollment**

A sociologist is studying various suggested programs that can aid in the education of preschool-age children in a certain city. The sociologist believes that x years after the beginning of a particular program, $f(x)$ thousand preschoolers will be enrolled, where

$$f(x) = \frac{10}{9}(12x - x^2) \quad 0 \le x \le 12$$

At what rate would enrollment change (a) after three years from the start of this program and (b) after nine years?

Solution: The rate of change of $f(x)$ is

$$f'(x) = \frac{10}{9}(12 - 2x)$$

a. After three years, the rate of change is

$$f'(3) = \frac{10}{9}(12 - 2(3)) = \frac{10}{9} \cdot 6 = \frac{20}{3} = 6\frac{2}{3}$$

Thus, enrollment would be increasing at the rate of $6\frac{2}{3}$ thousand preschoolers per year.

b. After nine years, the rate is

$$f'(9) = \frac{10}{9}(12 - 2(9)) = \frac{10}{9}(-6) = -\frac{20}{3} = -6\frac{2}{3}$$

Thus, enrollment would be *decreasing* at the rate of $6\frac{2}{3}$ thousand preschoolers per year.

Now Work Problem 9 ◁

Applications of Rate of Change to Economics

A manufacturer's **total-cost function,** $c = f(q)$, gives the total cost c of producing and marketing q units of a product. The rate of change of c with respect to q is called the **marginal cost.** Thus,

$$\text{marginal cost} = \frac{dc}{dq}$$

For example, suppose $c = f(q) = 0.1q^2 + 3$ is a cost function, where c is in dollars and q is in pounds. Then

$$\frac{dc}{dq} = 0.2q$$

The marginal cost when 4 lb are produced is dc/dq, evaluated when $q = 4$:

$$\left.\frac{dc}{dq}\right|_{q=4} = 0.2(4) = 0.80$$

This means that if production is increased by 1 lb, from 4 lb to 5 lb, then the change in cost is approximately \$0.80. That is, the additional pound costs about \$0.80. In general, *we interpret marginal cost as the approximate cost of one additional unit of output.* After all, the difference $f(q+1) - f(q)$ can be seen as a difference quotient

$$\frac{f(q+1) - f(q)}{1}$$

(the case where $h = 1$). Any difference quotient can be regarded as an approximation of the corresponding derivative and, conversely, any derivative can be regarded as an approximation of any of its corresponding difference quotients. Thus, for any function f of q we can always regard $f'(q)$ and $f(q+1) - f(q)$ as approximations of each other. In economics, the latter can usually be regarded as the exact value of the cost, or profit depending upon the function, of the $(q+1)$th item when q are produced. The derivative is often easier to compute than the exact value. [In the case at hand, the actual cost of producing one more pound beyond 4 lb is $f(5) - f(4) = 5.5 - 4.6 = \$0.90$.]

If c is the total cost of producing q units of a product, then the **average cost per unit,** $\bar{c}$, is

$$\bar{c} = \frac{c}{q} \tag{4}$$

For example, if the total cost of 20 units is \$100, then the average cost per unit is $\bar{c} = 100/20 = \$5$. By multiplying both sides of Equation (4) by q, we have

$$c = q\bar{c}$$

That is, total cost is the product of the number of units produced and the average cost per unit.

EXAMPLE 7 Marginal Cost

If a manufacturer's average-cost equation is

$$\bar{c} = 0.0001q^2 - 0.02q + 5 + \frac{5000}{q}$$

find the marginal-cost function. What is the marginal cost when 50 units are produced?

Solution:

Strategy The marginal-cost function is the derivative of the total-cost function c. Thus, we first find c by multiplying $\bar{c}$ by q. We have

$$\begin{aligned} c &= q\bar{c} \\ &= q\left(0.0001q^2 - 0.02q + 5 + \frac{5000}{q}\right) \\ c &= 0.0001q^3 - 0.02q^2 + 5q + 5000 \end{aligned}$$

Differentiating c, we have the marginal-cost function:

$$\frac{dc}{dq} = 0.0001(3q^2) - 0.02(2q) + 5(1) + 0$$
$$= 0.0003q^2 - 0.04q + 5$$

The marginal cost when 50 units are produced is

$$\left.\frac{dc}{dq}\right|_{q=50} = 0.0003(50)^2 - 0.04(50) + 5 = 3.75$$

If c is in dollars and production is increased by one unit, from $q = 50$ to $q = 51$, then the cost of the additional unit is approximately \$3.75. If production is increased by $\frac{1}{3}$ unit, from $q = 50$, then the cost of the additional output is approximately $\left(\frac{1}{3}\right)(3.75) = \$1.25$.

Now Work Problem 21 ◁

Suppose $r = f(q)$ is the **total-revenue function** for a manufacturer. The equation $r = f(q)$ states that the total dollar value received for selling q units of a product is r. The **marginal revenue** is defined as the rate of change of the total dollar value received with respect to the total number of units sold. Hence, marginal revenue is merely the derivative of r with respect to q:

$$\text{marginal revenue} = \frac{dr}{dq}$$

Marginal revenue indicates the rate at which revenue changes with respect to units sold. We interpret it as *the approximate revenue received from selling one additional unit of output.*

EXAMPLE 8 **Marginal Revenue**

Suppose a manufacturer sells a product at \$2 per unit. If q units are sold, the total revenue is given by

$$r = 2q$$

The marginal-revenue function is

$$\frac{dr}{dq} = \frac{d}{dq}(2q) = 2$$

which is a constant function. Thus, the marginal revenue is 2 regardless of the number of units sold. This is what we would expect, because the manufacturer receives \$2 for each unit sold.

Now Work Problem 23 ◁

Relative and Percentage Rates of Change

For the total-revenue function in Example 8, namely, $r = f(q) = 2q$, we have

$$\frac{dr}{dq} = 2$$

This means that revenue is changing at the rate of \$2 per unit, regardless of the number of units sold. Although this is valuable information, it may be more significant when compared to r itself. For example, if $q = 50$, then $r = 2(50) = 100$. Thus, the rate of change of revenue is $2/100 = 0.02$ *of* r. On the other hand, if $q = 5000$, then $r = 2(5000) = \$10{,}000$, so the rate of change of r is $2/10{,}000 = 0.0002$ *of* r.

Although r changes at the same rate at each level, compared to r itself, this rate is relatively smaller when $r = 10{,}000$ than when $r = 100$. By considering the ratio

$$\frac{dr/dq}{r}$$

we have a means of comparing the rate of change of r with r itself. This ratio is called the *relative rate of change* of r. We have shown that the relative rate of change when $q = 50$ is

$$\frac{dr/dq}{r} = \frac{2}{100} = 0.02$$

and when $q = 5000$, it is

$$\frac{dr/dq}{r} = \frac{2}{10{,}000} = 0.0002$$

CAUTION!

Percentages can be confusing! Remember that *percent* means "per hundred." Thus $100\% = \frac{100}{100} = 1$, $2\% = \frac{2}{100} = 0.02$, and so on.

By multiplying relative rates by 100%, we obtain the so-called *percentage rates of change*. The percentage rate of change when $q = 50$ is $(0.02)(100\%) = 2\%$; when $q = 5000$, it is $(0.0002)(100\%) = 0.02\%$. For example, if an additional unit beyond 50 is sold, then revenue increases by approximately 2%.

In general, for any function f, we have the following definition:

Definition

The ***relative rate of change*** of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

The ***percentage rate of change*** of $f(x)$ is

$$\frac{f'(x)}{f(x)} \cdot 100\%$$

APPLY IT ▶

5. The volume V enclosed by a capsule-shaped container with a cylindrical height of 4 feet and radius r is given by

$$V(r) = \frac{4}{3}\pi r^3 + 4\pi r^2$$

Determine the relative and percentage rates of change of volume with respect to the radius when the radius is 2 feet.

EXAMPLE 9 Relative and Percentage Rates of Change

Determine the relative and percentage rates of change of

$$y = f(x) = 3x^2 - 5x + 25$$

when $x = 5$.

Solution: Here

$$f'(x) = 6x - 5$$

Since $f'(5) = 6(5) - 5 = 25$ and $f(5) = 3(5)^2 - 5(5) + 25 = 75$, the relative rate of change of y when $x = 5$ is

$$\frac{f'(5)}{f(5)} = \frac{25}{75} \approx 0.333$$

Multiplying 0.333 by 100% gives the percentage rate of change: $(0.333)(100) = 33.3\%$.

Now Work Problem 35 ◁

PROBLEMS 11.3

1. Suppose that the position function of an object moving along a straight line is $s = f(t) = 2t^2 + 3t$, where t is in seconds and s is in meters. Find the average velocity $\Delta s/\Delta t$ over the interval $[1, 1 + \Delta t]$, where Δt is given in the following table:

Δt	1	0.5	0.2	0.1	0.01	0.001
$\Delta s/\Delta t$						

From your results, estimate the velocity when $t = 1$. Verify your estimate by using differentiation.

2. If $y = f(x) = \sqrt{2x+5}$, find the average rate of change of y with respect to x over the interval $[3, 3+\Delta x]$, where Δx is given in the following table:

Δx	1	0.5	0.2	0.1	0.01	0.001
$\Delta y/\Delta x$						

From your result, estimate the rate of change of y with respect to x when $x = 3$.

In each of Problems 3–8, a position function is given, where t is in seconds and s is in meters.
(a) *Find the position at the given t-value.*
(b) *Find the average velocity over the given interval.*
(c) *Find the velocity at the given t-value.*

3. $s = 2t^2 - 4t; [7, 7.5]; t = 7$

4. $s = \frac{1}{2}t + 1; [2, 2.1]; t = 2$

5. $s = 5t^3 + 3t + 24; [1, 1.01]; t = 1$

6. $s = -3t^2 + 2t + 1; [1, 1.25]; t = 1$

7. $s = t^4 - 2t^3 + t; [2, 2.1]; t = 2$

8. $s = 3t^4 - t^{7/2}; [0, \frac{1}{4}]; t = 0$

9. Income–Education Sociologists studied the relation between income and number of years of education for members of a particular urban group. They found that a person with x years of education before seeking regular employment can expect to receive an average yearly income of y dollars per year, where

$$y = 5x^{5/2} + 5900 \qquad 4 \le x \le 16$$

Find the rate of change of income with respect to number of years of education. Evaluation the expression when $x = 9$.

10. Find the rate of change of the volume V of a ball, with respect to its radius r, when $r = 1.5$ m. The volume V of a ball as a function of its radius r is given by

$$V = V(r) = \frac{4}{3}\pi r^3$$

11. Skin Temperature The approximate temperature T of the skin in terms of the temperature T_e of the environment is given by

$$T = 32.8 + 0.27(T_e - 20)$$

where T and T_e are in degrees Celsius.[3] Find the rate of change of T with respect to T_e.

12. Biology The volume V of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. Find the rate of change of volume with respect to the radius when $r = 6.3 \times 10^{-4}$ cm.

In Problems 13–18, cost functions are given, where c is the cost of producing q units of a product. In each case, find the marginal-cost function. What is the marginal cost at the given value(s) of q?

13. $c = 500 + 10q; q = 100$

14. $c = 5000 + 6q; q = 36$

15. $c = 0.2q^2 + 4q + 50; q = 10$

16. $c = 0.1q^2 + 3q + 2; q = 3$

17. $c = q^2 + 50q + 1000; q = 15, q = 16, q = 17$

18. $c = 0.04q^3 - 0.5q^2 + 4.4q + 7500; q = 5, q = 25, q = 1000$

In Problems 19–22, $\bar{c}$ represents average cost per unit, which is a function of the number q of units produced. Find the marginal-cost function and the marginal cost for the indicated values of q.

19. $\bar{c} = 0.01q + 5 + \dfrac{500}{q}; q = 50, q = 100$

20. $\bar{c} = 5 + \dfrac{2000}{q}; q = 25, q = 250$

21. $\bar{c} = 0.00002q^2 - 0.01q + 6 + \dfrac{20{,}000}{q}; q = 100, q = 500$

22. $\bar{c} = 0.002q^2 - 0.5q + 60 + \dfrac{7000}{q}; q = 15, q = 25$

In Problems 23–26, r represents total revenue and is a function of the number q of units sold. Find the marginal-revenue function and the marginal revenue for the indicated values of q.

23. $r = 0.8q; q = 9, q = 300, q = 500$

24. $r = q\left(15 - \frac{1}{30}q\right); q = 5, q = 15, q = 150$

25. $r = 240q + 40q^2 - 2q^3; q = 10; q = 15; q = 20$

26. $r = 2q(30 - 0.1q); q = 10, q = 20$

27. Hosiery Mill The total-cost function for a hosiery mill is estimated by Dean[4] to be

$$c = -10{,}484.69 + 6.750q - 0.000328q^2$$

where q is output in dozens of pairs and c is total cost in dollars. Find the marginal-cost function and the average cost function and evaluate each when $q = 2000$.

28. Light and Power Plant The total-cost function for an electric light and power plant is estimated by Nordin[5] to be

$$c = 32.07 - 0.79q + 0.02142q^2 - 0.0001q^3 \qquad 20 \le q \le 90$$

where q is the eight-hour total output (as a percentage of capacity) and c is the total fuel cost in dollars. Find the marginal-cost function and evaluate it when $q = 70$.

29. Urban Concentration Suppose the 100 largest cities in the United States in 1920 are ranked according to magnitude (areas of cities). From Lotka,[6] the following relation holds approximately:

$$PR^{0.93} = 5{,}000{,}000$$

Here, P is the population of the city having respective rank R. This relation is called the *law of urban concentration* for 1920. Solve for P in terms of R, and then find how fast the population is changing with respect to rank.

30. Depreciation Under the straight-line method of depreciation, the value v of a certain machine after t years have elapsed is given by

$$v = 120{,}000 - 15{,}500t$$

where $0 \le t \le 6$. How fast is v changing with respect to t when $t = 2$? $t = 4$? at any time?

---

[3] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[4] J. Dean, "Statistical Cost Functions of a Hosiery Mill," *Studies in Business Administration*, XI, no. 4 (Chicago: University of Chicago Press, 1941).

[5] J. A. Nordin, "Note on a Light Plant's Cost Curves," *Econometrica*, 15 (1947), 231–35.

[6] A. J. Lotka, *Elements of Mathematical Biology* (New York: Dover Publications, Inc., 1956).

31. Winter Moth A study of the winter moth was made in Nova Scotia (adapted from Embree).[7] The prepupae of the moth fall onto the ground from host trees. At a distance of x ft from the base of a host tree, the prepupal density (number of prepupae per square foot of soil) was y, where

$$y = 59.3 - 1.5x - 0.5x^2 \quad 1 \le x \le 9$$

(a) At what rate is the prepupal density changing with respect to distance from the base of the tree when $x = 6$?
(b) For what value of x is the prepupal density decreasing at the rate of 6 prepupae per square foot per foot?

32. Cost Function For the cost function

$$c = 0.4q^2 + 4q + 5$$

find the rate of change of c with respect to q when $q = 2$. Also, what is $\Delta c/\Delta q$ over the interval [2, 3]?

In Problems 33–38, find **(a)** *the rate of change of y with respect to x and* **(b)** *the relative rate of change of y. At the given value of x, find* **(c)** *the rate of change of y,* **(d)** *the relative rate of change of y, and* **(e)** *the percentage rate of change of y.*

33. $y = f(x) = x + 4; x = 5$ **34.** $y = f(x) = 7 - 3x; x = 6$

35. $y = 2x^2 + 5; x = 10$ **36.** $y = 5 - 3x^3; x = 1$

37. $y = 8 - x^3; x = 1$ **38.** $y = x^2 + 3x - 4; x = -1$

39. Cost Function For the cost function

$$c = 0.3q^2 + 3.5q + 9$$

how fast does c change with respect to q when $q = 10$? Determine the percentage rate of change of c with respect to q when $q = 10$.

40. Organic Matters/Species Diversity In a discussion of contemporary waters of shallows seas, Odum[8] claims that in such waters the total organic matter y (in milligrams per liter) is a function of species diversity x (in number of species per thousand individuals). If $y = 100/x$, at what rate is the total organic matter changing with respect to species diversity when $x = 10$? What is the percentage rate of change when $x = 10$?

41. Revenue For a certain manufacturer, the revenue obtained from the sale of q units of a product is given by

$$r = 30q - 0.3q^2$$

(a) How fast does r change with respect to q? When $q = 10$, (b) find the relative rate of change of r, and (c) to the nearest percent, find the percentage rate of change of r.

42. Revenue Repeat Problem 43 for the revenue function given by $r = 10q - 0.2q^2$ and $q = 25$.

43. Weight of Limb The weight of a limb of a tree is given by $W = 2t^{0.432}$, where t is time. Find the relative rate of change of W with respect to t.

44. Response to Shock A psychological experiment[9] was conducted to analyze human responses to electrical shocks (stimuli). The subjects received shocks of various intensities. The response R to a shock of intensity I (in microamperes) was to be a number that indicated the perceived magnitude relative to that of a "standard" shock. The standard shock was assigned a magnitude of 10. Two groups of subjects were tested under slightly different conditions. The responses R_1 and R_2 of the first and second groups to a shock of intensity I were given by

$$R_1 = \frac{I^{1.3}}{1855.24} \quad 800 \le I \le 3500$$

and

$$R_2 = \frac{I^{1.3}}{1101.29} \quad 800 \le I \le 3500$$

(a) For each group, determine the relative rate of change of response with respect to intensity.
(b) How do these changes compare with each other?
(c) In general, if $f(x) = C_1x^n$ and $g(x) = C_2x^n$, where C_1 and C_2 are constants, how do the relative rates of change of f and g compare?

45. Cost A manufacturer of mountain bikes has found that when 20 bikes are produced per day, the average cost is \$200 and the marginal cost is \$150. Based on that information, approximate the total cost of producing 21 bikes per day.

46. Marginal and Average Costs Suppose that the cost function for a certain product is $c = f(q)$. If the relative rate of change of c (with respect to q) is $\frac{1}{q}$, prove that the marginal-cost function and the average-cost function are equal.

In Problems 47 and 48, use the numerical derivative feature of your graphing calculator.

47. If the total-cost function for a manufacturer is given by

$$c = \frac{5q^2}{\sqrt{q^2 + 3}} + 5000$$

where c is in dollars, find the marginal cost when 10 units are produced. Round your answer to the nearest cent.

48. The population of a city t years from now is given by

$$P = 250{,}000e^{0.04t}$$

Find the rate of change of population with respect to time t three years from now. Round your answer to the nearest integer.

Objective

To find derivatives by applying the product and quotient rules, and to develop the concepts of marginal propensity to consume and marginal propensity to save.

11.4 The Product Rule and the Quotient Rule

The equation $F(x) = (x^2 + 3x)(4x + 5)$ expresses $F(x)$ as a product of two functions: $x^2 + 3x$ and $4x + 5$. To find $F'(x)$ by using only our previous rules, we first multiply

[7] D. G. Embree, "The Population Dynamics of the Winter Moth in Nova Scotia, 1954–1962," *Memoirs of the Entomological Society of Canada*, no. 46 (1965).

[8] H. T. Odum, "Biological Circuits and the Marine Systems of Texas," in *Pollution and Marine Biology*, eds T. A. Olsen and F. J. Burgess (New York: Interscience Publishers, 1967).

[9] H. Babkoff, "Magnitude Estimation of Short Electrocutaneous Pulses," *Psychological Research*, 39, no. 1 (1976), 39–49.

the functions. Then we differentiate the result, term by term:

$$F(x) = (x^2 + 3x)(4x + 5) = 4x^3 + 17x^2 + 15x$$
$$F'(x) = 12x^2 + 34x + 15 \tag{1}$$

However, in many problems that involve differentiating a product of functions, the multiplication is not as simple as it is here. At times, it is not even practical to attempt it. Fortunately, there is a rule for differentiating a product, and the rule avoids such multiplications. Since the derivative of a sum of functions is the sum of their derivatives, you might expect a similar rule for products. However, the situation is rather subtle.

COMBINING RULE 3 The Product Rule

If f and g are differentiable functions, then the product fg is differentiable, and

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

That is, the derivative of the product of two functions is the derivative of the first function times the second, plus the first function times the derivative of the second.

$$\frac{d}{dx}(\text{product}) = \begin{pmatrix}\text{derivative}\\ \text{of first}\end{pmatrix}(\text{second}) + (\text{first})\begin{pmatrix}\text{derivative}\\ \text{of second}\end{pmatrix}$$

Proof. If $F(x) = f(x)g(x)$, then, by the definition of the derivative of F,

$$F'(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now we use a "trick." Adding and subtracting $f(x)g(x+h)$ in the numerator, we have

$$F'(x) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x) + f(x)g(x+h) - f(x)g(x+h)}{h}$$

Regrouping gives

$$F'(x) = \lim_{h\to 0}\frac{(f(x+h)g(x+h) - f(x)g(x+h)) + (f(x)g(x+h) - f(x)g(x))}{h}$$
$$= \lim_{h\to 0}\frac{(f(x+h) - f(x))g(x+h) + f(x)(g(x+h) - g(x))}{h}$$
$$= \lim_{h\to 0}\frac{(f(x+h) - f(x))g(x+h)}{h} + \lim_{h\to 0}\frac{f(x)(g(x+h) - g(x))}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\cdot\lim_{h\to 0}g(x+h) + \lim_{h\to 0}f(x)\cdot\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

Since we assumed that f and g are differentiable,

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = f'(x)$$

and

$$\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = g'(x)$$

The differentiability of g implies that g is continuous, so, from Section 10.3,

$$\lim_{h\to 0}g(x+h) = g(x)$$

Thus,

$$F'(x) = f'(x)g(x) + f(x)g'(x)$$

EXAMPLE 1 **Applying the Product Rule**

If $F(x) = (x^2 + 3x)(4x + 5)$, find $F'(x)$.

Solution: We will consider F as a product of two functions:

$$F(x) = \underbrace{(x^2 + 3x)}_{f(x)}\underbrace{(4x + 5)}_{g(x)}$$

Therefore, we can apply the product rule:

$$\begin{aligned} F'(x) &= f'(x)g(x) + f(x)g'(x) \\ &= \underbrace{\frac{d}{dx}(x^2 + 3x)}_{\substack{\text{Derivative} \\ \text{of first}}}\underbrace{(4x + 5)}_{\text{Second}} + \underbrace{(x^2 + 3x)}_{\text{First}}\underbrace{\frac{d}{dx}(4x + 5)}_{\substack{\text{Derivative} \\ \text{of second}}} \\ &= (2x + 3)(4x + 5) + (x^2 + 3x)(4) \\ &= 12x^2 + 34x + 15 \qquad \text{simplifying} \end{aligned}$$

This agrees with our previous result. [See Equation (1).] Although there doesn't seem to be much advantage to using the product rule here, there are times when it is impractical to avoid it.

Now Work Problem 1 ◁

CAUTION!

It is worthwhile to repeat that the derivative of the product of two functions is somewhat subtle. Do not be tempted to make up a simpler rule.

EXAMPLE 2 **Applying the Product Rule**

If $y = (x^{2/3} + 3)(x^{-1/3} + 5x)$, find dy/dx.

Solution: Applying the product rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^{2/3} + 3)(x^{-1/3} + 5x) + (x^{2/3} + 3)\frac{d}{dx}(x^{-1/3} + 5x) \\ &= \left(\frac{2}{3}x^{-1/3}\right)(x^{-1/3} + 5x) + (x^{2/3} + 3)\left(\frac{-1}{3}x^{-4/3} + 5\right) \\ &= \frac{25}{3}x^{2/3} + \frac{1}{3}x^{-2/3} - x^{-4/3} + 15 \end{aligned}$$

Alternatively, we could have found the derivative without the product rule by first finding the product $(x^{2/3} + 3)(x^{-1/3} + 5x)$ and then differentiating the result, term by term.

Now Work Problem 15 ◁

APPLY IT ▷

6. A taco stand usually sells 225 tacos per day at \$2 each. A business student's research tells him that for every \$0.15 decrease in the price, the stand will sell 20 more tacos per day. The revenue function for the taco stand is $R(x) = (2 - 0.15x)(225 + 20x)$, where x is the number of \$0.15 reductions in price. Find $\frac{dR}{dx}$.

EXAMPLE 3 **Differentiating a Product of Three Factors**

If $y = (x + 2)(x + 3)(x + 4)$, find y'.

Solution:

Strategy We would like to use the product rule, but as given it applies only to *two* factors. By treating the first two factors as a single factor, we can consider y to be a product of two functions:

$$y = [(x + 2)(x + 3)](x + 4)$$

The product rule gives

$$\begin{aligned} y' &= \frac{d}{dx}[(x + 2)(x + 3)](x + 4) + [(x + 2)(x + 3)]\frac{d}{dx}(x + 4) \\ &= \frac{d}{dx}[(x + 2)(x + 3)](x + 4) + [(x + 2)(x + 3)](1) \end{aligned}$$

Applying the product rule again, we have

$$y' = \left(\frac{d}{dx}(x+2)(x+3) + (x+2)\frac{d}{dx}(x+3)\right)(x+4) + (x+2)(x+3)$$
$$= [(1)(x+3) + (x+2)(1)](x+4) + (x+2)(x+3)$$

After simplifying, we obtain

$$y' = 3x^2 + 18x + 26$$

Two other ways of finding the derivative are as follows:

1. Multiply the first two factors of y to obtain
$$y = (x^2 + 5x + 6)(x+4)$$
and then apply the product rule.
2. Multiply all three factors to obtain
$$y = x^3 + 9x^2 + 26x + 24$$
and then differentiate term by term.

Now Work Problem 19 ◁

It is sometimes helpful to remember differentiation rules in more streamlined notation. For example,

$$(fg)' = f'g + fg'$$

is a correct equality of functions that expresses the product rule. We can then calculate

$$(fgh)' = ((fg)h)'$$
$$= (fg)'h + (fg)h'$$
$$= (f'g + fg')h + (fg)h'$$
$$= f'gh + fg'h + fgh'$$

It is not suggested that you try to commit to memory derived rules like

$$(fgh)' = f'gh + fg'h + fgh'$$

Because $f'g + fg' = gf' + fg'$, using commutativity of the product of functions, we can express the product rule with the derivatives as second factors:

$$(fg)' = gf' + fg'$$

and using commutativity of addition

$$(fg)' = fg' + gf'$$

Some people prefer these forms.

APPLY IT ▶

7. One hour after x milligrams of a particular drug are given to a person, the change in body temperature $T(x)$, in degrees Fahrenheit, is given approximately by $T(x) = x^2\left(1 - \frac{x}{3}\right)$. The rate at which T changes with respect to the size of the dosage x, $T'(x)$, is called the *sensitivity* of the body to the dosage. Find the sensitivity when the dosage is 1 milligram. Do not use the product rule.

EXAMPLE 4 Using the Product Rule to Find Slope

Find the slope of the graph of $f(x) = (7x^3 - 5x + 2)(2x^4 + 7)$ when $x = 1$.

Solution:

Strategy We find the slope by evaluating the derivative when $x = 1$. Because f is a product of two functions, we can find the derivative by using the product rule.

We have

$$f'(x) = (7x^3 - 5x + 2)\frac{d}{dx}(2x^4 + 7) + (2x^4 + 7)\frac{d}{dx}(7x^3 - 5x + 2)$$
$$= (7x^3 - 5x + 2)(8x^3) + (2x^4 + 7)(21x^2 - 5)$$

Since we must compute $f'(x)$ when $x = 1$, *there is no need to simplify* $f'(x)$ *before evaluating it.* Substituting into $f'(x)$, we obtain

$$f'(1) = 4(8) + 9(16) = 176$$

Now Work Problem 49 ◁

Usually, we do not use the product rule when simpler ways are obvious. For example, if $f(x) = 2x(x + 3)$, then it is quicker to write $f(x) = 2x^2 + 6x$, from which $f'(x) = 4x + 6$. Similarly, we do not usually use the product rule to differentiate $y = 4(x^2 - 3)$. Since the 4 is a constant factor, by the constant-factor rule we have $y' = 4(2x) = 8x$.

The product rule (and quotient rule that follows) should not be applied when a more direct and efficient method is available.

The next rule is used for differentiating a *quotient* of two functions.

COMBINING RULE 4 The Quotient Rule

If f and g are differentiable functions and $g(x) \neq 0$, then the quotient f/g is also differentiable, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

With the understanding about the denominator not being zero, we can write

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

That is, the derivative of the quotient of two functions is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}(\text{quotient}) = \frac{(\text{denominator})\left(\begin{array}{c}\text{derivative}\\ \text{of numerator}\end{array}\right) - (\text{numerator})\left(\begin{array}{c}\text{derivative}\\ \text{of denominator}\end{array}\right)}{(\text{denominator})^2}$$

Proof. If $F(x) = \dfrac{f(x)}{g(x)}$, then

$$F(x)g(x) = f(x)$$

By the product rule,

$$F(x)g'(x) + g(x)F'(x) = f'(x)$$

Solving for $F'(x)$, we have

$$F'(x) = \frac{f'(x) - F(x)g'(x)}{g(x)}$$

But $F(x) = f(x)/g(x)$. Thus,

$$F'(x) = \frac{f'(x) - \dfrac{f(x)g'(x)}{g(x)}}{g(x)}$$

CAUTION!

The derivative of the quotient of two functions is trickier still than the product rule. We must remember where the minus sign goes!

Simplifying gives[10]

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

[10]The proof given assumes the existence of $F'(x)$. However, the rule can be proved without this assumption.

EXAMPLE 5 **Applying the Quotient Rule**

If $F(x) = \dfrac{4x^2+3}{2x-1}$, find $F'(x)$.

Solution:

Strategy We recognize F as a quotient, so we can apply the quotient rule.

Let $f(x) = 4x^2 + 3$ and $g(x) = 2x - 1$. Then

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{\overbrace{(2x-1)}^{\text{Denominator}} \overbrace{\frac{d}{dx}(4x^2+3)}^{\substack{\text{Derivative}\\ \text{of numerator}}} - \overbrace{(4x^2+3)}^{\text{Numerator}} \overbrace{\frac{d}{dx}(2x-1)}^{\substack{\text{Derivative of}\\ \text{numerator}}}}{\underbrace{(2x-1)^2}_{\substack{\text{Square of}\\ \text{denominator}}}}$$

$$= \frac{(2x-1)(8x) - (4x^2+3)(2)}{(2x-1)^2}$$

$$= \frac{8x^2 - 8x - 6}{(2x-1)^2} = \frac{2(2x+1)(2x-3)}{(2x-1)^2}$$

Now Work Problem 21 ◁

EXAMPLE 6 **Rewriting before Differentiating**

Differentiate $y = \dfrac{1}{x + \dfrac{1}{x+1}}$.

Solution:

Strategy To simplify the differentiation, we will rewrite the function so that no fraction appears in the denominator.

We have

$$y = \frac{1}{x + \dfrac{1}{x+1}} = \frac{1}{\dfrac{x(x+1)+1}{x+1}} = \frac{x+1}{x^2+x+1}$$

$$\frac{dy}{dx} = \frac{(x^2+x+1)(1) - (x+1)(2x+1)}{(x^2+x+1)^2} \qquad \text{quotient rule}$$

$$= \frac{(x^2+x+1) - (2x^2+3x+1)}{(x^2+x+1)^2}$$

$$= \frac{-x^2-2x}{(x^2+x+1)^2} = -\frac{x^2+2x}{(x^2+x+1)^2}$$

Now Work Problem 45 ◁

Although a function may have the form of a quotient, this does not necessarily mean that the quotient rule must be used to find the derivative. The next example illustrates some typical situations in which, although the quotient rule can be used, a simpler and more efficient method is available.

EXAMPLE 7 **Differentiating Quotients without Using the Quotient Rule**

Differentiate the following functions.

a. $f(x) = \dfrac{2x^3}{5}$

Solution: Rewriting, we have $f(x) = \frac{2}{5}x^3$. By the constant-factor rule,

$$f'(x) = \frac{2}{5}(3x^2) = \frac{6x^2}{5}$$

b. $f(x) = \dfrac{4}{7x^3}$

Solution: Rewriting, we have $f(x) = \frac{4}{7}(x^{-3})$. Thus,

$$f'(x) = \frac{4}{7}(-3x^{-4}) = -\frac{12}{7x^4}$$

c. $f(x) = \dfrac{5x^2 - 3x}{4x}$

Solution: Rewriting, we have $f(x) = \dfrac{1}{4}\left(\dfrac{5x^2 - 3x}{x}\right) = \dfrac{1}{4}(5x - 3)$ for $x \neq 0$. Thus,

$$f'(x) = \frac{1}{4}(5) = \frac{5}{4} \qquad \text{for } x \neq 0$$

Since the function f is not defined for $x = 0$, f' is not defined for $x = 0$ either.

Now Work Problem 17 ◁

CAUTION!

To differentiate $f(x) = \dfrac{1}{x^2 - 2}$, we might be tempted first to rewrite the quotient as $(x^2 - 2)^{-1}$. Currently it would be a mistake to do this because we do not yet have a rule for differentiating that form. In short, we have no choice now but to use the quotient rule. However, in the next section we will develop a rule that allows us to differentiate $(x^2 - 2)^{-1}$ in a direct and efficient way.

EXAMPLE 8 **Marginal Revenue**

If the demand equation for a manufacturer's product is

$$p = \frac{1000}{q + 5}$$

where p is in dollars, find the marginal-revenue function and evaluate it when $q = 45$.

Solution:

Strategy First we must find the revenue function. The revenue r received for selling q units when the price per unit is p is given by

$$\textbf{revenue} = \textbf{(price)(quantity)}; \quad \text{that is,} \quad r = pq$$

Using the demand equation, we will express r in terms of q only. Then we will differentiate to find the marginal-revenue function, dr/dq.

The revenue function is

$$r = \left(\frac{1000}{q + 5}\right)q = \frac{1000q}{q + 5}$$

Thus, the marginal-revenue function is given by

$$\frac{dr}{dq} = \frac{(q + 5)\dfrac{d}{dq}(1000q) - (1000q)\dfrac{d}{dq}(q + 5)}{(q + 5)^2}$$

$$= \frac{(q + 5)(1000) - (1000q)(1)}{(q + 5)^2} = \frac{5000}{(q + 5)^2}$$

and

$$\left.\frac{dr}{dq}\right|_{q=45} = \frac{5000}{(45+5)^2} = \frac{5000}{2500} = 2$$

This means that selling one additional unit beyond 45 results in approximately \$2 more in revenue.

Now Work Problem 59 ◁

Consumption Function

A function that plays an important role in economic analysis is the **consumption function.** The consumption function $C = f(I)$ expresses a relationship between the total national income I and the total national consumption C. Usually, both I and C are expressed in billions of dollars and I is restricted to some interval. The *marginal propensity to consume* is defined as the rate of change of consumption with respect to income. It is merely the derivative of C with respect to I:

$$\text{Marginal propensity to consume} = \frac{dC}{dI}$$

If we assume that the difference between income I and consumption C is savings S, then

$$S = I - C$$

Differentiating both sides with respect to I gives

$$\frac{dS}{dI} = \frac{d}{dI}(I) - \frac{d}{dI}(C) = 1 - \frac{dC}{dI}$$

We define dS/dI as the **marginal propensity to save.** Thus, the marginal propensity to save indicates how fast savings change with respect to income, and

$$\begin{matrix}\text{Marginal propensity}\\ \text{to save}\end{matrix} = 1 - \begin{matrix}\text{Marginal propensity}\\ \text{to consume}\end{matrix}$$

EXAMPLE 9 **Finding Marginal Propensities to Consume and to Save**

If the consumption function is given by

$$C = \frac{5(2\sqrt{I^3}+3)}{I+10}$$

determine the marginal propensity to consume and the marginal propensity to save when $I = 100$.

Solution:

$$\frac{dC}{dI} = 5\left(\frac{(I+10)\frac{d}{dI}(2I^{3/2}+3) - (2\sqrt{I^3}+3)\frac{d}{dI}(I+10)}{(I+10)^2}\right)$$

$$= 5\left(\frac{(I+10)(3I^{1/2}) - (2\sqrt{I^3}+3)(1)}{(I+10)^2}\right)$$

When $I = 100$, the marginal propensity to consume is

$$\left.\frac{dC}{dI}\right|_{I=100} = 5\left(\frac{1297}{12{,}100}\right) \approx 0.536$$

The marginal propensity to save when $I = 100$ is $1 - 0.536 = 0.464$. This means that if a current income of \$100 billion increases by \$1 billion, the nation consumes approximately 53.6% (536/1000) and saves 46.4% (464/1000) of that increase.

Now Work Problem 69 ◁

PROBLEMS 11.4

In Problems 1–48, differentiate the functions.

1. $f(x) = (4x + 1)(6x + 3)$
2. $f(x) = (3x - 1)(7x + 2)$
3. $s(t) = (5 - 3t)(t^3 - 2t^2)$
4. $Q(x) = (x^2 + 3x)(7x^2 - 5)$
5. $f(r) = (3r^2 - 4)(r^2 - 5r + 1)$
6. $C(I) = (2I^2 - 3)(3I^2 - 4I + 1)$
7. $f(x) = x^2(2x^2 - 5)$
8. $f(x) = 3x^3(x^2 - 2x + 2)$
9. $y = (x^2 + 5x - 7)(6x^2 - 5x + 4)$
10. $\phi(x) = (3 - 5x + 2x^2)(2 + x - 4x^2)$
11. $f(w) = (w^2 + 3w - 7)(2w^3 - 4)$
12. $f(x) = (3x - x^2)(3 - x - x^2)$
13. $y = (x^2 - 1)(3x^3 - 6x + 5) - 4(4x^2 + 2x + 1)$
14. $h(x) = 5(x^7 + 4) + 4(5x^3 - 2)(4x^2 + 7x)$
15. $F(p) = \frac{3}{2}(5\sqrt{p} - 2)(3p - 1)$
16. $g(x) = (\sqrt{x} + 5x - 2)(\sqrt[3]{x} - 3\sqrt{x})$
17. $y = 7 \cdot \frac{2}{3}$
18. $y = (x - 1)(x - 2)(x - 3)$
19. $y = (5x + 3)(2x - 5)(7x + 9)$
20. $y = \dfrac{2x - 3}{4x + 1}$
21. $f(x) = \dfrac{5x}{x - 1}$
22. $H(x) = \dfrac{-5x}{5 - x}$
23. $f(x) = \dfrac{-13}{3x^5}$
24. $f(x) = \dfrac{3(5x^2 - 7)}{4}$
25. $y = \dfrac{x + 2}{x - 1}$
26. $h(w) = \dfrac{3w^2 + 5w - 1}{w - 3}$
27. $h(z) = \dfrac{6 - 2z}{z^2 - 4}$
28. $z = \dfrac{2x^2 + 5x - 2}{3x^2 + 5x + 3}$
29. $y = \dfrac{4x^2 + 3x + 2}{3x^2 - 2x + 1}$
30. $f(x) = \dfrac{x^3 - x^2 + 1}{x^2 + 1}$
31. $y = \dfrac{x^2 - 4x + 3}{2x^2 - 3x + 2}$
32. $F(z) = \dfrac{z^4 + 4}{3z}$
33. $g(x) = \dfrac{1}{x^{100} + 7}$
34. $y = \dfrac{-8}{7x^6}$
35. $u(v) = \dfrac{v^3 - 8}{v}$
36. $y = \dfrac{x - 5}{8\sqrt{x}}$
37. $y = \dfrac{3x^2 - x - 1}{\sqrt[3]{x}}$
38. $y = \dfrac{x^{0.3} - 2}{2x^{2.1} + 1}$
39. $y = 1 - \dfrac{5}{2x + 5} + \dfrac{2x}{3x + 1}$
40. $q(x) = 2x^3 + \dfrac{5x + 1}{3x - 5} - \dfrac{2}{x^3}$
41. $y = \dfrac{x - 5}{(x + 2)(x - 4)}$
42. $y = \dfrac{(9x - 1)(3x + 2)}{4 - 5x}$
43. $s(t) = \dfrac{t^2 + 3t}{(t^2 - 1)(t^3 + 7)}$
44. $f(s) = \dfrac{17}{s(4s^3 + 5s - 23)}$
45. $y = 3x - \dfrac{\frac{2}{x} - \frac{3}{x - 1}}{x - 2}$
46. $y = 3 - 12x^3 + \dfrac{1 - \frac{5}{x^2 + 2}}{x^2 + 5}$
47. $f(x) = \dfrac{a + x}{a - x}$, where a is a constant
48. $f(x) = \dfrac{x^{-1} + a^{-1}}{x^{-1} - a^{-1}}$, where a is a constant
49. Find the slope of the curve $y = (2x^2 - x + 3)(x^3 + x + 1)$ at $(1, 12)$.
50. Find the slope of the curve $y = \dfrac{x^3}{x^4 + 1}$ at $(-1, -\frac{1}{2})$.

In Problems 51–54, find an equation of the tangent line to the curve at the given point.

51. $y = \dfrac{6}{x - 1}; (3, 3)$
52. $y = \dfrac{x + 5}{x^2}; (1, 6)$
53. $y = (2x + 3)[2(x^4 - 5x^2 + 4)]; (0, 24)$
54. $y = \dfrac{x - 1}{x(x^2 + 1)}; (2, \frac{1}{10})$

In Problems 55 and 56, determine the relative rate of change of y with respect to x for the given value of x.

55. $y = \dfrac{x}{2x - 6}; x = 1$
56. $y = \dfrac{1 - x}{1 + x}; x = 5$

57. Motion The position function for an object moving in a straight line is

$$s = \frac{2}{t^3 + 1}$$

where t is in seconds and s is in meters. Find the position and velocity of the object at $t = 1$.

58. Motion The position function for an object moving in a straight-line path is

$$s = \frac{t + 3}{t^2 + 7}$$

where t is in seconds and s is in meters. Find the positive value(s) of t for which the velocity of the object is 0.

In Problems 59–62, each equation represents a demand function for a certain product, where p denotes the price per unit for q units. Find the marginal-revenue function in each case. Recall that revenue = pq.

59. $p = 80 - 0.02q$
60. $p = 500/q$
61. $p = \dfrac{108}{q + 2} - 3$
62. $p = \dfrac{q + 750}{q + 50}$

63. Consumption Function For the United States (1922–1942), the consumption function is estimated by[11]

$$C = 0.672I + 113.1$$

Find the marginal propensity to consume.

64. Consumption Function Repeat Problem 63 for $C = 0.836I + 127.2$.

In Problems 65–68, each equation represents a consumption function. Find the marginal propensity to consume and the marginal propensity to save for the given value of I.

65. $C = 3 + \sqrt{I} + 2\sqrt[3]{I}; I = 1$
66. $C = 6 + \dfrac{3I}{4} - \dfrac{\sqrt{I}}{3}; I = 25$

[11] T. Haavelmo, "Methods of Measuring the Marginal Propensity to Consume," *Journal of the American Statistical Association*, XLII (1947), 105–22.

67. $C = \dfrac{16\sqrt{I} + 0.8\sqrt{I^3} - 0.2I}{\sqrt{I} + 4}; I = 36$

68. $C = \dfrac{20\sqrt{I} + 0.5\sqrt{I^3} - 0.4I}{\sqrt{I} + 5}; I = 100$

69. Consumption Function Suppose that a country's consumption function is given by

$$C = \frac{9\sqrt{I} + 0.8\sqrt{I^3} - 0.3I}{\sqrt{I}}$$

where C and I are expressed in billions of dollars.

(a) Find the marginal propensity to save when income is \$25 billion.

(b) Determine the relative rate of change of C with respect to I when income is \$25 billion.

70. Marginal Propensities to Consume and to Save Suppose that the savings function of a country is

$$S = \frac{I - 2\sqrt{I} - 8}{\sqrt{I} + 2}$$

where the national income (I) and the national savings (S) are measured in billions of dollars. Find the country's marginal propensity to consume and its marginal propensity to save when the national income is \$150 billion. (*Hint:* It may be helpful to first factor the numerator.)

71. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{6q^2}{q + 2} + 6000$$

find the marginal-cost function.

72. Marginal and Average Costs Given the cost function $c = f(q)$, show that if $\dfrac{d}{dq}(\bar{c}) = 0$, then the marginal-cost function and average-cost function are equal.

73. Host–Parasite Relation For a particular host–parasite relationship, it is determined that when the host density (number of hosts per unit of area) is x, the number of hosts that are parasitized is y, where

$$y = \frac{900x}{10 + 45x}$$

At what rate is the number of hosts parasitized changing with respect to host density when $x = 2$?

74. Acoustics The persistence of sound in a room after the source of the sound is turned off is called *reverberation*. The *reverberation time* RT of the room is the time it takes for the intensity level of the sound to fall 60 decibels. In the acoustical design of an auditorium, the following formula may be used to compute the RT of the room:[12]

$$\text{RT} = \frac{0.05V}{A + xV}$$

Here V is the room volume, A is the total room absorption, and x is the air absorption coefficient. Assuming that A and x are positive constants, show that the rate of change of RT with respect to V is always positive. If the total room volume increases by one unit, does the reverberation time increase or decrease?

75. Predator–Prey In a predator–prey experiment,[13] it was statistically determined that the number of prey consumed, y, by an individual predator was a function of the prey density x (the number of prey per unit of area), where

$$y = \frac{0.7355x}{1 + 0.02744x}$$

Determine the rate of change of prey consumed with respect to prey density.

76. Social Security Benefits In a discussion of social security benefits, Feldstein[14] differentiates a function of the form

$$f(x) = \frac{a(1 + x) - b(2 + n)x}{a(2 + n)(1 + x) - b(2 + n)x}$$

where a, b, and n are constants. He determines that

$$f'(x) = \frac{-1(1 + n)ab}{(a(1 + x) - bx)^2(2 + n)}$$

Verify this. (*Hint:* For convenience, let $2 + n = c$.) Next observe that Feldstein's function f is of the form

$$g(x) = \frac{A + Bx}{C + Dx}, \quad \text{where } A, B, C, \text{ and } D \text{ are constants}$$

Show that $g'(x)$ is a constant divided by a nonnegative function of x. What does this mean?

77. Business The manufacturer of a product has found that when 20 units are produced per day, the average cost is \$150 and the marginal cost is \$125. What is the relative rate of change of average cost with respect to quantity when $q = 20$?

78. Use the result $(fgh)' = f'gh + fg'h + fgh'$ to find dy/dx if

$$y = (3x + 1)(2x - 1)(x - 4)$$

Objective

To introduce and apply the chain rule, to derive a special case of the chain rule, and to develop the concept of the marginal-revenue product as an application of the chain rule.

11.5 The Chain Rule

Our next rule, the *chain rule*, is ultimately the most important rule for finding derivatives. It involves a situation in which y is a function of the variable u, but u is a function of x,

[12] L. L. Doelle, *Environmental Acoustics* (New York: McGraw-Hill Book Company, 1972).

[13] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

[14] M. Feldstein, "The Optimal Level of Social Security Benefits," *The Quarterly Journal of Economics*, C, no. 2 (1985), 303–20.

and we want to find the derivative of y with respect to x. For example, the equations

$$y = u^2 \quad \text{and} \quad u = 2x + 1$$

define y as a function of u and u as a function of x. If we substitute $2x + 1$ for u in the first equation, we can consider y to be a function of x:

$$y = (2x + 1)^2$$

To find dy/dx, we first expand $(2x + 1)^2$:

$$y = 4x^2 + 4x + 1$$

Then

$$\frac{dy}{dx} = 8x + 4$$

From this example, you can see that finding dy/dx by first performing a substitution *could* be quite involved. For instance, if originally we had been given $y = u^{100}$ instead of $y = u^2$, we wouldn't even want to try substituting. Fortunately, the chain rule will allow us to handle such situations with ease.

COMBINING RULE 5 The Chain Rule

If y is a differentiable function of u and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

We can show you why the chain rule is reasonable by considering rates of change. Suppose

$$y = 8u + 5 \quad \text{and} \quad u = 2x - 3$$

Let x change by one unit. How does u change? To answer this question, we differentiate and find $du/dx = 2$. But for *each* one-unit change in u, there is a change in y of $dy/du = 8$. Therefore, what is the change in y if x changes by one unit; that is, what is dy/dx? The answer is $8 \cdot 2$, which is $\frac{dy}{du} \cdot \frac{du}{dx}$. Thus, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

We will now use the chain rule to redo the problem at the beginning of this section. If

$$y = u^2 \quad \text{and} \quad u = 2x + 1$$

then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(u^2) \cdot \frac{d}{dx}(2x + 1)$$
$$= (2u)2 = 4u$$

Replacing u by $2x + 1$ gives

$$\frac{dy}{dx} = 4(2x + 1) = 8x + 4$$

which agrees with our previous result.

APPLY IT ▶

8. If an object moves horizontally according to $x = 6t$, where t is in seconds, and vertically according to $y = 4x^2$, find its vertical velocity $\frac{dy}{dt}$.

EXAMPLE 1 Using the Chain Rule

a. If $y = 2u^2 - 3u - 2$ and $u = x^2 + 4$, find dy/dx.

Solution: By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(2u^2 - 3u - 2) \cdot \frac{d}{dx}(x^2 + 4)$$
$$= (4u - 3)(2x)$$

We can write our answer in terms of x alone by replacing u by $x^2 + 4$.

$$\frac{dy}{dx} = [4(x^2 + 4) - 3](2x) = [4x^2 + 13](2x) = 8x^3 + 26x$$

b. If $y = \sqrt{w}$ and $w = 7 - t^3$, find dy/dt.

Solution: Here, y is a function of w and w is a function of t, so we can view y as a function of t. By the chain rule,

$$\frac{dy}{dt} = \frac{dy}{dw} \cdot \frac{dw}{dt} = \frac{d}{dw}(\sqrt{w}) \cdot \frac{d}{dt}(7 - t^3)$$
$$= \left(\frac{1}{2}w^{-1/2}\right)(-3t^2) = \frac{1}{2\sqrt{w}}(-3t^2)$$
$$= -\frac{3t^2}{2\sqrt{w}} = -\frac{3t^2}{2\sqrt{7 - t^3}}$$

Now Work Problem 1 ◁

EXAMPLE 2 Using the Chain Rule

If $y = 4u^3 + 10u^2 - 3u - 7$ and $u = 4/(3x - 5)$, find dy/dx when $x = 1$.

Solution: By the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{d}{du}(4u^3 + 10u^2 - 3u - 7) \cdot \frac{d}{dx}\left(\frac{4}{3x - 5}\right)$$
$$= (12u^2 + 20u - 3) \cdot \frac{(3x - 5)\frac{d}{dx}(4) - 4\frac{d}{dx}(3x - 5)}{(3x - 5)^2}$$
$$= (12u^2 + 20u - 3) \cdot \frac{-12}{(3x - 5)^2}$$

When x is replaced by a, $u = u(x)$ must be replaced by $u(a)$.

Even though dy/dx is in terms of x's and u's, we can evaluate it when $x = 1$ if we determine the corresponding value of u. When $x = 1$,

$$u = \frac{4}{3(1) - 5} = -2$$

Thus,

$$\left.\frac{dy}{dx}\right|_{x=1} = [12(-2)^2 + 20(-2) - 3] \cdot \frac{-12}{[3(1) - 5]^2}$$
$$= 5 \cdot (-3) = -15$$

Now Work Problem 5 ◁

The chain rules states that if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Actually, the chain rule applies to a composite function, because

$$y = f(u) = f(g(x)) = (f \circ g)(x)$$

Thus y, as a function of x, is $f \circ g$. This means that we can use the chain rule to differentiate a function when we recognize the function as a composition. However, we must first break down the function into composite parts.

For example, to differentiate

$$y = (x^3 - x^2 + 6)^{100}$$

we think of the function as a composition. Let

$$y = f(u) = u^{100} \quad \text{and} \quad u = g(x) = x^3 - x^2 + 6$$

Then $y = (x^3 - x^2 + 6)^{100} = (g(x))^{100} = f(g(x))$. Now that we have a composite, we differentiate. Since $y = u^{100}$ and $u = x^3 - x^2 + 6$, by the chain rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= (100u^{99})(3x^2 - 2x) \\ &= 100(x^3 - x^2 + 6)^{99}(3x^2 - 2x)\end{aligned}$$

We have just used the chain rule to differentiate $y = (x^3 - x^2 + 6)^{100}$, which is a power of a *function* of x, not simply a power of x. The following rule, called the **power rule,** generalizes our result and is a special case of the chain rule:

The Power Rule $$\frac{d}{dx}(u^a) = au^{a-1}\frac{du}{dx}$$

where it is understood that u is a differentiable function of x and a is a real number.

Proof. Let $y = u^a$. Since y is a differentiable function of u and u is a differentiable function of x, the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

But $dy/du = au^{a-1}$. Thus,

$$\frac{dy}{dx} = au^{a-1}\frac{du}{dx}$$

which is the power rule.

EXAMPLE 3 Using the Power Rule

If $y = (x^3 - 1)^7$, find y'.

Solution: Since y is a power of a *function* of x, the power rule applies. Letting $u(x) = x^3 - 1$ and $a = 7$, we have

$$\begin{aligned}y' &= a[u(x)]^{a-1}u'(x) \\ &= 7(x^3 - 1)^{7-1}\frac{d}{dx}(x^3 - 1) \\ &= 7(x^3 - 1)^6(3x^2) = 21x^2(x^3 - 1)^6\end{aligned}$$

Now Work Problem 9 ◁

EXAMPLE 4 Using the Power Rule

If $y = \sqrt[3]{(4x^2 + 3x - 2)^2}$, find dy/dx when $x = -2$.

Solution: Since $y = (4x^2 + 3x - 2)^{2/3}$, we use the power rule with

$$u = 4x^2 + 3x - 2$$

and $a = \frac{2}{3}$. We have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{3}(4x^2 + 3x - 2)^{(2/3)-1}\frac{d}{dx}(4x^2 + 3x - 2) \\ &= \frac{2}{3}(4x^2 + 3x - 2)^{-1/3}(8x + 3) \\ &= \frac{2(8x + 3)}{3\sqrt[3]{4x^2 + 3x - 2}}\end{aligned}$$

Thus,

$$\left.\frac{dy}{dx}\right|_{x=-2} = \frac{2(-13)}{3\sqrt[3]{8}} = -\frac{13}{3}$$

Now Work Problem 19 ◁

EXAMPLE 5 **Using the Power Rule**

If $y = \dfrac{1}{x^2 - 2}$, find $\dfrac{dy}{dx}$.

The technique used in Example 5 is frequently used when the numerator of a quotient is a constant and the denominator is not.

Solution: Although the quotient rule can be used here, a more efficient approach is to treat the right side as the power $(x^2 - 2)^{-1}$ and use the power rule. Let $u = x^2 - 2$. Then $y = u^{-1}$, and

$$\begin{aligned}\frac{dy}{dx} &= (-1)(x^2 - 2)^{-1-1}\frac{d}{dx}(x^2 - 2)\\ &= (-1)(x^2 - 2)^{-2}(2x)\\ &= -\frac{2x}{(x^2 - 2)^2}\end{aligned}$$

Now Work Problem 27 ◁

EXAMPLE 6 **Differentiating a Power of a Quotient**

If $z = \left(\dfrac{2s + 5}{s^2 + 1}\right)^4$, find $\dfrac{dz}{ds}$.

The problem here is to recognize the form of the function to be differentiated. In this case it is a power, not a quotient.

Solution: Since z is a power of a function, we first use the power rule:

$$\frac{dz}{ds} = 4\left(\frac{2s + 5}{s^2 + 1}\right)^{4-1}\frac{d}{ds}\left(\frac{2s + 5}{s^2 + 1}\right)$$

Now we use the quotient rule:

$$\frac{dz}{ds} = 4\left(\frac{2s + 5}{s^2 + 1}\right)^3\left(\frac{(s^2 + 1)(2) - (2s + 5)(2s)}{(s^2 + 1)^2}\right)$$

Simplifying, we have

$$\begin{aligned}\frac{dz}{ds} &= 4 \cdot \frac{(2s + 5)^3}{(s^2 + 1)^3}\left(\frac{-2s^2 - 10s + 2}{(s^2 + 1)^2}\right)\\ &= -\frac{8(s^2 + 5s - 1)(2s + 5)^3}{(s^2 + 1)^5}\end{aligned}$$

Now Work Problem 41 ◁

EXAMPLE 7 **Differentiating a Product of Powers**

If $y = (x^2 - 4)^5(3x + 5)^4$, find y'.

Solution: Since y is a product, we first apply the product rule:

$$y' = (x^2 - 4)^5\frac{d}{dx}((3x + 5)^4) + (3x + 5)^4\frac{d}{dx}((x^2 - 4)^5)$$

Now we use the power rule:

$$\begin{aligned}y' &= (x^2 - 4)^5(4(3x + 5)^3(3)) + (3x + 5)^4(5(x^2 - 4)^4(2x))\\ &= 12(x^2 - 4)^5(3x + 5)^3 + 10x(3x + 5)^4(x^2 - 4)^4\end{aligned}$$

In differentiating a product in which at least one factor is a power, simplifying the derivative usually involves factoring.

To simplify, we first remove common factors:

$$y' = 2(x^2 - 4)^4(3x + 5)^3[6(x^2 - 4) + 5x(3x + 5)]$$
$$= 2(x^2 - 4)^4(3x + 5)^3(21x^2 + 25x - 24)$$

Now Work Problem 39 ◁

Usually, the power rule should be used to differentiate $y = [u(x)]^n$. Although a function such as $y = (x^2+2)^2$ can be written $y = x^4+4x^2+4$ and differentiated easily, this method is impractical for a function such as $y = (x^2+2)^{1000}$. Since $y = (x^2+2)^{1000}$ is of the form $y = [u(x)]^n$, we have

$$y' = 1000(x^2 + 2)^{999}(2x)$$

Marginal-Revenue Product

Let us now use our knowledge of calculus to develop a concept relevant to economic studies. Suppose a manufacturer hires m employees who produce a total of q units of a product per day. We can think of q as a function of m. If r is the total revenue the manufacturer receives for selling these units, then r can also be considered a function of m. Thus, we can look at dr/dm, the rate of change of revenue with respect to the number of employees. The derivative dr/dm is called the **marginal-revenue product.** It approximates the change in revenue that results when a manufacturer hires an extra employee.

EXAMPLE 8 Marginal-Revenue Product

A manufacturer determines that m employees will produce a total of q units of a product per day, where

$$q = \frac{10m^2}{\sqrt{m^2 + 19}} \tag{1}$$

If the demand equation for the product is $p = 900/(q + 9)$, determine the marginal-revenue product when $m = 9$.

Solution: We must find dr/dm, where r is revenue. Note that, by the chain rule,

$$\frac{dr}{dm} = \frac{dr}{dq} \cdot \frac{dq}{dm}$$

Thus, we must find both dr/dq and dq/dm when $m = 9$. We begin with dr/dq. The revenue function is given by

$$r = pq = \left(\frac{900}{q+9}\right) q = \frac{900q}{q+9} \tag{2}$$

so, by the quotient rule,

$$\frac{dr}{dq} = \frac{(q+9)(900) - 900q(1)}{(q+9)^2} = \frac{8100}{(q+9)^2}$$

In order to evaluate this expression when $m = 9$, we first use the given equation $q = 10m^2/\sqrt{m^2 + 19}$ to find the corresponding value of q:

$$q = \frac{10(9)^2}{\sqrt{9^2 + 19}} = 81$$

Hence,

$$\left.\frac{dr}{dq}\right|_{m=9} = \left.\frac{dr}{dq}\right|_{q=81} = \frac{8100}{(81+9)^2} = 1$$

Now we turn to dq/dm. From the quotient and power rules, we have

$$\frac{dq}{dm} = \frac{d}{dm}\left(\frac{10m^2}{\sqrt{m^2+19}}\right)$$

$$= \frac{(m^2+19)^{1/2}\dfrac{d}{dm}(10m^2) - (10m^2)\dfrac{d}{dm}[(m^2+19)^{1/2}]}{[(m^2+19)^{1/2}]^2}$$

$$= \frac{(m^2+19)^{1/2}(20m) - (10m^2)[\frac{1}{2}(m^2+19)^{-1/2}(2m)]}{m^2+19}$$

so

$$\left.\frac{dq}{dm}\right|_{m=9} = \frac{(81+19)^{1/2}(20\cdot 9) - (10\cdot 81)[\frac{1}{2}(81+19)^{-1/2}(2\cdot 9)]}{81+19}$$

$$= 10.71$$

A direct formula for the marginal-revenue product is

$$\frac{dr}{dm} = \frac{dq}{dm}\left(p + q\frac{dp}{dq}\right)$$

Therefore, from the chain rule,

$$\left.\frac{dr}{dm}\right|_{m=9} = (1)(10.71) = 10.71$$

This means that if a tenth employee is hired, revenue will increase by approximately \$10.71 per day.

Now Work Problem 80 ◁

PROBLEMS 11.5

In Problems 1–8, use the chain rule.

1. If $y = u^2 - 2u$ and $u = x^2 - x$, find dy/dx.
2. If $y = 2u^3 - 8u$ and $u = 7x - x^3$, find dy/dx.
3. If $y = \dfrac{1}{w}$ and $w = 3x - 5$, find dy/dx.
4. If $y = \sqrt[4]{z}$ and $z = x^5 - x^4 + 3$, find dy/dx.
5. If $w = u^3$ and $u = \dfrac{t-1}{t+1}$, find dw/dt when $t = 1$.
6. If $z = u^2 + \sqrt{u} + 9$ and $u = 2s^2 - 1$, find dz/ds when $s = -1$.
7. If $y = 3w^2 - 8w + 4$ and $w = 2x^2 + 1$, find dy/dx when $x = 0$.
8. If $y = 2u^3 + 3u^2 + 5u - 1$ and $u = 3x + 1$, find dy/dx when $x = 1$.

In Problems 9–52, find y'.

9. $y = (3x+2)^6$
10. $y = (x^2-4)^4$
11. $y = (3+2x^3)^5$
12. $y = (x^2+x)^4$
13. $y = 5(x^3 - 3x^2 + 2x)^{100}$
14. $y = \dfrac{(2x^2+1)^4}{2}$
15. $y = (x^2-2)^{-3}$
16. $y = (2x^3 - 8x)^{-12}$
17. $y = 2(x^2 + 5x - 2)^{-5/7}$
18. $y = 3(5x - 2x^3)^{-5/3}$
19. $y = \sqrt{5x^2 - x}$
20. $y = \sqrt{3x^2 - 7}$
21. $y = \sqrt[4]{2x-1}$
22. $y = \sqrt[3]{8x^2 - 1}$
23. $y = 4\sqrt[7]{(x^2+1)^3}$
24. $y = 7\sqrt[3]{(x^5-3)^5}$
25. $y = \dfrac{6}{2x^2 - x + 1}$
26. $y = \dfrac{3}{x^4+2}$
27. $y = \dfrac{1}{(x^2-3x)^2}$
28. $y = \dfrac{1}{(3+5x)^3}$
29. $y = \dfrac{4}{\sqrt{9x^2+1}}$
30. $y = \dfrac{3}{(3x^2-x)^{2/3}}$
31. $y = \sqrt[3]{7x} + \sqrt[3]{7x}$
32. $y = \sqrt{2x} + \dfrac{1}{\sqrt{2x}}$
33. $y = x^3(2x+3)^7$
34. $y = x(x+4)^4$
35. $y = 4x^2\sqrt{5x+1}$
36. $y = 4x^3\sqrt{1-x^2}$
37. $y = (x^2 + 2x - 1)^3(5x)$
38. $y = x^4(x^4-1)^5$
39. $y = (8x-1)^3(2x+1)^4$
40. $y = (3x+2)^5(4x-5)^2$
41. $y = \left(\dfrac{x-3}{x+2}\right)^{12}$
42. $y = \left(\dfrac{2x}{x+2}\right)^4$
43. $y = \sqrt{\dfrac{x+1}{x-5}}$
44. $y = \sqrt[3]{\dfrac{8x^2-3}{x^2+2}}$
45. $y = \dfrac{2x-5}{(x^2+4)^3}$
46. $y = \dfrac{(4x-2)^4}{3x^2+7}$
47. $y = \dfrac{(8x-1)^5}{(3x-1)^3}$
48. $y = \sqrt[3]{(x-3)^3(x+5)}$
49. $y = 6(5x^2+2)\sqrt{x^4+5}$
50. $y = 6 + 3x - 4x(7x+1)^2$
51. $y = 8t + \dfrac{t-1}{t+4} - \left(\dfrac{8t-7}{4}\right)^2$
52. $y = \dfrac{(2x^3+6)(7x-5)}{(2x+4)^2}$

In Problems 53 and 54, use the quotient rule and power rule to find y'. Do not simplify your answer.

53. $y = \dfrac{(3x+2)^3(x+1)^4}{(x^2-7)^3}$
54. $y = \dfrac{\sqrt{x+2}(4x^2-1)^2}{9x-3}$
55. If $y = (5u+6)^3$ and $u = (x^2+1)^4$, find dy/dx when $x = 0$.
56. If $z = 2y^2 - 4y + 5$, $y = 6x - 5$, and $x = 2t$, find dz/dt when $t = 1$.
57. Find the slope of the curve $y = (x^2 - 7x - 8)^3$ at the point $(8, 0)$.

58. Find the slope of the curve $y = \sqrt{x+2}$ at the point $(7, 3)$.

In Problems 59–62, find an equation of the tangent line to the curve at the given point.

59. $y = \sqrt[3]{(x^2-8)^2}$; $(3, 1)$

60. $y = (x+3)^3$; $(-1, 8)$

61. $y = \dfrac{\sqrt{7x+2}}{x+1}$; $\left(1, \dfrac{3}{2}\right)$

62. $y = \dfrac{-3}{(3x^2+1)^3}$; $(0, -3)$

In Problems 63 and 64, determine the percentage rate of change of y with respect to x for the given value of x.

63. $y = (x^2+1)^4$; $x = 1$

64. $y = \dfrac{1}{(x^2-1)^3}$; $x = 2$

In Problems 65–68, q is the total number of units produced per day by m employees of a manufacturer, and p is the price per unit at which the q units are sold. In each case, find the marginal-revenue product for the given value of m.

65. $q = 5m$, $p = -0.4q + 50$; $m = 6$

66. $q = (200m - m^2)/20$, $p = -0.1q + 70$; $m = 40$

67. $q = 10m^2/\sqrt{m^2+9}$, $p = 525/(q+3)$; $m = 4$

68. $q = 50m/\sqrt{m^2+11}$, $p = 100/(q+10)$; $m = 5$

69. Demand Equation Suppose $p = 100 - \sqrt{q^2+20}$ is a demand equation for a manufacturer's product.

(a) Find the rate of change of p with respect to q.
(b) Find the relative rate of change of p with respect to q.
(c) Find the marginal-revenue function.

70. Marginal-Revenue Product If $p = k/q$, where k is a constant, is the demand equation for a manufacturer's product and $q = f(m)$ defines a function that gives the total number of units produced per day by m employees, show that the marginal-revenue product is always zero.

71. Cost Function The cost c of producing q units of a product is given by

$$c = 5500 + 12q + 0.2q^2$$

If the price per unit p is given by the equation

$$q = 900 - 1.5p$$

use the chain rule to find the rate of change of cost with respect to price per unit when $p = 85$.

72. Hospital Discharges A governmental health agency examined the records of a group of individuals who were hospitalized with a particular illness. It was found that the total proportion that had been discharged at the end of t days of hospitalization was given by

$$f(t) = 1 - \left(\frac{250}{250+t}\right)^3$$

Find $f'(100)$ and interpret your answer.

73. Marginal Cost If the total-cost function for a manufacturer is given by

$$c = \frac{4q^2}{\sqrt{q^2+2}} + 6000$$

find the marginal-cost function.

74. Salary/Education For a certain population, if E is the number of years of a person's education and S represents average annual salary in dollars, then for $E \geq 7$,

$$S = 340E^2 - 4360E + 42{,}800$$

(a) How fast is salary changing with respect to education when $E = 16$?
(b) At what level of education does the rate of change of salary equal \$5000 per year of education?

75. Biology The volume of a spherical cell is given by $V = \frac{4}{3}\pi r^3$, where r is the radius. At time t seconds, the radius (in centimeters) is given by

$$r = 10^{-8}t^2 + 10^{-7}t$$

Use the chain rule to find dV/dt when $t = 10$.

76. Pressure in Body Tissue Under certain conditions, the pressure p developed in body tissue by ultrasonic beams is given as a function of the beam's intensity via the equation[15]

$$p = (2\rho VI)^{1/2}$$

where ρ (a Greek letter read "rho") is density of the affected tissue and V is the velocity of propagation of the beam. Here ρ and V are constants. (a) Find the rate of change of p with respect to I. (b) Find the relative rate of change of p with respect to I.

77. Demography Suppose that, for a certain group of 20,000 births, the number of people surviving to age x years is

$$l_x = -0.000354x^4 + 0.00452x^3 + 0.848x^2 - 34.9x + 20{,}000$$
$$0 \leq x \leq 95.2$$

(a) Find the rate of change of l_x with respect to x, and evaluate your answer for $x = 65$.
(b) Find the relative rate of change and the percentage rate of change of l_x when $x = 65$. Round your answers to three decimal places.

78. Muscle Contraction A muscle has the ability to shorten when a load, such as a weight, is imposed on it. The equation

$$(P + a)(v + b) = k$$

is called the "fundamental equation of muscle contraction."[16] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Express v as a function of P. Use your result to find dv/dP.

79. Economics Suppose $pq = 100$ is the demand equation for a manufacturer's product. Let c be the total cost, and assume that the marginal cost is 0.01 when $q = 200$. Use the chain rule to find dc/dp when $q = 200$.

80. Marginal-Revenue Product A monopolist who employs m workers finds that they produce

$$q = 2m(2m+1)^{3/2}$$

units of product per day. The total revenue r (in dollars) is given by

$$r = \frac{50q}{\sqrt{1000+3q}}$$

---

[15] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).
[16] Ibid.

(a) What is the price per unit (to the nearest cent) when there are 12 workers?
(b) Determine the marginal revenue when there are 12 workers.
(c) Determine the marginal-revenue product when $m = 12$.

81. Suppose $y = f(x)$, where $x = g(t)$. Given that $g(2) = 3, g'(2) = 4, f(2) = 5, f'(2) = 6, g(3) = 7, g'(3) = 8,$ $f(3) = 9$, and $f'(3) = 10$, determine the value of $\frac{dy}{dt}\Big|_{t=2}$.

82. Business A manufacturer has determined that, for his product, the daily average cost (in hundreds of dollars) is given by

$$\bar{c} = \frac{324}{\sqrt{q^2+35}} + \frac{5}{q} + \frac{19}{18}$$

(a) As daily production increases, the average cost approaches a constant dollar amount. What is this amount?
(b) Determine the manufacturer's marginal cost when 17 units are produced per day.
(c) The manufacturer determines that if production (and sales) were increased to 18 units per day, revenue would increase by \$275. Should this move be made? Why?

83. If

$$y = (u+2)\sqrt{u+3}$$

and

$$u = x(x^2+3)^3$$

find dy/dx when $x = 0.1$. Round your answer to two decimal places.

84. If

$$y = \frac{2u+3}{u^3-2}$$

and

$$u = \frac{x+4}{(2x+3)^3}$$

find dy/dx when $x = -1$. Round your answer to two decimal places.

Chapter 11 Review

Important Terms and Symbols

Examples

Section 11.1 The Derivative

secant line — tangent line — slope of a curve — Ex. 1, p. 494

derivative — $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ — $\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$ — Ex. 2, p. 495

difference quotient — $f'(x)$ — y' — $\frac{d}{dx}(f(x))$ — $\frac{dy}{dx}$ — Ex. 4, p. 496

Section 11.2 Rules for Differentiation

power function — constant factor rule — sum or difference rule — Ex. 5, p. 505

Section 11.3 The Derivative as a Rate of Change

position function — Δx — velocity — rate of change — Ex. 1, p. 510

total-cost function — marginal cost — average cost — Ex. 7, p. 513

total-revenue function — marginal revenue — Ex. 8, p. 514

relative rate of change — percentage rate of change — Ex. 9, p. 515

Section 11.4 The Product Rule and the Quotient Rule

product rule — quotient rule — Ex. 5, p. 522

consumption function — marginal propensity to consume and to save — Ex. 9, p. 524

Section 11.5 The Chain Rule

chain rule — power rule — marginal-revenue product — Ex. 8, p. 531

Summary

The tangent line (or tangent) to a curve at point P is the limiting position of secant lines PQ as Q approaches P along the curve. The slope of the tangent at P is called the slope of the curve at P.

If $y = f(x)$, the derivative of f at x is the function $f'(x)$ defined by the limit in the equation

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Geometrically, the derivative gives the slope of the curve $y = f(x)$ at the point $(x, f(x))$. An equation of the tangent line at a particular point $(a, f(a))$ is obtained by evaluating $f'(a)$, which is the slope of the tangent line, and using the point-slope form of a line: $y - f(a) = f'(a)(x - a)$. Any function that is differentiable at a point must also be continuous there.

The rules for finding derivatives, discussed so far, are as follows, where we assume that all functions are differentiable:

$$\frac{d}{dx}(c) = 0, \text{ where } c \text{ is any constant}$$

$$\frac{d}{dx}(x^a) = ax^{a-1}, \text{ where } a \text{ is any real number}$$

$$\frac{d}{dx}(cf(x)) = cf'(x), \text{ where } c \text{ is a constant}$$

$$\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$$

$$\frac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$$

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x)+f(x)g'(x)$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$$

$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$, where y is a function of u and u is a function of x

$\frac{d}{dx}(u^a) = au^{a-1}\frac{du}{dx}$, where u is a function of x and a is any real number

The derivative dy/dx can also be interpreted as giving the (instantaneous) rate of change of y with respect to x:

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{\text{change in } y}{\text{change in } x}$$

In particular, if $s = f(t)$ is a position function, where s is position at time t, then

$$\frac{ds}{dt} = \text{velocity at time } t$$

In economics, the term *marginal* is used to describe derivatives of specific types of functions. If $c = f(q)$ is a total-cost function (c is the total cost of q units of a product), then the rate of change

$$\frac{dc}{dq} \text{ is called marginal cost}$$

We interpret marginal cost as the approximate cost of one additional unit of output. (Average cost per unit, $\bar{c}$, is related to total cost c by $\bar{c} = c/q$, equivalently, $c = \bar{c}q$.)

A total-revenue function $r = f(q)$ gives a manufacturer's revenue r for selling q units of product. (Revenue r and price p are related by $r = pq$.) The rate of change

$$\frac{dr}{dq} \text{ is called marginal revenue}$$

which is interpreted as the approximate revenue obtained from selling one additional unit of output.

If r is the revenue that a manufacturer receives when the total output of m employees is sold, then the derivative $dr/dm = dr/dq \cdot dq/dm$ is called the marginal-revenue product and gives the approximate change in revenue that results when the manufacturer hires an extra employee.

If $C = f(I)$ is a consumption function, where I is national income and C is national consumption, then

$$\frac{dC}{dI} \text{ is marginal propensity to consume}$$

and

$$1-\frac{dC}{dI} \text{ is marginal prospensity to save}$$

For any function, the relative rate of change of $f(x)$ is

$$\frac{f'(x)}{f(x)}$$

which compares the rate of change of $f(x)$ with $f(x)$ itself. The percentage rate of change is

$$\frac{f'(x)}{f(x)}\cdot 100\%$$

Review Problems

In Problems 1–4, use the definition of the derivative to find $f'(x)$.

1. $f(x) = 2 - x^2$

2. $f(x) = 5x^3 - 2x + 1$

3. $f(x) = \sqrt{3x}$

4. $f(x) = \dfrac{2}{1+4x}$

In Problems 5–38, differentiate.

5. $y = 7^4$

6. $y = ex$

7. $y = \pi x^4 - \sqrt{2}x^3 + 2x^2 + 4$

8. $y = 4(x^2+5) - 7x$

9. $f(s) = s^2(s^2+2)$

10. $y = \sqrt{x+3}$

11. $y = \dfrac{x^2+1}{5}$

12. $y = -\dfrac{1}{nx^n}$

13. $y = (x^3+7x^2)(x^3-x^2+5)$

14. $y = (x^2+1)^{100}(x-6)$

15. $f(x) = (2x^2+4x)^{100}$

16. $f(w) = w\sqrt{w} + w^2$

17. $y = \dfrac{c}{ax+b}$

18. $y = \dfrac{5x^2-8x}{2x}$

19. $y = (8+2x)(x^2+1)^4$

20. $g(z) = (2z)^{3/5} + 5$

21. $f(z) = \dfrac{z^2-1}{z^2+4}$

22. $y = \dfrac{ax+b}{cx+d}$

23. $y = \sqrt[3]{4x-1}$

24. $f(x) = (1+2^3)^{12}$

25. $y = \dfrac{1}{\sqrt{1-x^2}}$

26. $y = \dfrac{x(x+1)}{2x^2+3}$

27. $h(x) = (ax+b)^m(cx+d)^n$

28. $y = \dfrac{(x+3)^5}{x}$

29. $y = \dfrac{5x-4}{x+6}$

30. $f(x) = 5x^3\sqrt{3+2x^4}$

31. $y = 2x^{-3/8} + (2x)^{-3/8}$

32. $y = \sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{a}{x}}$

33. $y = \dfrac{x^2+6}{\sqrt{x^2+5}}$

34. $y = \sqrt[3]{(7-3x^2)^2}$

35. $y = (x^3+6x^2+9)^{3/5}$

36. $z = 0.4x^2(x+1)^{-3} + 0.5$

37. $g(z) = \dfrac{-3z}{(z-2)^{-3}}$

38. $g(z) = \dfrac{-3}{4(z^5+2z-5)^4}$

12 Additional Differentiation Topics

12.1 Derivatives of Logarithmic Functions

12.2 Derivatives of Exponential Functions

12.3 Elasticity of Demand

12.4 Implicit Differentiation

12.5 Logarithmic Differentiation

12.6 Newton's Method

12.7 Higher-Order Derivatives

Chapter 12 Review

EXPLORE & EXTEND
Economic Order Quantity

After an uncomfortable trip in a vehicle, passengers sometimes describe the ride as "jerky." But what is jerkiness, exactly? What does it mean for, say, an engineer designing a new transportation system?

Travel in a straight line at a constant speed is called *uniform motion,* and there is nothing jerky about it. But if either the path or the speed changes, the ride may become jerky. Change in velocity over time is the derivative of velocity. Called acceleration, the change in velocity is the *second derivative* of position with respect to time—the derivative of the derivative of position. One of the important concepts covered in this chapter is that of a higher-order derivative, of which acceleration is an example.

But is acceleration responsible for jerkiness? The feeling of being jerked back and forth on a roller coaster is certainly related to acceleration. On the other hand, automotive magazines often praise a car for having *smooth* acceleration. So apparently acceleration has something to do with jerkiness but is not itself the cause.

The derivative of acceleration is the *third* derivative of position with respect to time. When this third derivative is large, the acceleration is changing rapidly. A roller coaster in a steady turn to the left is undergoing steady leftward acceleration. But when the coaster changes abruptly from a hard left turn to a hard right turn, the acceleration changes directions—and the riders experience a jerk. The third derivative of position is, in fact, so apt a measure of jerkiness that it is customarily called the *jerk,* just as the second derivative is called the acceleration.

Jerk has implications not only for passenger comfort in vehicles but also for equipment reliability. Engineers designing equipment for spacecraft, for instance, follow guidelines about the maximum jerk the equipment must be able to survive without damage to its internal components.

Objective

To develop a differentiation formula for $y = \ln u$, to apply the formula, and to use it to differentiate a logarithmic function to a base other than e.

12.1 Derivatives of Logarithmic Functions

So far, the only derivatives we have been able to calculate are those of functions that are constructed from power functions using multiplication by a constant, arithmetic operations, and composition. (As pointed out in Problem 65 of Section 11.6, we can calculate the derivative of a constant function c by writing $c = cx^0$; then

$$\frac{d}{dx}(c) = \frac{d}{dx}(cx^0) = c\frac{d}{dx}(x^0) = c \cdot 0x^{-1} = 0$$

Thus, we really have only one *basic* differention formula so far.) The logarithmic functions $\log_b x$ and the exponential functions b^x *cannot* be constructed from power functions using multiplication by a constant, arithmetic operations, and composition. It follows that we will need at least another truly *basic* differentiation formula.

In this section, we develop formulas for differentiating logarithmic functions. We begin with the derivative of $\ln x$, commenting further on the numbered steps at the end of the calculation.

$$\frac{d}{dx}(\ln x) \overset{(1)}{=} \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h} \qquad \text{definition of derivative}$$

$$\overset{(2)}{=} \lim_{h\to 0}\frac{\ln\left(\dfrac{x+h}{x}\right)}{h} \qquad \text{since } \ln m - \ln n = \ln(m/n)$$

$$\overset{(3)}{=} \lim_{h\to 0}\left(\frac{1}{h}\ln\left(1+\frac{h}{x}\right)\right) \qquad \text{algebra}$$

$$\overset{(4)}{=} \lim_{h\to 0}\left(\frac{1}{x}\cdot\frac{x}{h}\ln\left(1+\frac{h}{x}\right)\right) \qquad \text{writing } \frac{1}{h} = \frac{1}{x}\cdot\frac{x}{h}$$

$$\overset{(5)}{=} \lim_{h\to 0}\left(\frac{1}{x}\ln\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \text{since } r\ln m = \ln m^r$$

$$\overset{(6)}{=} \frac{1}{x}\cdot\lim_{h\to 0}\left(\ln\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \text{by limit property 1 in Section 10.1}$$

$$\overset{(7)}{=} \frac{1}{x}\cdot\ln\left(\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \ln \text{ is continuous}$$

$$\overset{(8)}{=} \frac{1}{x}\cdot\ln\left(\lim_{h/x\to 0}\left(1+\frac{h}{x}\right)^{x/h}\right) \qquad \text{for fixed } x > 0$$

$$\overset{(9)}{=} \frac{1}{x}\cdot\ln\left(\lim_{k\to 0}(1+k)^{1/k}\right) \qquad \text{setting } k = h/x$$

$$\overset{(10)}{=} \frac{1}{x}\cdot\ln(e) \qquad \text{as shown in Section 10.1}$$

$$\overset{(11)}{=} \frac{1}{x} \qquad \text{since } \ln e = 1$$

The calculation is long but following it step by step allows for review of many important ideas. Step (1) is the key definition introduced in Section 11.1. Steps (2), (5), and (11) involve properties found in 4.3. In step (3), labeled simply *algebra*, we use properties of fractions first given in 0.2. Step (4) is admittedly a *trick* that requires experience to discover. Note that necessarily $x \neq 0$ since x is in the domain of ln, which is $(0, \infty)$. To understand the justification for Step (6), we must observe that x, and hence $1/x$, is constant with respect to the limit variable h. We have already remarked in Section 10.3 that logarithmic functions are continuous and this is what allows us to interchange the processes of applying the ln function and taking a limit in (7). In (8) the point is that, for fixed

$x > 0$, h/x goes to 0 when h goes to 0 and conversely h goes to 0 when h/x goes to 0. Thus, we can regard h/x as a new limit variable, k, and this we do in step (9).

In conclusion, we have derived the following:

BASIC RULE 2 Derivative of $\ln x$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \text{for } x > 0$$

Some care is required with this rule because while the left-hand side is defined only for $x > 0$, the right-hand side is defined for all $x \neq 0$. For $x < 0$, $\ln(-x)$ is defined and by the chain rule we have

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{-1}{-x} = \frac{1}{x} \quad \text{for } x < 0$$

We can combine the last two equations by using the absolute function to get

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \quad \text{for } x \neq 0 \tag{1}$$

EXAMPLE 1 Differentiating Functions Involving $\ln x$

a. Differentiate $f(x) = 5\ln x$.

Solution: Here f is a constant (5) times a function ($\ln x$), so by Basic Rule 2, we have

$$f'(x) = 5\frac{d}{dx}(\ln x) = 5 \cdot \frac{1}{x} = \frac{5}{x} \quad \text{for } x > 0$$

b. Differentiate $y = \dfrac{\ln x}{x^2}$.

Solution: By the quotient rule and Basic Rule 2,

$$y' = \frac{x^2\dfrac{d}{dx}(\ln x) - (\ln x)\dfrac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2\left(\dfrac{1}{x}\right) - (\ln x)(2x)}{x^4} = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3} \quad \text{for } x > 0$$

Now Work Problem 1 ◁

The chain rule is used to develop the differentiation formula for $\ln|u|$.

We will now extend Equation (1) to cover a broader class of functions. Let $y = \ln|u|$, where u is a differentiable function of x. By the chain rule,

$$\frac{d}{dx}(\ln|u|) = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{d}{du}(\ln|u|)\cdot\frac{du}{dx} = \frac{1}{u}\cdot\frac{du}{dx} \quad \text{for } u \neq 0$$

Thus,

$$\frac{d}{du}(\ln|u|) = \frac{1}{u}\cdot\frac{du}{dx} \quad \text{for } u \neq 0 \tag{2}$$

Of course, Equation (2) gives us $\dfrac{d}{du}(\ln u) = \dfrac{1}{u}\cdot\dfrac{du}{dx}$ for $u > 0$.

APPLY IT ▷

1. The supply of q units of a product at a price of p dollars per unit is given by $q(p) = 25 + 2\ln(3p^2 + 4)$. Find the rate of change of supply with respect to price, $\dfrac{dq}{dp}$.

EXAMPLE 2 Differentiating Functions Involving $\ln u$

a. Differentiate $y = \ln(x^2 + 1)$.

Solution: This function has the form $\ln u$ with $u = x^2 + 1$, and since $x^2 + 1 > 0$, for all x, $y = \ln(x^2 + 1)$ is defined for all x. Using Equation (2), we have

$$\frac{dy}{dx} = \frac{1}{x^2+1}\frac{d}{dx}(x^2 + 1) = \frac{1}{x^2+1}(2x) = \frac{2x}{x^2+1}$$

b. Differentiate $y = x^2 \ln(4x+2)$.

Solution: Using the product rule gives

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\ln(4x+2)) + (\ln(4x+2))\frac{d}{dx}(x^2)$$

By Equation (2) with $u = 4x+2$,

$$\frac{dy}{dx} = x^2\left(\frac{1}{4x+2}\right)(4) + (\ln(4x+2))(2x)$$

$$= \frac{2x^2}{2x+1} + 2x\ln(4x+2) \quad \text{for } 4x+2>0$$

Since $4x+2>0$ exactly when $x > -1/2$, we have

$$\frac{d}{dx}(x^2\ln(4x+2)) = \frac{2x^2}{2x+1} + 2x\ln(4x+2) \quad \text{for } x > -1/2$$

c. Differentiate $y = \ln|\ln|x||$.

Solution: This has the form $y = \ln|u|$ with $u = \ln|x|$. Using Equation (2), we obtain

$$y' = \frac{1}{\ln|x|}\frac{d}{dx}(\ln|x|) = \frac{1}{\ln|x|}\left(\frac{1}{x}\right) = \frac{1}{x\ln|x|} \quad \text{for } x, u \neq 0$$

Since $\ln|x| = 0$ when $x = -1, 1$, we have

$$\frac{d}{dx}(\ln|\ln|x||) = \frac{1}{x\ln|x|} \quad \text{for } x \neq -1, 0, 1$$

Now Work Problem 9 ◁

Frequently, we can reduce the work involved in differentiating the logarithm of a product, quotient, or power by using properties of logarithms to rewrite the logarithm *before* differentiating. The next example will illustrate.

EXAMPLE 3 **Rewriting Logarithmic Functions before Differentiating**

a. Find $\frac{dy}{dx}$ if $y = \ln(2x+5)^3$.

Solution: Here we have the logarithm of a power. First we simplify the right side by using properties of logarithms. Then we differentiate. We have

$$y = \ln(2x+5)^3 = 3\ln(2x+5) \quad \text{for } 2x+5>0$$

$$\frac{dy}{dx} = 3\left(\frac{1}{2x+5}\right)(2) = \frac{6}{2x+5} \quad \text{for } x > -5/2$$

Comparing both methods, we note that the easier one is to simplify first and then differentiate.

Alternatively, if the simplification were not performed first, we would write

$$\frac{dy}{dx} = \frac{1}{(2x+5)^3}\frac{d}{dx}((2x+5)^3)$$

$$= \frac{1}{(2x+5)^3}(3)(2x+5)^2(2) = \frac{6}{2x+5}$$

b. Find $f'(p)$ if $f(p) = \ln((p+1)^2(p+2)^3(p+3)^4)$.

Solution: We simplify the right side and then differentiate:

$$f(p) = 2\ln(p+1) + 3\ln(p+2) + 4\ln(p+3)$$

$$f'(p) = 2\left(\frac{1}{p+1}\right)(1) + 3\left(\frac{1}{p+2}\right)(1) + 4\left(\frac{1}{p+3}\right)(1)$$

$$= \frac{2}{p+1} + \frac{3}{p+2} + \frac{4}{p+3}$$

Now Work Problem 5 ◁

EXAMPLE 4 **Differentiating Functions Involving Logarithms**

a. Find $f'(w)$ if $f(w) = \ln\sqrt{\dfrac{1+w^2}{w^2-1}}$.

Solution: We simplify by using properties of logarithms and then differentiate:

$$f(w) = \frac{1}{2}(\ln(1+w^2) - \ln(w^2-1))$$

$$f'(w) = \frac{1}{2}\left(\frac{1}{1+w^2}(2w) - \frac{1}{w^2-1}(2w)\right)$$

$$= \frac{w}{1+w^2} - \frac{w}{w^2-1} = -\frac{2w}{w^4-1}$$

b. Find $f'(x)$ if $f(x) = \ln^3(2x+5)$.

Solution: The exponent 3 refers to the cubing of $\ln(2x+5)$. That is,

$$f(x) = \ln^3(2x+5) = [\ln(2x+5)]^3$$

By the power rule,

$$f'(x) = 3(\ln(2x+5))^2 \frac{d}{dx}(\ln(2x+5))$$

$$= 3(\ln(2x+5))^2\left(\frac{1}{2x+5}(2)\right)$$

$$= \frac{6}{2x+5}(\ln(2x+5))^2$$

CAUTION!

Do not confuse $\ln^3(2x+5)$ with $\ln(2x+5)^3$, which occurred in Example 3(a). It is advisable to write $\ln^3(2x+5)$ explicitly as $[\ln(2x+5)]^3$ and avoid $\ln^3(2x+5)$.

Now Work Problem 39 ◁

Derivatives of Logarithmic Functions to the Base b

To differentiate a logarithmic function to a base different from e, we can first convert the logarithm to natural logarithms via the change-of-base formula and then differentiate the resulting expression. For example, consider $y = \log_b u$, where u is a differentiable function of x. By the change-of-base formula,

$$y = \log_b u = \frac{\ln u}{\ln b} \quad \text{for } u > 0$$

Differentiating, we have

$$\frac{d}{dx}(\log_b u) = \frac{d}{dx}\left(\frac{\ln u}{\ln b}\right) = \frac{1}{\ln b}\frac{d}{dx}(\ln u) = \frac{1}{\ln b}\cdot\frac{1}{u}\frac{du}{dx}$$

CAUTION!

Note that $\ln b$ is just a constant!

Summarizing,

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u}\cdot\frac{du}{dx} \quad \text{for } u > 0$$

Rather than memorize this rule, we suggest that you remember the procedure used to obtain it.

Procedure to Differentiate $\log_b u$

Convert $\log_b u$ to natural logarithms to obtain $\dfrac{\ln u}{\ln b}$, and then differentiate.

EXAMPLE 5 **Differentiating a Logarithmic Function to the Base 2**

Differentiate $y = \log_2 x$.

Solution: Following the foregoing procedure, we have

$$\frac{d}{dx}(\log_2 x) = \frac{d}{dx}\left(\frac{\ln x}{\ln 2}\right) = \frac{1}{\ln 2}\frac{d}{dx}(\ln x) = \frac{1}{(\ln 2)x}$$

It is worth mentioning that we can write our answer in terms of the original base. Because

$$\frac{1}{\ln b} = \frac{1}{\dfrac{\log_b b}{\log_b e}} = \frac{\log_b e}{1} = \log_b e$$

we can express $\dfrac{1}{(\ln 2)x}$ as $\dfrac{\log_2 e}{x}$. More generally, $\dfrac{d}{dx}(\log_b u) = \dfrac{\log_b e}{u} \cdot \dfrac{du}{dx}$.

Now Work Problem 15 ◁

APPLY IT ▷

2. The intensity of an earthquake is measured on the Richter scale. The reading is given by $R = \log \dfrac{I}{I_0}$, where I is the intensity and I_0 is a standard minimum intensity. If $I_0 = 1$, find $\dfrac{dR}{dI}$, the rate of change of the Richter-scale reading with respect to the intensity.

EXAMPLE 6 **Differentiating a Logarithmic Function to the Base 10**

If $y = \log(2x + 1)$, find the rate of change of y with respect to x.

Solution: The rate of change is dy/dx, and the base involved is 10. Therefore, we have

$$\frac{dy}{dx} = \frac{d}{dx}(\log(2x+1)) = \frac{d}{dx}\left(\frac{\ln(2x+1)}{\ln 10}\right)$$
$$= \frac{1}{\ln 10} \cdot \frac{1}{2x+1}(2) = \frac{2}{\ln 10(2x+1)}$$

◁

PROBLEMS 12.1

In Problems 1–44, differentiate the functions. If possible, first use properties of logarithms to simplify the given function.

1. $y = a \ln x$
2. $y = \dfrac{5 \ln x}{9}$
3. $y = \ln(3x - 7)$
4. $y = \ln(5x - 6)$
5. $y = \ln x^2$
6. $y = \ln(5x^3 + 3x^2 + 2x + 1)$
7. $y = \ln(1 - x^2)$
8. $y = \ln(-x^2 + 6x)$
9. $f(X) = \ln(4X^6 + 2X^3)$
10. $f(r) = \ln(2r^4 - 3r^2 + 2r + 1)$
11. $f(t) = t \ln t - t$
12. $y = x^2 \ln x$
13. $y = x^3 \ln(2x + 5)$
14. $y = (ax + b)^3 \ln(ax + b)$
15. $y = \log_3(8x - 1)$
16. $f(w) = \log(w^2 + 2w + 1)$
17. $y = x^2 + \log_2(x^2 + 4)$
18. $y = x^2 \log_2 x$
19. $f(z) = \dfrac{\ln z}{z}$
20. $y = \dfrac{x^2}{\ln x}$
21. $y = \dfrac{x^4 + 3x^2 + x}{\ln x}$
22. $y = \ln x^{100}$
23. $y = \ln(x^2 + 4x + 5)^3$
24. $y = 6 \ln \sqrt[3]{x}$
25. $y = 9 \ln \sqrt{1 + x^2}$
26. $f(t) = \ln\left(\dfrac{t^4}{1 + 6t + t^2}\right)$
27. $f(l) = \ln\left(\dfrac{1 + l}{1 - l}\right)$
28. $y = \ln\left(\dfrac{2x + 3}{3x - 4}\right)$
29. $y = \ln \sqrt[4]{\dfrac{1 + x^2}{1 - x^2}}$
30. $y = \ln \sqrt[3]{\dfrac{x^3 - 1}{x^3 + 1}}$
31. $y = \ln[(ax^2 + bx + c)^p (hx^2 + kx + l)^q]$
32. $y = \ln[(5x + 2)^4 (8x - 3)^6]$
33. $y = 13 \ln\left(x^2 \sqrt[3]{5x + 2}\right)$
34. $y = 6 \ln \dfrac{x}{\sqrt{2x + 1}}$
35. $y = (x^2 + 1) \ln(2x + 1)$
36. $y = (ax^2 + bx + c) \ln(hx^2 + kx + l)$
37. $y = \ln x^3 + \ln^3 x$
38. $y = x^{\ln 2}$
39. $y = \ln^4(ax)$
40. $y = \ln^2(2x + 11)$
41. $y = \ln \sqrt{f(x)}$
42. $y = \ln\left(x^3 \sqrt[4]{2x + 1}\right)$
43. $y = \sqrt{4 + 3 \ln x}$
44. $y = \ln\left(x + \sqrt{1 + x^2}\right)$
45. Find an equation of the tangent line to the curve
$$y = \ln(x^2 - 3x - 3)$$
when $x = 4$.
46. Find an equation of the tangent line to the curve
$$y = x \ln x - x$$
at the point where $x = 1$.
47. Find the slope of the curve $y = \dfrac{x}{\ln x}$ when $x = 3$.
48. **Marginal Revenue** Find the marginal-revenue function if the demand function is $p = 25/\ln(q + 2)$.
49. **Marginal Cost** A total-cost function is given by
$$c = 25 \ln(q + 1) + 12$$
Find the marginal cost when $q = 6$.

50. Marginal Cost A manufacturer's average-cost function, in dollars, is given by

$$\bar{c} = \frac{500}{\ln(q+20)}$$

Find the marginal cost (rounded to two decimal places) when $q = 50$.

51. Supply Change The supply of q units of a product at a price of p dollars per unit is given by $q(p) = 27 + 11\ln(2p+1)$. Find the rate of change of supply with respect to price, $\frac{dq}{dp}$.

52. Sound Perception The loudness of sound L, measured in decibels, perceived by the human ear depends upon intensity levels I according to $L = 10\log\frac{I}{I_0}$, where I_0 is the standard threshold of audibility. If $I_0 = 17$, find $\frac{dL}{dI}$, the rate of change of the loudness with respect to the intensity.

53. Biology In a certain experiment with bacteria, it is observed that the relative activeness of a given bacteria colony is described by

$$A = 6\ln\left(\frac{T}{a-T} - a\right)$$

where a is a constant and T is the surrounding temperature. Find the rate of change of A with respect to T.

54. Show that the relative rate of change of $y = f(x)$ with respect to x is equal to the derivative of $y = \ln f(x)$.

55. Show that $\frac{d}{dx}(\log_b u) = \frac{1}{u}(\log_b e)\frac{du}{dx}$.

In Problems 56 and 57, use differentiation rules to find $f'(x)$. Then use your graphing calculator to find all roots of $f'(x) = 0$. Round your answers to two decimal places.

56. $f(x) = x^3\ln x$

57. $f(x) = \frac{\ln(x^2)}{x^2}$

Objective

To develop a differentiation formula for $y = e^u$, to apply the formula, and to use it to differentiate an exponential function with a base other than e.

12.2 Derivatives of Exponential Functions

As we pointed out in Section 12.1, the exponential functions cannot be constructed from power functions using multiplication by a constant, arithmetic operations, and composition. However, the functions b^x, for $b > 0$ and $b \neq 1$, are inverse to the functions $\log_b(x)$, and if an invertible function f is differentiable, it is fairly easy to see that its inverse is differentiable. The key idea is that the graph of the inverse of a function is obtained by reflecting the graph of the original function in the line $y = x$. This reflection process preserves smoothness so that if the graph of an invertible function is smooth, then so is the graph of its inverse. Differentiating $f(f^{-1}(x)) = x$, we have

$$\frac{d}{dx}(f(f^{-1}(x))) = \frac{d}{dx}(x)$$

$$f'(f^{-1}(x))\frac{d}{dx}(f^{-1}(x)) = 1 \qquad \text{Chain Rule}$$

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

Thus we have

COMBINING RULE 6 Inverse Function Rule

If f is an invertible, differentiable function, then f^{-1} is differentiable and

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}$$

As with the chain rule, Leibniz notation is well suited for inverse functions. Indeed, if $y = f^{-1}(x)$, then $\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x))$ and since $f(y) = x$, $f'(y) = \frac{dx}{dy}$. When we substitute these in Combining Rule 6, we get

$$\frac{dy}{dx} = \frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(y)} = \frac{1}{\frac{dx}{dy}}$$

so that Combining Rule 6 can be rewritten as

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \tag{1}$$

In the immediate case of interest, with $y = e^x$ so that $x = \ln y$ and $dx/dy = 1/y = 1/e^x$, we have

$$\frac{d}{dx}(e^x) = \frac{1}{\frac{1}{e^x}} = e^x$$

which we record as

$$\frac{d}{dx}(e^x) = e^x \tag{2}$$

CAUTION!

The power rule does not apply to e^x and other exponential functions, b^x. The power rule applies to power functions, x^a. Note the location of the variable.

For u a differentiable function of x, an application of the Chain Rule gives

$$\frac{d}{dx}(e^u) = e^u \frac{du}{dx} \tag{3}$$

EXAMPLE 1 Differentiating Functions Involving e^x

a. Find $\frac{d}{dx}(3e^x)$. Since 3 is a constant factor,

$$\begin{aligned}\frac{d}{dx}(3e^x) &= 3\frac{d}{dx}(e^x) \\ &= 3e^x \qquad \text{by Equation (2)}\end{aligned}$$

If a quotient can be easily rewritten as a product, then we can use the somewhat simpler product rule rather than the quotient rule.

b. If $y = \frac{x}{e^x}$, find $\frac{dy}{dx}$.

Solution: We could use first the quotient rule and then Equation (2), but it is a little easier first to rewrite the function as $y = xe^{-x}$ and use the product rule and Equation (3):

$$\frac{dy}{dx} = e^{-x}\frac{d}{dx}(x) + x\frac{d}{dx}(e^{-x}) = e^{-x}(1) + x(e^{-x})(-1) = e^{-x}(1-x) = \frac{1-x}{e^x}$$

c. If $y = e^2 + e^x + \ln 3$, find y'.

Solution: Since e^2 and $\ln 3$ are constants, $y' = 0 + e^x + 0 = e^x$.

Now Work Problem 1 ◁

APPLY IT ▷

3. When an object is moved from one environment to another, the change in temperature of the object is given by $T = Ce^{kt}$, where C is the temperature difference between the two environments, t is the time in the new environment, and k is a constant. Find the rate of change of temperature with respect to time.

EXAMPLE 2 Differentiating Functions Involving e^u

a. Find $\frac{d}{dx}\left(e^{x^3+3x}\right)$.

Solution: The function has the form e^u with $u = x^3 + 3x$. From Equation (2),

$$\begin{aligned}\frac{d}{dx}\left(e^{x^3+3x}\right) &= e^{x^3+3x}\frac{d}{dx}(x^3+3x) = e^{x^3+3x}(3x^2+3) \\ &= 3(x^2+1)e^{x^3+3x}\end{aligned}$$

$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$. Don't forget the $\frac{du}{dx}$.

b. Find $\frac{d}{dx}(e^{x+1}\ln(x^2+1))$.

Solution: By the product rule,

$$\begin{aligned}\frac{d}{dx}(e^{x+1}\ln(x^2+1)) &= e^{x+1}\frac{d}{dx}(\ln(x^2+1)) + (\ln(x^2+1))\frac{d}{dx}(e^{x+1}) \\ &= e^{x+1}\left(\frac{1}{x^2+1}\right)(2x) + (\ln(x^2+1))e^{x+1}(1) \\ &= e^{x+1}\left(\frac{2x}{x^2+1} + \ln(x^2+1)\right)\end{aligned}$$

Now Work Problem 3 ◁

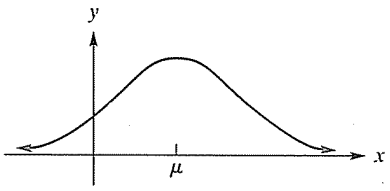


FIGURE 12.1 The normal-distribution density function.

EXAMPLE 3 **The Normal-Distribution Density Function**

An important function used in the social sciences is the **normal-distribution density function**

$$y = f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)((x-\mu)/\sigma)^2}$$

where σ (a Greek letter read "sigma") and μ (a Greek letter read "mu") are constants. The graph of this function, called the normal curve, is bell shaped. (See Figure 12.1.) Determine the rate of change of y with respect to x when $x = \mu + \sigma$.

Solution: The rate of change of y with respect to x is dy/dx. We note that the factor $\dfrac{1}{\sigma\sqrt{2\pi}}$ is a constant and the second factor has the form e^u, where

$$u = -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

Thus,

$$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-(1/2)((x-\mu)/\sigma)^2}\right)\left(-\frac{1}{2}(2)\left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\right)$$

Evaluating dy/dx when $x = \mu + \sigma$, we obtain

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{x=\mu+\sigma} &= \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-(1/2)((\mu+\sigma-\mu)/\sigma)^2}\right)\left(-\frac{\mu+\sigma-\mu}{\sigma}\right)\left(\frac{1}{\sigma}\right)\\ &= \frac{1}{\sigma\sqrt{2\pi}}\left(e^{-(1/2)}\right)\left(-\frac{1}{\sigma}\right)\\ &= \frac{-e^{-(1/2)}}{\sigma^2\sqrt{2\pi}} = \frac{-1}{\sigma^2\sqrt{2\pi e}}\end{aligned}$$

◁

Differentiating Exponential Functions to the Base b

Now that we are familiar with the derivative of e^u, we consider the derivative of the more general exponential function b^u. Because $b = e^{\ln b}$, we can express b^u as an exponential function with the base e, a form we can differentiate. We have

$$\begin{aligned}\frac{d}{dx}(b^u) = \frac{d}{dx}((e^{\ln b})^u) &= \frac{d}{dx}(e^{(\ln b)u})\\ &= e^{(\ln b)u}\frac{d}{dx}((\ln b)u)\\ &= e^{(\ln b)u}(\ln b)\left(\frac{du}{dx}\right)\\ &= b^u(\ln b)\frac{du}{dx} \qquad \text{since } e^{(\ln b)u} = b^u\end{aligned}$$

Summarizing,

$$\frac{d}{dx}(b^u) = b^u(\ln b)\frac{du}{dx} \tag{4}$$

Note that if $b = e$, then the factor $\ln b$ in Equation (4) is 1. Thus, if exponential functions to the base e are used, we have a simpler differentiation formula with which to work. This is the reason natural exponential functions are used extensively in calculus. Rather than memorizing Equation (4), we advocate remembering the procedure for obtaining it.

Procedure to Differentiate b^u

Convert b^u to a natural exponential function by using the property that $b = e^{\ln b}$, and then differentiate.

The next example will illustrate this procedure.

EXAMPLE 4 Differentiating an Exponential Function with Base 4

Find $\frac{d}{dx}(4^x)$.

Solution: Using the preceding procedure, we have

$$\begin{aligned}\frac{d}{dx}(4^x) &= \frac{d}{dx}((e^{\ln 4})^x) \\ &= \frac{d}{dx}\left(e^{(\ln 4)x}\right) && \text{form : } \frac{d}{dx}(e^u) \\ &= e^{(\ln 4)x}(\ln 4) && \text{by Equation (2)} \\ &= 4^x(\ln 4)\end{aligned}$$

Now Work Problem 15 ◁

Verify the result by using Equation (4) directly.

EXAMPLE 5 Differentiating Different Forms

Find $\frac{d}{dx}\left(e^2 + x^e + 2^{\sqrt{x}}\right)$.

Solution: Here we must differentiate three different forms; do not confuse them! The first (e^2) is a constant base to a constant power, so it is a constant itself. Thus, its derivative is zero. The second (x^e) is a variable base to a constant power, so the power rule applies. The third ($2^{\sqrt{x}}$) is a constant base to a variable power, so we must differentiate an exponential function. Taken all together, we have

$$\begin{aligned}\frac{d}{dx}\left(e^2 + x^e + 2^{\sqrt{x}}\right) &= 0 + ex^{e-1} + \frac{d}{dx}\left[e^{(\ln 2)\sqrt{x}}\right] \\ &= ex^{e-1} + \left[e^{(\ln 2)\sqrt{x}}\right](\ln 2)\left(\frac{1}{2\sqrt{x}}\right) \\ &= ex^{e-1} + \frac{2^{\sqrt{x}}\ln 2}{2\sqrt{x}}\end{aligned}$$

Now Work Problem 17 ◁

EXAMPLE 6 Differentiating Power Functions Again

We have often used the rule $d/dx(x^a) = ax^{a-1}$, but we have only *proved* it for a a positive integer and a few other special cases. At least for $x > 0$, we can now improve our understanding of power functions, using Equation (2).

For $x > 0$, we can write $x^a = e^{a \ln x}$. So we have

$$\frac{d}{dx}(x^a) = \frac{d}{dx}e^{a\ln x} = e^{a\ln x}\frac{d}{dx}(a\ln x) = x^a(ax^{-1}) = ax^{a-1}$$

Now Work Problem 19 ◁

PROBLEMS 12.2

In Problems 1–28, differentiate the functions.

1. $y = 5e^x$
2. $y = \dfrac{ae^x}{b}$
3. $y = e^{2x^2+3}$
4. $y = e^{2x^2+5}$
5. $y = e^{9-5x}$
6. $f(q) = e^{-q^3+6q-1}$
7. $f(r) = e^{4r^3+5r^2+2r+6}$
8. $y = e^{x^2+6x^3+1}$
9. $y = xe^x$
10. $y = 3x^4e^{-x}$
11. $y = x^2e^{-x^2}$
12. $y = xe^{ax}$
13. $y = \dfrac{e^x + e^{-x}}{3}$
14. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
15. $y = 5^{2x^3}$
16. $y = 2^x x^2$
17. $f(w) = \dfrac{e^{aw}}{w^2 + w + 1}$
18. $y = e^{x-\sqrt{x}}$
19. $y = e^{1+\sqrt{x}}$
20. $y = (e^{2x} + 1)^3$
21. $y = x^5 - 5^x$
22. $f(z) = e^{1/z}$
23. $y = \dfrac{e^x - 1}{e^x + 1}$
24. $y = e^{2x}(x + 6)$
25. $y = \ln e^x$
26. $y = e^{-x} \ln x$
27. $y = x^x$
28. $y = \ln e^{4x+1}$
29. If $f(x) = ee^xe^{x^2}$, find $f'(-1)$.
30. If $f(x) = 5^{x^2 \ln x}$, find $f'(1)$.
31. Find an equation of the tangent line to the curve $y = e^x$ when $x = -2$.
32. Find an equation of the tangent line to the curve $y = e^x$ at the point $(1, e)$. Show that this tangent line passes through $(0, 0)$ and show that it is the only tangent line to $y = e^x$ that passes through $(0, 0)$.

For each of the demand equations in Problems 33 and 34, find the rate of change of price p with respect to quantity q. What is the rate of change for the indicated value of q?

33. $p = 15e^{-0.001q}; q = 500$
34. $p = 9e^{-5q/750}; q = 300$

In Problems 35 and 36, $\bar{c}$ is the average cost of producing q units of a product. Find the marginal-cost function and the marginal cost for the given values of q.

35. $\bar{c} = \dfrac{7000e^{q/700}}{q}; q = 350, q = 700$
36. $\bar{c} = \dfrac{850}{q} + 4000\dfrac{e^{(2q+6)/800}}{q}; q = 97, q = 197$
37. If $w = e^{x^2}$ and $x = \dfrac{t+1}{t-1}$, find $\dfrac{dw}{dt}$ when $t = 2$.
38. If $f'(x) = x^3$ and $u = e^x$, show that

$$\frac{d}{dx}[f(u)] = e^{4x}$$

39. Determine the value of the positive constant c if

$$\frac{d}{dx}(c^x - x^c)\bigg|_{x=1} = 0$$

40. Calculate the relative rate of change of

$$f(x) = 10^{-x} + \ln(8 + x) + 0.01e^{x-2}$$

when $x = 2$. Round your answer to four decimal places.

41. **Production Run** For a firm, the daily output on the tth day of a production run is given by

$$q = 500(1 - e^{-0.2t})$$

Find the rate of change of output q with respect to t on the tenth day.

42. **Normal-Density Function** For the normal-density function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

find $f'(-1)$.

43. **Population** The population, in millions, of the greater Seattle area t years from 1970 is estimated by $P = 1.92e^{0.0176t}$. Show that $dP/dt = kP$, where k is a constant. This means that the rate of change of population at any time is proportional to the population at that time.

44. **Market Penetration** In a discussion of diffusion of a new process into a market, Hurter and Rubenstein[1] refer to an equation of the form

$$Y = k\alpha^{\beta^t}$$

where Y is the cumulative level of diffusion of the new process at time t and k, α, and β are positive constants. Verify their claim that

$$\frac{dY}{dt} = k\alpha^{\beta^t}(\beta^t \ln \alpha) \ln \beta$$

45. **Finance** After t years, the value S of a principal of P dollars invested at the annual rate of r compounded continuously is given by $S = Pe^{rt}$. Show that the relative rate of change of S with respect to t is r.

46. **Predator–Prey Relationship** In an article concerning predators and prey, Holling[2] refers to an equation of the form

$$y = K(1 - e^{-ax})$$

where x is the prey density, y is the number of prey attacked, and K and a are constants. Verify his statement that

$$\frac{dy}{dx} = a(K - y)$$

47. **Earthquakes** According to Richter,[3] the number of earthquakes of magnitude M or greater per unit of time is given by $N = 10^A 10^{-bM}$, where A and b are constants. Find dN/dM.

---

[1] A. P. Hurter, Jr., A. H. Rubenstein, et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

[2] C. S. Holling, "Some Characteristics of Simple Types of Predation and Parasitism," *The Canadian Entomologist*, XCI, no. 7 (1959), 385–98.

[3] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, Publishers, 1958).

48. Psychology Short-term retention was studied by Peterson and Peterson.[4] The two researchers analyzed a procedure in which an experimenter verbally gave a subject a three-letter consonant syllable, such as CHJ, followed by a three-digit number, such as 309. The subject then repeated the number and counted backward by 3's, such as 309, 306, 303, ... After a period of time, the subject was signaled by a light to recite the three-letter consonant syllable. The time between the experimenter's completion of the last consonant to the onset of the light was called the *recall interval*. The time between the onset of the light and the completion of a response was referred to as *latency*. After many trials, it was determined that, for a recall interval of t seconds, the approximate proportion of correct recalls with latency below 2.83 seconds was

$$p = 0.89[0.01 + 0.99(0.85)^t]$$

(a) Find dp/dt and interpret your result.
(b) Evaluate dp/dt when $t = 2$. Round your answer to two decimal places.

49. Medicine Suppose a tracer, such as a colored dye, is injected instantly into the heart at time $t = 0$ and mixes uniformly with blood inside the heart. Let the initial concentration of the tracer in the heart be C_0, and assume that the heart has constant volume V. Also assume that, as fresh blood flows into the heart, the diluted mixture of blood and tracer flows out at the constant positive rate r. Then the concentration of the tracer in the heart at time t is given by

$$C(t) = C_0 e^{-(r/V)t}$$

Show that $dC/dt = (-r/V)C(t)$.

50. Medicine In Problem 49, suppose the tracer is injected at a constant rate R. Then the concentration at time t is

$$C(t) = \frac{R}{r}\left[1 - e^{-(r/V)t}\right]$$

(a) Find $C(0)$.
(b) Show that $\dfrac{dC}{dt} = \dfrac{R}{V} - \dfrac{r}{V}C(t)$.

51. Schizophrenia Several models have been used to analyze the length of stay in a hospital. For a particular group of schizophrenics, one such model is[5]

$$f(t) = 1 - e^{-0.008t}$$

where $f(t)$ is the proportion of the group that was discharged at the end of t days of hospitalization. Find the rate of discharge (the proportion discharged per day) at the end of 100 days. Round your answer to four decimal places.

52. Savings and Consumption A country's savings S (in billions of dollars) is related to its national income I (in billions of dollars) by the equation

$$S = \ln \frac{3}{2 + e^{-I}}$$

(a) Find the marginal propensity to consume as a function of income.
(b) To the nearest million dollars, what is the national income when the marginal propensity to save is $\frac{1}{7}$?

In Problems 53 and 54, use differentiation rules to find $f'(x)$. Then use your graphing calculator to find all real zeros of $f'(x)$. Round your answers to two decimal places.

53. $f(x) = e^{2x^3 + x^2 - 3x}$ **54.** $f(x) = x + e^{-x}$

Objective

To give a mathematical analysis of the economic concept of elasticity.

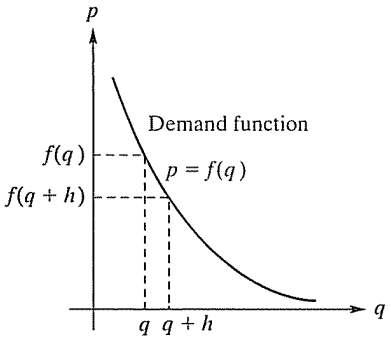


FIGURE 12.2 Change in demand.

12.3 Elasticity of Demand

Elasticity of demand is a means by which economists measure how a change in the price of a product will affect the quantity demanded. That is, it refers to consumer response to price changes. Loosely speaking, elasticity of demand is the ratio of the resulting percentage change in quantity demanded to a given percentage change in price:

$$\frac{\text{percentage change in quantity}}{\text{percentage change in price}}$$

For example, if, for a price increase of 5%, quantity demanded were to decrease by 2%, we would loosely say that elasticity of demand is $-2/5$.

To be more general, suppose $p = f(q)$ is the demand function for a product. Consumers will demand q units at a price of $f(q)$ per unit and will demand $q + h$ units at a price of $f(q + h)$ per unit (Figure 12.2). The *percentage* change in quantity demanded from q to $q + h$ is

$$\frac{(q+h) - q}{q} \cdot 100\% = \frac{h}{q} \cdot 100\%$$

The corresponding percentage change in price per unit is

$$\frac{f(q+h) - f(q)}{f(q)} \cdot 100\%$$

[4] L. R. Peterson and M. J. Peterson, "Short-Term Retention of Individual Verbal Items," *Journal of Experimental Psychology*, 58 (1959), 193–98.

[5] W. W. Eaton and G. A. Whitmore, "Length of Stay as a Stochastic Process: A General Approach and Application to Hospitalization for Schizophrenia," *Journal of Mathematical Sociology*, 5 (1977), 273–92.

The ratio of these percentage changes is

$$\frac{\dfrac{h}{q}\cdot 100\%}{\dfrac{f(q+h)-f(q)}{f(q)}\cdot 100\%} = \frac{h}{q}\cdot\frac{f(q)}{f(q+h)-f(q)}$$

$$= \frac{f(q)}{q}\cdot\frac{h}{f(q+h)-f(q)}$$

$$= \frac{\dfrac{f(q)}{q}}{\dfrac{f(q+h)-f(q)}{h}} \tag{1}$$

If f is differentiable, then as $h \to 0$, the limit of $[f(q+h)-f(q)]/h$ is $f'(q) = dp/dq$. Thus, the limit of (1) is

$$\frac{\dfrac{f(q)}{q}}{f'(q)} = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}} \qquad \text{since } p = f(q)$$

which is called the *point elasticity of demand.*

Definition

If $p = f(q)$ is a differentiable demand function, the ***point elasticity of demand,*** denoted by the Greek letter η (eta), at (q, p) is given by

$$\eta = \eta(q) = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}}$$

CAUTION!

Since p is a function of q, dp/dq is a function of q and thus the ratio that defines η is a function of q. That is why we write $\eta = \eta(q)$.

To illustrate, let us find the point elasticity of demand for the demand function $p = 1200 - q^2$. We have

$$\eta = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}} = \frac{\dfrac{1200-q^2}{q}}{-2q} = -\frac{1200-q^2}{2q^2} = -\left(\frac{600}{q^2} - \frac{1}{2}\right) \tag{2}$$

For example, if $q = 10$, then $\eta = -\left((600/10^2) - \frac{1}{2}\right) = -5\frac{1}{2}$. Since

$$\eta \approx \frac{\%\text{ change in demand}}{\%\text{ change in price}}$$

we have

$$(\%\text{ change in price})(\eta) \approx \%\text{ change in demand}$$

Thus, if price were increased by 1% when $q = 10$, then quantity demanded would change by approximately

$$(1\%)\left(-5\frac{1}{2}\right) = -5\frac{1}{2}\%$$

That is, demand would decrease $5\frac{1}{2}\%$. Similarly, decreasing price by $\frac{1}{2}\%$ when $q = 10$ results in a change in demand of approximately

$$\left(-\frac{1}{2}\%\right)\left(-5\frac{1}{2}\right) = 2\frac{3}{4}\%$$

Hence, demand increases by $2\frac{3}{4}\%$.

Note that when elasticity is evaluated, no units are attached to it—it is nothing more than a real number. In fact, the 100%'s arising from the word *percentage* cancel, so that elasticity is really an approximation of the ratio

$$\frac{\text{relative change in quantity}}{\text{relative change in price}}$$

and each of the relative changes is no more than a real number. For usual behavior of demand, an increase (decrease) in price corresponds to a decrease (increase) in quantity. This means that if price is plotted as a function of quantity then the graph will have a negative slope at each point. Thus, dp/dq will typically be negative, and since p and q are positive, η will typically be negative too. Some economists disregard the minus sign; in the preceding situation, they would consider the elasticity to be $5\frac{1}{2}$. We will not adopt this practice.

There are three categories of elasticity:

1. When $|\eta| > 1$, demand is *elastic*.
2. When $|\eta| = 1$, demand has *unit elasticity*.
3. When $|\eta| < 1$, demand is *inelastic*.

For example, in Equation (2), since $|\eta| = 5\frac{1}{2}$ when $q = 10$, demand is elastic. If $q = 20$, then $|\eta| = \left|-\left[(600/20^2) - \frac{1}{2}\right]\right| = 1$ so demand has unit elasticity. If $q = 25$, then $|\eta| = \left|-\frac{23}{50}\right|$, and demand is inelastic.

Loosely speaking, for a given percentage change in price, there is a greater percentage change in quantity demanded if demand is elastic, a smaller percentage change if demand is inelastic, and an equal percentage change if demand has unit elasticity. To better understand elasticity, it is helpful to think of typical examples. Demand for an essential utilty such as electricity tends to be inelastic through a wide range of prices. If electricity prices are increased by 10%, consumers can be expected to reduce their consumption somewhat, but a full 10% decrease may not be possible if most of their electricity usage is for essentials of life such as heating and food preparation. On the other hand, demand for luxury goods tends to be quite elastic. A 10% increase in the price of jewelry, for example, may result in a 50% decrease in demand.

EXAMPLE 1 Finding Point Elasticity of Demand

Determine the point elasticity of the demand equation

$$p = \frac{k}{q}, \quad \text{where } k > 0 \text{ and } q > 0$$

Solution: From the definition, we have

$$\eta = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}} = \frac{\dfrac{k}{q^2}}{\dfrac{-k}{q^2}} = -1$$

Thus, the demand has unit elasticity for all $q > 0$. The graph of $p = k/q$ is called an *equilateral hyperbola* and is often found in economics texts in discussions of elasticity. (See Figure 2.11 for a graph of such a curve.)

Now Work Problem 1 ◁

If we are given $p = f(q)$ for our demand equation, as in our discussion thus far, then it is usually straightforward to calculate $dp/dq = f'(q)$. However, if instead we are given q as a function of p, then we will have $q = f^{-1}(p)$ and, from Section 12.2,

$$\frac{dp}{dq} = \frac{1}{\dfrac{dq}{dp}}$$

It follows that

$$\eta = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}} = \frac{p}{q} \cdot \frac{dq}{dp} \tag{3}$$

provides another useful expression for η. Notice too that if $q = g(p)$, then

$$\eta = \eta(p) = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{g(p)} \cdot g'(p) = p \cdot \frac{g'(p)}{g(p)}$$

and thus

$$\text{elasticity} = \text{price} \cdot \text{relative rate of change of quantity as a function of price} \tag{4}$$

EXAMPLE 2 **Finding Point Elasticity of Demand**

Determine the point elasticity of the demand equation

$$q = p^2 - 40p + 400, \quad \text{where } q > 0$$

Solution: Here we have q given as a function of p and it is easy to see that $dq/dp = 2p - 40$. Thus,

$$\eta(p) = \frac{p}{q} \cdot \frac{dq}{dp} = \frac{p}{q(p)}(2p - 40)$$

For example, if $p = 15$, then $q = q(15) = 25$; hence, $\eta(15) = (15(-10))/25 = -6$, so demand is elastic for $p = 15$.

Now Work Problem 13 ◁

Here we analyze elasticity for linear demand.

Point elasticity for a *linear* demand equation is quite interesting. Suppose the equation has the form

$$p = mq + b, \quad \text{where } m < 0 \text{ and } b > 0$$

(See Figure 12.3.) We assume that $q > 0$; thus, $p < b$. The point elasticity of demand is

$$\eta = \frac{\dfrac{p}{q}}{\dfrac{dp}{dq}} = \frac{\dfrac{p}{q}}{m} = \frac{p}{mq} = \frac{p}{p - b}$$

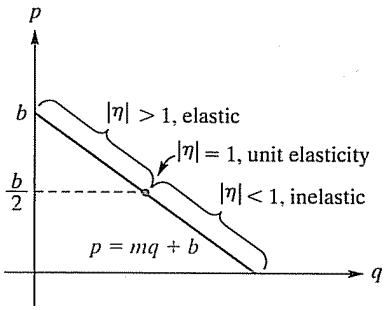


FIGURE 12.3 Elasticity for linear demand.

By considering $d\eta/dp$, we will show that η is a decreasing function of p. By the quotient rule,

$$\frac{d\eta}{dp} = \frac{(p - b) - p}{(p - b)^2} = -\frac{b}{(p - b)^2}$$

Since $b > 0$ and $(p - b)^2 > 0$, it follows that $d\eta/dp < 0$, meaning that the graph of $\eta = \eta(p)$ has a negative slope. This means that as price p increases, elasticity η decreases. However, p ranges between 0 and b, and at the midpoint of this range, $b/2$,

$$\eta = \eta(b) = \frac{\dfrac{b}{2}}{\dfrac{b}{2} - b} = \frac{\dfrac{b}{2}}{-\dfrac{b}{2}} = -1$$

Therefore, if $p < b/2$, then $\eta > -1$; if $p > b/2$, then $\eta < -1$. Because we typically have $\eta < 0$, we can state these facts another way: When $p < b/2, |\eta| < 1$, and demand is inelastic; when $p = b/2, |\eta| = 1$, and demand has unit elasticity; when $p > b/2, |\eta| > 1$ and demand is elastic. This shows that the slope of a demand curve is not a measure of elasticity. The slope of the line in Figure 12.3 is m everywhere, but elasticity varies with the point on the line. Of course, this is in accord with Equation (4).

Elasticity and Revenue

Here we analyze the relationship between elasticity and the rate of change of revenue.

Turning to a different situation, we can relate how elasticity of demand affects changes in revenue (marginal revenue). If $p = f(q)$ is a manufacturer's demand function, the total revenue is given by

$$r = pq$$

To find the marginal revenue, dr/dq, we differentiate r by using the product rule:

$$\frac{dr}{dq} = p + q\frac{dp}{dq}. \tag{5}$$

Factoring the right side of Equation (5), we have

$$\frac{dr}{dq} = p\left(1 + \frac{q}{p}\frac{dp}{dq}\right)$$

But

$$\frac{q}{p}\frac{dp}{dq} = \frac{\dfrac{dp}{dq}}{\dfrac{p}{q}} = \frac{1}{\eta}$$

Thus,

$$\frac{dr}{dq} = p\left(1 + \frac{1}{\eta}\right) \tag{6}$$

If demand is elastic, then $\eta < -1$, so $1 + \dfrac{1}{\eta} > 0$. If demand is inelastic, then $\eta > -1$, so $1 + \dfrac{1}{\eta} < 0$. We can assume that $p > 0$. From Equation (6) we can conclude that $dr/dq > 0$ on intervals for which demand is elastic. As we will soon see, a function is increasing on intervals for which its derivative is positive and a function is decreasing on intervals for which its derivative is negative. Hence, total revenue r is increasing on intervals for which demand is elastic and total revenue is decreasing on intervals for which demand is inelastic.

Thus, we conclude from the preceding argument that as more units are sold, a manufacturer's total revenue increases if demand is elastic, but decreases if demand is inelastic. That is, if demand is elastic, a lower price will increase revenue. This means that a lower price will cause a large enough increase in demand to actually increase revenue. If demand is inelastic, a lower price will decrease revenue. For unit elasticity, a lower price leaves total revenue unchanged.

If we solve the demand equation to obtain the form $q = g(p)$, rather than $p = f(q)$, then a similar analysis gives

$$\frac{dr}{dp} = q(1 + \eta) \tag{7}$$

and the conclusions of the last paragraph follow even more directly.

PROBLEMS 12.3

In Problems 1–14, find the point elasticity of the demand equations for the indicated values of q or p, and determine whether demand is elastic, is inelastic, or has unit elasticity.

1. $p = 40 - 2q;\ q = 5$

2. $p = 10 - 0.04q;\ q = 100$

3. $p = \dfrac{3000}{q};\ q = 300$

4. $p = \dfrac{500}{q^2};\ q = 52$

5. $p = \dfrac{500}{q+2};\ q = 104$

6. $p = \dfrac{800}{2q+1};\ q = 24$

7. $p = 150 - e^{q/100};\ q = 100$

8. $p = 250e^{-q/50};\ q = 50$

9. $q = 1200 - 150p;\ p = 4$

10. $q = 100 - p;\ p = 50$

11. $q = \sqrt{500 - p};\ p = 400$

12. $q = \sqrt{2500 - p^2};\ p = 20$

13. $q = (p - 50)^2;\ p = 10$

14. $q = p^2 - 50p + 850;\ p = 20$

15. For the linear demand equation $p = 13 - 0.05q$, verify that demand is elastic when $p = 10$, is inelastic when $p = 3$, and has unit elasticity when $p = 6.50$.

16. For what value (or values) of q do the following demand equations have unit elasticity?

(a) $p = 36 - 0.25q$
(b) $p = 300 - q^2$

17. The demand equation for a product is

$$q = 500 - 40p + p^2$$

where p is the price per unit (in dollars) and q is the quantity of units demanded (in thousands). Find the point elasticity of demand when $p = 15$. If this price of 15 is increased by $\frac{1}{2}\%$, what is the approximate change in demand?

18. The demand equation for a certain product is

$$q = \sqrt{3000 - p^2}$$

where p is in dollars. Find the point elasticity of demand when $p = 40$, and use this value to compute the percentage change in demand if the price of \$40 is increased by 7%.

19. For the demand equation $p = 500 - 2q$, verify that demand is elastic and total revenue is increasing for $0 < q < 125$. Verify that demand is inelastic and total revenue is decreasing for $125 < q < 250$.

20. Verify that $\dfrac{dr}{dq} = p\left(1 + \dfrac{1}{\eta}\right)$ if $p = 50 - 3q$.

21. Repeat Problem 20 for $p = \dfrac{1000}{q^2}$.

22. Suppose $p = mq + b$ is a linear demand equation, where $m \neq 0$ and $b > 0$.
(a) Show that $\lim_{p \to b^-} \eta = -\infty$.
(b) Show that $\eta = 0$ when $p = 0$.

23. The demand equation for a manufacturer's product has the form

$$q = a\sqrt{b - cp^2}$$

where a, b, and c are positive constants.
(a) Show that elasticity does not depend on a.
(b) Determine the interval of prices for which demand is elastic.
(c) For which price is there unit elasticity?

24. Given the demand equation $q^2(1 + p)^2 = p$, determine the point elasticity of demand when $p = 9$.

25. The demand equation for a product is

$$q = \frac{60}{p} + \ln(65 - p^3)$$

(a) Determine the point elasticity of demand when $p = 4$, and classify the demand as elastic, inelastic, or of unit elasticity at this price level.
(b) If the price is lowered by 2% (from \$4.00 to \$3.92), use the answer to part (a) to estimate the corresponding percentage change in quantity sold.
(c) Will the changes in part (b) result in an increase or decrease in revenue? Explain.

26. The demand equation for a manufacturer's product is

$$p = 50(151 - q)^{0.02\sqrt{q+19}}$$

(a) Find the value of dp/dq when 150 units are demanded.
(b) Using the result in part (a), determine the point elasticity of demand when 150 units are demanded. At this level, is demand elastic, inelastic, or of unit elasticity?
(c) Use the result in part (b) to approximate the price per unit if demand decreases from 150 to 140 units.
(d) If the current demand is 150 units, should the manufacturer increase or decrease price in order to increase revenue? (Justify your answer.)

27. A manufacturer of aluminum doors currently is able to sell 500 doors per week at a price of \$80 each. If the price were lowered to \$75 each, an additional 50 doors per week could be sold. Estimate the current elasticity of demand for the doors, and also estimate the current value of the manufacturer's marginal-revenue function.

28. Given the demand equation

$$p = 2000 - q^2$$

where $5 \leq q \leq 40$, for what value of q is $|\eta|$ a maximum? For what value is it a minimum?

29. Repeat Problem 28 for

$$p = \frac{200}{q + 5}$$

such that $5 \leq q \leq 95$.

Objective

To discuss the notion of a function defined implicitly and to determine derivatives by means of implicit differentiation.

12.4 Implicit Differentiation

Implicit differentiation is a technique for differentiating functions that are not given in the usual form $y = f(x)$ [nor in the form $x = g(y)$]. To introduce this technique, we will find the slope of a tangent line to a circle. Let us take the circle of radius 2 whose center is at the origin (Figure 12.4). Its equation is

$$x^2 + y^2 = 4$$
$$x^2 + y^2 - 4 = 0 \tag{1}$$

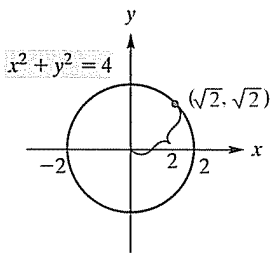


FIGURE 12.4 The circle $x^2 + y^2 = 4$.

The point $(\sqrt{2}, \sqrt{2})$ lies on the circle. To find the slope at this point, we need to find dy/dx there. Until now, we have always had y given explicitly (directly) in terms of x before determining y'; that is, our equation was always in the form $y = f(x)$ [or in the form $x = g(y)$]. In Equation (1), this is not so. We say that Equation (1) has the form $F(x, y) = 0$, where $F(x, y)$ denotes a function of two variables as introduced in Section 2.8. The obvious thing to do is solve Equation (1) for y in terms of x:

$$x^2 + y^2 - 4 = 0$$
$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2} \tag{2}$$

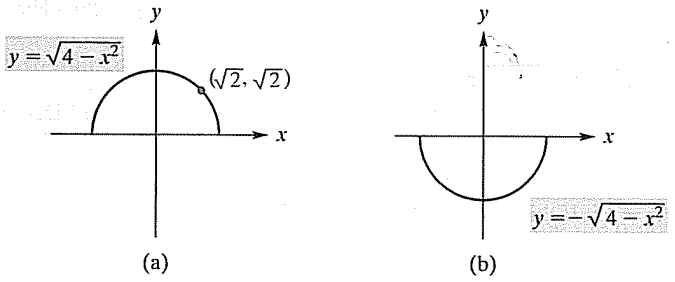


FIGURE 12.5 $x^2 + y^2 = 4$ gives rise to two different functions.

A problem now occurs: Equation (2) may give two values of y for a value of x. It does not define y explicitly as a function of x. We can, however, suppose that Equation (1) defines y as one of two different functions of x,

$$y = +\sqrt{4 - x^2} \quad \text{and} \quad y = -\sqrt{4 - x^2}$$

whose graphs are given in Figure 12.5. Since the point $(\sqrt{2}, \sqrt{2})$ lies on the graph of $y = \sqrt{4 - x^2}$, we should differentiate that function:

$$\begin{aligned} y &= \sqrt{4 - x^2} \\ \frac{dy}{dx} &= \frac{1}{2}(4 - x^2)^{-1/2}(-2x) \\ &= -\frac{x}{\sqrt{4 - x^2}} \end{aligned}$$

So

$$\left.\frac{dy}{dx}\right|_{x=\sqrt{2}} = -\frac{\sqrt{2}}{\sqrt{4 - 2}} = -1$$

Thus, the slope of the circle $x^2 + y^2 - 4 = 0$ at the point $(\sqrt{2}, \sqrt{2})$ is -1.

Let us summarize the difficulties we had. First, y was not originally given explicitly in terms of x. Second, after we tried to find such a relation, we ended up with more than one function of x. In fact, depending on the equation given, it may be very complicated or even impossible to find an explicit expression for y. For example, it would be difficult to solve $ye^x + \ln(x + y) = 0$ for y. We will now consider a method that avoids such difficulties.

An equation of the form $F(x, y) = 0$, such as we had originally, is said to express y *implicitly* as a function of x. The word *implicitly* is used, since y is not given explicitly as a function of x. However, it is assumed or *implied* that the equation defines y as at least one differentiable function of x. Thus, we assume that Equation (1), $x^2 + y^2 - 4 = 0$, defines some differentiable function of x, say, $y = f(x)$. Next, we treat y as a function of x and differentiate both sides of Equation (1) with respect to x. Finally, we solve the result for dy/dx. Applying this procedure, we obtain

$$\begin{aligned} \frac{d}{dx}(x^2 + y^2 - 4) &= \frac{d}{dx}(0) \\ \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) - \frac{d}{dx}(4) &= \frac{d}{dx}(0) \end{aligned} \tag{3}$$

We know that $\dfrac{d}{dx}(x^2) = 2x$ and that both $\dfrac{d}{dx}(4)$ and $\dfrac{d}{dx}(0)$ are 0. But $\dfrac{d}{dx}(y^2)$ is **not** $2y$, because we are differentiating with respect to x, not y. That is, y is not the independent variable. Since y is assumed to be a function of x, y^2 has the form u^n, where y plays the role of u. Just as the power rule states that $\dfrac{d}{dx}(u^2) = 2u\dfrac{du}{dx}$, we have $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$. Hence, Equation (3) becomes

$$2x + 2y\frac{dy}{dx} = 0$$

Solving for dy/dx gives

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{for } y \neq 0 \tag{4}$$

Notice that the expression for dy/dx involves the variable y as well as x. This means that to find dy/dx at a point, both coordinates of the point must be substituted into dy/dx. Thus,

$$\left.\frac{dy}{dx}\right|_{(\sqrt{2},\sqrt{2})} = -\frac{\sqrt{2}}{\sqrt{2}} = -1$$

as before. This method of finding dy/dx is called **implicit differentiation.** We note that Equation (4) is not defined when $y = 0$. Geometrically, this is clear, since the tangent line to the circle at either $(2, 0)$ or $(-2, 0)$ is vertical, and the slope is not defined.

Here are the steps to follow when differentiating implicitly:

Implicit Differentiation Procedure

For an equation that we assume defines y implicitly as a differentiable function of x, the derivative $\dfrac{dy}{dx}$ can be found as follows:

1. Differentiate both sides of the equation with respect to x.
2. Collect all terms involving $\dfrac{dy}{dx}$ on one side of the equation, and collect all other terms on the other side.
3. Factor $\dfrac{dy}{dx}$ from the side involving the $\dfrac{dy}{dx}$ terms.
4. Solve for $\dfrac{dy}{dx}$, noting any restrictions.

EXAMPLE 1 **Implicit Differentiation**

Find $\dfrac{dy}{dx}$ by implicit differentiation if $y + y^3 - x = 7$.

Solution: Here y is not given as an explicit function of x [that is, not in the form $y = f(x)$]. Thus, we assume that y is an implicit (differentiable) function of x and apply the preceding four-step procedure:

1. Differentiating both sides with respect to x, we have

$$\frac{d}{dx}(y + y^3 - x) = \frac{d}{dx}(7)$$

$$\frac{d}{dx}(y) + \frac{d}{dx}(y^3) - \frac{d}{dx}(x) = \frac{d}{dx}(7)$$

CAUTION!
The derivative of y^3 with respect to x is $3y^2\dfrac{dy}{dx}$, not $3y^2$.

Now, $\dfrac{d}{dx}(y)$ can be written $\dfrac{dy}{dx}$, and $\dfrac{d}{dx}(x) = 1$. By the power rule,

$$\frac{d}{dx}(y^3) = 3y^2\frac{dy}{dx}$$

Hence, we obtain

$$\frac{dy}{dx} + 3y^2\frac{dy}{dx} - 1 = 0$$

2. Collecting all $\dfrac{dy}{dx}$ terms on the left side and all other terms on the right side gives

$$\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 1$$

3. Factoring $\frac{dy}{dx}$ from the left side, we have

$$\frac{dy}{dx}(1+3y^2)=1$$

In an implicit-differentiation problem, we are able to find the derivative of a function without knowing the function.

4. We solve for $\frac{dy}{dx}$ by dividing both sides by $1+3y^2$:

$$\frac{dy}{dx}=\frac{1}{1+3y^2}$$

Because step 4 of the process often involves division by an expression involving the variables, the answer obtained must often be restricted to exclude those values of the variables that would make the denominator zero. Here the denominator is always greater than or equal to 1, so there is no restriction.

Now Work Problem 3 ◁

APPLY IT ▷

4. Suppose that P, the proportion of people affected by a certain disease, is described by $\ln\left(\frac{P}{1-P}\right)=0.5t$, where t is the time in months. Find $\frac{dP}{dt}$, the rate at which P grows with respect to time.

EXAMPLE 2 Implicit Differentiation

Find $\frac{dy}{dx}$ if $x^3+4xy^2-27=y^4$.

Solution: Since y is not given explicitly in terms of x, we will use the method of implicit differentiation:

1. Assuming that y is a function of x and differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^3+4xy^2-27)=\frac{d}{dx}(y^4)$$

$$\frac{d}{dx}(x^3)+4\frac{d}{dx}(xy^2)-\frac{d}{dx}(27)=\frac{d}{dx}(y^4)$$

To find $\frac{d}{dx}(xy^2)$, we use the product rule:

$$3x^2+4\left[x\frac{d}{dx}(y^2)+y^2\frac{d}{dx}(x)\right]-0=4y^3\frac{dy}{dx}$$

$$3x^2+4\left[x\left(2y\frac{dy}{dx}\right)+y^2(1)\right]=4y^3\frac{dy}{dx}$$

$$3x^2+8xy\frac{dy}{dx}+4y^2=4y^3\frac{dy}{dx}$$

2. Collecting $\frac{dy}{dx}$ terms on the left side and other terms on the right gives

$$8xy\frac{dy}{dx}-4y^3\frac{dy}{dx}=-3x^2-4y^2$$

3. Factoring $\frac{dy}{dx}$ from the left side yields

$$\frac{dy}{dx}(8xy-4y^3)=-3x^2-4y^2$$

4. Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx}=\frac{-3x^2-4y^2}{8xy-4y^3}=\frac{3x^2+4y^2}{4y^3-8xy}$$

which gives the value of dy/dx at points (x,y) for which $4y^3-8xy\neq 0$.

Now Work Problem 11 ◁

APPLY IT ▶

5. The volume V enclosed by a spherical balloon of radius r is given by the equation $V = \frac{4}{3}\pi r^3$. If the radius is increasing at a rate of 5 inches/minute (that is, $\frac{dr}{dt} = 5$), then find $\left.\frac{dV}{dt}\right|_{r=12}$, the rate of increase of the volume, when the radius is 12 inches.

EXAMPLE 3 Implicit Differentiation

Find the slope of the curve $x^3 = (y - x^2)^2$ at $(1, 2)$.

Solution: The slope at $(1, 2)$ is the value of dy/dx at that point. Finding dy/dx by implicit differentiation, we have

$$\frac{d}{dx}(x^3) = \frac{d}{dx}[(y - x^2)^2]$$

$$3x^2 = 2(y - x^2)\left(\frac{dy}{dx} - 2x\right)$$

$$3x^2 = 2\left(y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} + 2x^3\right)$$

$$3x^2 = 2y\frac{dy}{dx} - 4xy - 2x^2\frac{dy}{dx} + 4x^3$$

$$3x^2 + 4xy - 4x^3 = 2y\frac{dy}{dx} - 2x^2\frac{dy}{dx}$$

$$3x^2 + 4xy - 4x^3 = 2\frac{dy}{dx}(y - x^2)$$

$$\frac{dy}{dx} = \frac{3x^2 + 4xy - 4x^3}{2(y - x^2)} \quad \text{for } y - x^2 \neq 0$$

For the point $(1, 2)$, $y - x^2 = 2 - 1^2 = 1 \neq 0$. Thus, the slope of the curve at $(1, 2)$ is

$$\left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3(1)^2 + 4(1)(2) - 4(1)^3}{2(2 - (1)^2)} = \frac{7}{2}$$

Now Work Problem 25 ◁

APPLY IT ▶

6. A 10-foot ladder is placed against a vertical wall. Suppose the bottom of the ladder slides away from the wall at a constant rate of 3 ft/s. (That is, $\frac{dx}{dt} = 3$.) How fast is the top of the ladder sliding down the wall when the top of the ladder is 8 feet from the ground (that is, when $y = 8$)? (That is, what is $\frac{dy}{dt}$?) (Use the Pythagorean theorem for right triangles, $x^2 + y^2 = z^2$, where x and y are the legs of the triangle and z is the hypotenuse.)

EXAMPLE 4 Implicit Differentiation

If $q - p = \ln q + \ln p$, find dq/dp.

Solution: We assume that q is a function of p and differentiate both sides with respect to p:

$$\frac{d}{dp}(q) - \frac{d}{dp}(p) = \frac{d}{dp}(\ln q) + \frac{d}{dp}(\ln p)$$

$$\frac{dq}{dp} - 1 = \frac{1}{q}\frac{dq}{dp} + \frac{1}{p}$$

$$\frac{dq}{dp} - \frac{1}{q}\frac{dq}{dp} = \frac{1}{p} + 1$$

$$\frac{dq}{dp}\left(1 - \frac{1}{q}\right) = \frac{1}{p} + 1$$

$$\frac{dq}{dp}\left(\frac{q - 1}{q}\right) = \frac{1 + p}{p}$$

$$\frac{dq}{dp} = \frac{(1 + p)q}{p(q - 1)} \quad \text{for } p(q - 1) \neq 0$$

Now Work Problem 19 ◁

PROBLEMS 12.4

In Problems 1–24, find dy/dx by implicit differentiation.

1. $x^2 + 4y^2 = 4$
2. $3x^2 + 6y^2 = 1$
3. $2y^3 - 7x^2 = 5$
4. $5y^2 - 2x^2 = 10$
5. $\sqrt[3]{x} + \sqrt[3]{y} = 3$
6. $x^{1/5} + y^{1/5} = 4$
7. $x^{3/4} + y^{3/4} = 5$
8. $y^3 = 4x$
9. $xy = 36$
10. $x^2 + xy - 2y^2 = 0$
11. $xy - y - 11x = 5$
12. $x^3 - y^3 = 3x^2y - 3xy^2$
13. $2x^3 + y^3 - 12xy = 0$
14. $5x^3 + 6xy + 7y^3 = 0$
15. $x = \sqrt{y} + \sqrt[4]{y}$
16. $x^3y^3 + x = 9$
17. $5x^3y^4 - x + y^2 = 25$
18. $y^2 + y = \ln x$
19. $\ln(xy) = e^{xy}$
20. $\ln(xy) + x = 4$
21. $xe^y + y = 13$
22. $4x^2 + 9y^2 = 16$
23. $(1 + e^{3x})^2 = 3 + \ln(x + y)$
24. $e^{x-y} = \ln(x - y)$
25. If $x + xy + y^2 = 7$, find dy/dx at $(1, 2)$.
26. If $x\sqrt{y+1} = y\sqrt{x+1}$, find dy/dx at $(3, 3)$.
27. Find the slope of the curve $4x^2 + 9y^2 = 1$ at the point $\left(0, \frac{1}{3}\right)$; at the point (x_0, y_0).
28. Find the slope of the curve $(x^2 + y^2)^2 = 4y^2$ at the point $(0, 2)$.
29. Find equations of the tangent lines to the curve
$$x^3 + xy + y^3 = -1$$
at the points $(-1, -1)$, $(-1, 0)$, and $(-1, 1)$.
30. Repeat Problem 29 for the curve
$$y^2 + xy - x^2 = 5$$
at the point $(4, 3)$.

For the demand equations in Problems 31–34, find the rate of change of q with respect to p.

31. $p = 100 - q^2$
32. $p = 400 - \sqrt{q}$
33. $p = \dfrac{20}{(q+5)^2}$
34. $p = \dfrac{3}{q^2 + 1}$

35. Radioactivity The relative activity I/I_0 of a radioactive element varies with elapsed time according to the equation
$$\ln\left(\frac{I}{I_0}\right) = -\lambda t$$
where λ (a Greek letter read "lambda") is the disintegration constant and I_0 is the initial intensity (a constant). Find the rate of change of the intensity I with respect to the elapsed time t.

36. Earthquakes The magnitude M of an earthquake and its energy E are related by the equation[6]
$$1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$$
Here M is given in terms of Richter's preferred scale of 1958 and E is in ergs. Determine the rate of change of energy with respect to magnitude and the rate of change of magnitude with respect to energy.

37. Physical Scale The relationship among the speed (v), frequency (f), and wavelength (λ) of any wave is given by
$$v = f\lambda$$
Find $df/d\lambda$ by differentiating implicitly. (Treat v as a constant.) Then show that the same result is obtained if you first solve the equation for f and then differentiate with respect to λ.

38. Biology The equation $(P + a)(v + b) = k$ is called the "fundamental equation of muscle contraction."[7] Here P is the load imposed on the muscle, v is the velocity of the shortening of the muscle fibers, and a, b, and k are positive constants. Use implicit differentiation to show that, in terms of P,
$$\frac{dv}{dP} = -\frac{k}{(P + a)^2}$$

39. Marginal Propensity to Consume A country's savings S is defined implicitly in terms of its national income I by the equation
$$S^2 + \frac{1}{4}I^2 = SI + I$$
where both S and I are in billions of dollars. Find the marginal propensity to consume when $I = 16$ and $S = 12$.

40. Technological Substitution New products or technologies often tend to replace old ones. For example, today most commercial airlines use jet engines rather than prop engines. In discussing the forecasting of technological substitution, Hurter and Rubenstein[8] refer to the equation
$$\ln\frac{f(t)}{1 - f(t)} + \sigma\frac{1}{1 - f(t)} = C_1 + C_2 t$$
where $f(t)$ is the market share of the substitute over time t and C_1, C_2, and σ (a Greek letter read "sigma") are constants. Verify their claim that the rate of substitution is
$$f'(t) = \frac{C_2 f(t)[1 - f(t)]^2}{\sigma f(t) + [1 - f(t)]}$$

12.5 Logarithmic Differentiation

Objective

To describe the method of logarithmic differentiation and to show how to differentiate a function of the form u^v.

A technique called **logarithmic differentiation** often simplifies the differentiation of $y = f(x)$ when $f(x)$ involves products, quotients, or powers. The procedure is as

[6] K. E. Bullen, *An Introduction to the Theory of Seismology* (Cambridge, U.K.: Cambridge at the University Press, 1963).

[7] R. W. Stacy et al., *Essentials of Biological and Medical Physics* (New York: McGraw-Hill Book Company, 1955).

[8] A. P. Hurter, Jr., A. H. Rubenstein et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

follows:

Logarithmic Differentiation

To differentiate $y = f(x)$,

1. Take the natural logarithm of both sides. This results in
$$\ln y = \ln (f(x))$$
2. Simplify $\ln (f(x))$ by using properties of logarithms.
3. Differentiate both sides with respect to x.
4. Solve for $\dfrac{dy}{dx}$.
5. Express the answer in terms of x only. This requires substituting $f(x)$ for y.

There are a couple of points worth noting. First, irrespective of any simplification, the procedure produces

$$\frac{y'}{y} = \frac{d}{dx}(\ln (f(x))$$

so that

$$\frac{dy}{dx} = y\frac{d}{dx}(\ln (f(x))$$

is a formula that you can memorize, if you prefer. Second, the quantity $\dfrac{f'(x)}{f(x)}$, which results from differentiating $\ln (f(x))$, is what was called the *relative rate of change of* $f(x)$ in Section 11.3.

The next example illustrates the procedure.

EXAMPLE 1 **Logarithmic Differentiation**

Find y' if $y = \dfrac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}$.

Solution: Differentiating this function in the usual way is messy because it involves the quotient, power, and product rules. Logarithmic differentiation makes the work less of a chore.

1. We take the natural logarithm of both sides:
$$\ln y = \ln \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}$$
2. Simplifying by using properties of logarithms, we have
$$\begin{aligned}\ln y &= \ln (2x-5)^3 - \ln \left(x^2\sqrt[4]{x^2+1}\right)\\ &= 3\ln (2x-5) - (\ln x^2 + \ln (x^2+1)^{1/4})\\ &= 3\ln (2x-5) - 2\ln x - \frac{1}{4}\ln (x^2+1)\end{aligned}$$
3. Differentiating with respect to x gives
$$\begin{aligned}\frac{y'}{y} &= 3\left(\frac{1}{2x-5}\right)(2) - 2\left(\frac{1}{x}\right) - \frac{1}{4}\left(\frac{1}{x^2+1}\right)(2x)\\ &= \frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\end{aligned}$$
4. Solving for y' yields
$$y' = y\left(\frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\right)$$

CAUTION!

Since y is a function of x, differentiating $\ln y$ with respect to x gives $\dfrac{y'}{y}$.

5. Substituting the original expression for y gives y' in terms of x only:

$$y' = \frac{(2x-5)^3}{x^2\sqrt[4]{x^2+1}}\left[\frac{6}{2x-5} - \frac{2}{x} - \frac{x}{2(x^2+1)}\right]$$

Now Work Problem 1 ◁

Logarithmic differentiation can also be used to differentiate a function of the form $y = u^v$, where both u and v are differentiable functions of x. Because neither the base nor the exponent is necessarily a constant, the differentiation techniques for u^n and a^u do not apply here.

EXAMPLE 2 **Differentiating the Form u^v**

Differentiate $y = x^x$ by using logarithmic differentiation.

Solution: This example is a good candidate for the *formula* approach to logarithmic differentiation.

$$y' = y\frac{d}{dx}(\ln x^x) = x^x\frac{d}{dx}(x\ln x) = x^x\left((1)(\ln x) + (x)\left(\frac{1}{x}\right)\right) = x^x(\ln x + 1)$$

It is worthwhile mentioning that an alternative technique for differentiating a function of the form $y = u^v$ is to convert it to an exponential function to the base e. To illustrate, for the function in this example, we have

$$y = x^x = (e^{\ln x})^x = e^{x\ln x}$$

$$y' = e^{x\ln x}\left(1\ln x + x\frac{1}{x}\right) = x^x(\ln x + 1)$$

Now Work Problem 15 ◁

EXAMPLE 3 **Relative Rate of Change of a Product**

Show that the relative rate of change of a product is the sum of the relative rates of change of its factors. Use this result to express the percentage rate of change in revenue in terms of the percentage rate of change in price.

Solution: Recall that the relative rate of change of a function r is $\frac{r'}{r}$. We are to show that if $r = pq$, then $\frac{r'}{r} = \frac{p'}{p} + \frac{q'}{q}$. From $r = pq$ we have $\ln r = \ln p + \ln q$, which, when both sides are differentiated, gives

$$\frac{r'}{r} = \frac{p'}{p} + \frac{q'}{q}$$

as required. Multiplying both sides by 100% gives an expression for the percentage rate of change of r in terms of those of p and q:

$$\frac{r'}{r}100\% = \frac{p'}{p}100\% + \frac{q'}{q}100\%$$

If p is *price* per item and q is *quantity* sold, then $r = pq$ is total *revenue*. In this case we take differentiation to be with respect to p and note that now $\frac{q'}{q} = \eta\frac{p'}{p}$, where η is the elasticity of demand as in Section 12.3. It follows that in this case we have

$$\frac{r'}{r}100\% = (1+\eta)\frac{p'}{p}100\%$$

expressing the percentage rate of change in revenue in terms of the percentage rate of change in price. For example, if at a given price and quantity, $\eta = -5$, then a 1% increase in price will result in a $(1-5)\% = -4\%$ increase in revenue, which is to say a 4% *decrease* in revenue, while a 3% decrease in price—that is, a -3% *increase* in price—will result in a $(1-5)(-3)\% = 12\%$ increase in revenue. It is also clear that

at points at which there is unit elasticity ($\eta = -1$), any percentage change in price produces no percentage change in revenue.

Now Work Problem 29 ◁

EXAMPLE 4 Differentiating the Form u^v

Find the derivative of $y = (1 + e^x)^{\ln x}$.

Solution: This has the form $y = u^v$, where $u = 1 + e^x$ and $v = \ln x$. Using logarithmic differentiation, we have

$$\ln y = \ln\left((1 + e^x)^{\ln x}\right)$$

$$\ln y = (\ln x)\ln(1 + e^x)$$

$$\frac{y'}{y} = \left(\frac{1}{x}\right)(\ln(1 + e^x)) + (\ln x)\left(\frac{1}{1 + e^x} \cdot e^x\right)$$

$$\frac{y'}{y} = \frac{\ln(1 + e^x)}{x} + \frac{e^x \ln x}{1 + e^x}$$

$$y' = y\left(\frac{\ln(1 + e^x)}{x} + \frac{e^x \ln x}{1 + e^x}\right)$$

$$y' = (1 + e^x)^{\ln x}\left(\frac{\ln(1 + e^x)}{x} + \frac{e^x \ln x}{1 + e^x}\right)$$

Now Work Problem 17 ◁

Alternatively, we can differentiate even a general function of the form $y = u(x)^{v(x)}$ with $u(x) > 0$ by using the equation

$$u^v = e^{v \ln u}$$

Indeed, if $y = u(x)^{v(x)} = e^{v(x)\ln u(x)}$ for $u(x) > 0$, then

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{v(x)\ln u(x)}\right) = e^{v(x)\ln u(x)}\frac{d}{dx}(v(x)\ln u(x)) = u^v\left(v'(x)\ln u(x) + v(x)\frac{u'(x)}{u(x)}\right)$$

which could be summarized as

$$(u^v)' = u^v\left(v' \ln u + v\frac{u'}{u}\right)$$

As is often the case, there is no suggestion that the preceding formula should be memorized. The point here is that we have shown that *any* function of the form u^v can be differentiated using the equation $u^v = e^{v \ln u}$. The same result will be obtained from logarithmic differentiation:

$$\ln y = \ln(u^v)$$

$$\ln y = v \ln u$$

$$\frac{y'}{y} = v' \ln u + v\frac{u'}{u}$$

$$y' = y\left(v' \ln u + v\frac{u'}{u}\right)$$

$$(u^v)' = u^v\left(v' \ln u + v\frac{u'}{u}\right)$$

After completing this section, we understand how to differentiate each of the following forms:

$$y = \begin{cases} (f(x))^a & \text{(a)} \\ b^{f(x)} & \text{(b)} \\ (f(x))^{g(x)} & \text{(c)} \end{cases}$$

For type (a), use the power rule. For type (b), use the differentiation formula for exponential functions. [If $b \neq e$, first convert $b^{f(x)}$ to an e^u function.] For type (c), use logarithmic differentiation or first convert to an e^u function. Do not apply a rule in a situation where the rule does not apply. For example, the power rule does not apply to x^x.

PROBLEMS 12.5

In Problems 1–12, find y' by using logarithmic differentiation.

1. $y = (x+1)^2(x-2)(x^2+3)$

2. $y = (3x+4)(8x-1)^2(3x^2+1)^4$

3. $y = (3x^3-1)^2(2x+5)^3$

4. $y = (2x^2+1)\sqrt{8x^2-1}$

5. $y = \sqrt{x+1}\sqrt{x-1}\sqrt{x^2+1}$

6. $y = (2x+1)\sqrt{x^3+2}\sqrt[3]{2x+5}$

7. $y = \dfrac{\sqrt{1-x^2}}{1-2x}$

8. $y = \sqrt{\dfrac{x^2+5}{x+9}}$

9. $y = \dfrac{(2x^2+2)^2}{(x+1)^2(3x+2)}$

10. $y = \dfrac{x^2(1+x^2)}{\sqrt{x^2+4}}$

11. $y = \sqrt{\dfrac{(x+3)(x-2)}{2x-1}}$

12. $y = \sqrt[3]{\dfrac{6(x^3+1)^2}{x^6e^{-4x}}}$

In Problems 13–20, find y'.

13. $y = x^{x^2+1}$

14. $y = (2x)^{\sqrt{x}}$

15. $y = x^{\sqrt{x}}$

16. $y = \left(\dfrac{3}{x^2}\right)^x$

17. $y = (3x+1)^{2x}$

18. $y = (x^2+1)^{x+1}$

19. $y = 4e^x x^{3x}$

20. $y = (\sqrt{x})^x$

21. If $y = (4x-3)^{2x+1}$, find dy/dx when $x = 1$.

22. If $y = (\ln x)^{\ln x}$, find dy/dx when $x = e$.

23. Find an equation of the tangent line to

$$y = (x+1)(x+2)^2(x+3)^2$$

at the point where $x = 0$.

24. Find an equation of the tangent line to the graph of

$$y = x^x$$

at the point where $x = 1$.

25. Find an equation of the tangent line to the graph of

$$y = x^x$$

at the point where $x = e$.

26. If $y = x^x$, find the relative rate of change of y with respect to x when $x = 1$.

27. If $y = (3x)^{-2x}$, find the value of x for which the *percentage* rate of change of y with respect to x is 60.

28. Suppose $f(x)$ is a positive differentiable function and g is a differentiable function and $y = (f(x))^{g(x)}$. Use logarithmic differentiation to show that

$$\frac{dy}{dx} = (f(x))^{g(x)}\left(f'(x)\frac{g(x)}{f(x)} + g'(x)\ln(f(x))\right)$$

29. The demand equation for a compact disc is

$$q = 500 - 40p + p^2$$

If the price of \$15 is increased by 1/2%, find the corresponding percentage change in revenue.

30. Repeat Problem 29 with the same information except for a 5% *decrease* in price.

Objective

To approximate real roots of an equation by using calculus. The method shown is suitable for calculators.

12.6 Newton's Method

It is easy to solve equations of the form $f(x) = 0$ when f is a linear or quadratic function. For example, we can solve $x^2+3x-2 = 0$ by the quadratic formula. However, if $f(x)$ has a degree greater than 2 (or is not a polynomial), it may be difficult, or even impossible, to find solutions (or roots) of $f(x) = 0$ by the methods to which you are accustomed. For this reason, we may settle for approximate solutions, which can be obtained in a variety of efficient ways. For example, a graphing calculator can be used to estimate the real roots of $f(x) = 0$. In this section, we will study how the derivative can be so used (provided that f is differentiable). The procedure we will develop, called *Newton's method,* is well suited to a calculator or computer.

Newton's method requires an initial estimate for a root of $f(x) = 0$. One way of obtaining this estimate is by making a rough sketch of the graph of $y = f(x)$ and estimating the root from the graph. A point on the graph where $y = 0$ is an x-intercept, and the x-value of this point is a root of $f(x) = 0$. Another way of locating a root is based on the following fact:

> If f is continuous on the interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root between a and b.

Figure 12.6 depicts this situation. The x-intercept between a and b corresponds to a root of $f(x) = 0$, and we can use either a or b to approximate this root.

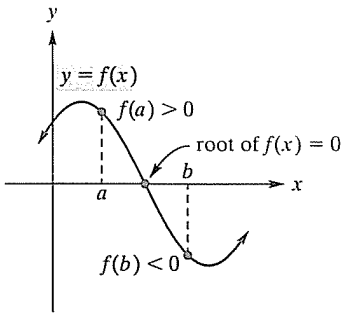


FIGURE 12.6 Root of $f(x) = 0$ between a and b, where $f(a)$ and $f(b)$ have opposite signs.

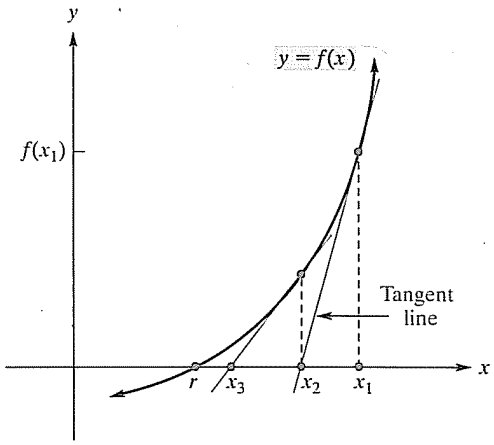


FIGURE 12.7 Improving approximation of root via tangent line.

Assuming that we have an estimated (but incorrect) value for a root, we turn to a way of getting a better approximation. In Figure 12.7, we see that $f(r) = 0$, so r is a root of the equation $f(x) = 0$. Suppose x_1 is an initial approximation to r (and one that is close to r). Observe that the tangent line to the curve at $(x_1, f(x_1))$ intersects the x-axis at the point $(x_2, 0)$, and x_2 is a better approximation to r than is x_1.

We can find x_2 from the equation of the tangent line. The slope of the tangent line is $f'(x_1)$, so a point-slope form for this line is

$$y - f(x_1) = f'(x_1)(x - x_1) \tag{1}$$

Since $(x_2, 0)$ is on the tangent line, its coordinates must satisfy Equation (1). This gives

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

$$-\frac{f(x_1)}{f'(x_1)} = x_2 - x_1 \qquad \text{if } f'(x_1) \neq 0$$

Thus,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \tag{2}$$

To get a better approximation to r, we again perform the procedure described, but this time we use x_2 as our starting point. This gives the approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \tag{3}$$

Repeating (or *iterating*) this computation over and over, we hope to obtain better approximations, in the sense that the sequence of values

$$x_1, x_2, x_3, \ldots$$

will approach r. In practice, we terminate the process when we have reached a desired degree of accuracy.

Analyzing Equations (2) and (3), we see how x_2 is obtained from x_1 and how x_3 is obtained from x_2. In general, x_{n+1} is obtained from x_n by means of the following general formula, called **Newton's method:**

Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 1, 2, 3, \ldots \tag{4}$$

TO REVIEW recursively defined sequences, see Section 1.6.

A formula, like Equation (4), that indicates how one number in a sequence is obtained from the preceding one is called a **recursion formula,** or an *iteration equation.*

APPLY IT ▶

7. If the total profit (in dollars) from the sale of x televisions is $P(x) = 20x - 0.01x^2 - 850 + 3\ln(x)$, use Newton's method to approximate the break-even quantities. (*Note:* There are two break-even quantities; one is between 10 and 50, and the other is between 1900 and 2000.) Give the x-value to the nearest integer.

In the event that a root lies between a and b and $f(a)$ and $f(b)$ are equally close to 0, choose either a or b as the first approximation.

EXAMPLE 1 **Approximating a Root by Newton's Method**

Approximate the root of $x^4 - 4x + 1 = 0$ that lies between 0 and 1. Continue the approximation procedure until two successive approximations differ by less than 0.0001.

Solution: Letting $f(x) = x^4 - 4x + 1$, we have

$$f(0) = 0 - 0 + 1 = 1$$

and

$$f(1) = 1 - 4 + 1 = -2$$

(Note the change in sign.) Since $f(0)$ is closer to 0 than is $f(1)$, we choose 0 to be our first approximation, x_1. Now,

$$f'(x) = 4x^3 - 4$$

so

$$f(x_n) = x_n^4 - 4x_n + 1 \quad \text{and} \quad f'(x_n) = 4x_n^3 - 4$$

Substituting into Equation (4) gives the recursion formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 4x_n + 1}{4x_n^3 - 4}$$

$$= \frac{4x_n^4 - 4x_n - x_n^4 + 4x_n - 1}{4x_n^3 - 4}$$

so

$$x_{n+1} = \frac{3x_n^4 - 1}{4x_n^3 - 4} \tag{5}$$

Since $x_1 = 0$, letting $n = 1$ in Equation (5) gives

$$x_2 = \frac{3x_1^4 - 1}{4x_1^3 - 4} = \frac{3(0)^4 - 1}{4(0)^3 - 4} = 0.25$$

Letting $n = 2$ in Equation (5) gives

$$x_3 = \frac{3x_2^4 - 1}{4x_2^3 - 4} = \frac{3(0.25)^4 - 1}{4(0.25)^3 - 4} \approx 0.25099$$

Letting $n = 3$ in Equation (5) gives

$$x_4 = \frac{3x_3^4 - 1}{4x_3^3 - 4} = \frac{3(0.25099)^4 - 1}{4(0.25099)^3 - 4} \approx 0.25099$$

Table 12.1

n	x_n	x_{n+1}
1	0.00000	0.25000
2	0.25000	0.25099
3	0.25099	0.25099

The data obtained thus far are displayed in Table 12.1. Since the values of x_3 and x_4 differ by less than 0.0001, we take the root to be 0.25099 (that is, x_4).

Now Work Problem 1 ◁

EXAMPLE 2 **Approximating a Root by Newton's Method**

Approximate the root of $x^3 = 3x - 1$ that lies between -1 and -2. Continue the approximation procedure until two successive approximations differ by less than 0.0001.

Solution: Letting $f(x) = x^3 - 3x + 1$ [we need the form $f(x) = 0$], we find that

$$f(-1) = (-1)^3 - 3(-1) + 1 = 3$$

and

$$f(-2) = (-2)^3 - 3(-2) + 1 = -1$$

(Note the change in sign.) Since $f(-2)$ is closer to 0 than is $f(-1)$, we choose -2 to be our first approximation, x_1. Now,

$$f'(x) = 3x^2 - 3$$

so

$$f(x_n) = x_n^3 - 3x_n + 1 \quad \text{and} \quad f'(x_n) = 3x_n^2 - 3$$

Substituting into Equation (4) gives the recursion formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3}$$

so

$$x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 - 3} \tag{6}$$

Since $x_1 = -2$, letting $n = 1$ in Equation (6) gives

$$x_2 = \frac{2x_1^3 - 1}{3x_1^2 - 3} = \frac{2(-2)^3 - 1}{3(-2)^2 - 3} \approx -1.88889$$

Continuing in this way, we obtain Table 12.2. Because the values of x_3 and x_4 differ by 0.00006, which is less than 0.0001, we take the root to be -1.87939 (that is, x_4).

Table 12.2

n	x_n	$x_n + 1$
1	−2.00000	−1.88889
2	−1.88889	−1.87945
3	−1.87945	−1.87939

Now Work Problem 3 ◁

If our choice of x_1 has $f'(x_1) = 0$, then Newton's method will fail to produce a value for x_2. This occurs in Problems 2 and 8 of Problems 12.6. When this happens, the choice of x_1 must be rejected and a different number, close to the desired root, must be chosen for x_1. A graph of f can be helpful in this situation. Finally, there are times when the sequence of approximations does not approach the root. A discussion of such situations is beyond the scope of this book.

TECHNOLOGY

Figure 12.8 gives a short TI-83 Plus program for Newton's method. Before the program is executed, the first approximation to the root of $f(x) = 0$ is stored as X, and $f(x)$ and $f'(x)$ are stored as Y_1 and Y_2, respectively.

```
PROGRAM:NEWTON
:Lbl A
:X-Y1(X)/Y2(X)
:Ans→X
:Disp X
:Pause
:Goto A
```

FIGURE 12.8 Calculator program for Newton's method.

When executed, the program computes the first iteration and pauses. Successive iterations are obtained by successively pressing ENTER. Figure 12.9 shows the iterations for the problem in Example 2.

```
-2→X                    -2
prgmNEWTON
      -1.888888889
      -1.879451567
      -1.879385245
```

FIGURE 12.9 Iterations for problem in Example 2.

PROBLEMS 12.6

In Problems 1–10, use Newton's method to approximate the indicated root of the given equation. Continue the approximation procedure until the difference of two successive approximations is less than 0.0001.

1. $x^3 - 5x + 1 = 0$; root between 0 and 1
2. $x^3 + 2x^2 - 1 = 0$; root between 0 and 1
3. $x^3 - x - 1 = 0$; root between 1 and 2
4. $x^3 - 9x + 6 = 0$; root between 2 and 3
5. $x^3 + x + 1 = 0$; root between -1 and 0
6. $x^3 = 2x + 6$; root between 2 and 3
7. $x^4 = 3x - 1$; root between 0 and 1
8. $x^4 + 4x - 1 = 0$; root between -2 and -1
9. $x^4 - 2x^3 + x^2 - 3 = 0$; root between 1 and 2

10. $x^4 - x^3 + x - 2 = 0$; root between 1 and 2

11. Estimate, to three-decimal-place accuracy, the cube root of 73. [*Hint:* Show that the problem is equivalent to finding a root of $f(x) = x^3 - 73 = 0$.] Choose 4 as the initial estimate. Continue the iteration until two successive approximations, rounded to three decimal places, are the same.

12. Estimate $\sqrt[4]{19}$, to two-decimal-place accuracy. Use 2 as your initial estimate.

13. Find, to two-decimal-place accuracy, all real solutions of the equation $e^x = x + 5$. (*Hint:* A rough sketch of the graphs of $y = e^x$ and $y = x + 5$ makes it clear how many solutions there are. Use nearby integer values for your initial estimates.)

14. Find, to three-decimal-place accuracy, all real solutions of the equation $\ln x = 5 - x$.

15. Break-Even Quantity The cost of manufacturing q tons of a certain product is given by

$$c = 250 + 2q - 0.1q^3$$

and the revenue obtained by selling the q tons is given by

$$r = 3q$$

Approximate, to two-decimal-place accuracy, the break-even quantity. (*Hint:* Approximate a root of $r - c = 0$ by choosing 13 as your initial estimate.)

16. Break-Even Quantity The total cost of manufacturing q hundred pencils is c dollars, where

$$c = 50 + 4q + \frac{q^2}{1000} + \frac{1}{q}$$

Pencils are sold for \$8 per hundred.

(a) Show that the break-even quantity is a solution of the equation

$$f(q) = \frac{q^3}{1000} - 4q^2 + 50q + 1 = 0$$

(b) Use Newton's method to approximate the solution of $f(q) = 0$, where $f(q)$ is given in part (a). Use 10 as your initial approximation, and give your answer to two-decimal-place accuracy.

17. Equilibrium Given the supply equation $p = 2q + 5$ and the demand equation $p = \dfrac{100}{q^2 + 1}$, use Newton's method to estimate the market equilibrium quantity. Give your answer to three-decimal-place accuracy.

18. Equilibrium Given the supply equation

$$p = 0.2q^3 + 0.5q + 2$$

and the demand equation $p = 10 - q$, use Newton's method to estimate the market equilibrium quantity, and find the corresponding equilibrium price. Use 5 as an initial estimate for the required value of q, and give your answer to two-decimal-place accuracy.

19. Use Newton's method to approximate (to two-decimal-place accuracy) a critical value of the function

$$f(x) = \frac{x^3}{3} - x^2 - 5x + 1$$

on the interval $[3, 4]$.

12.7 Higher-Order Derivatives

Objective

To find higher-order derivatives both directly and implicitly.

We know that the derivative of a function $y = f(x)$ is itself a function, $f'(x)$. If we differentiate $f'(x)$, the resulting function $(f')'(x)$ is called the **second derivative** of f at x. It is denoted $f''(x)$, which is read "f double prime of x." Similarly, the derivative of the second derivative is called the **third derivative,** written $f'''(x)$. Continuing in this way, we get *higher-order derivatives*. Some notations for higher-order derivatives are given in Table 12.3. To avoid clumsy notation, primes are not used beyond the third derivative.

Table 12.3

First derivative:	y'	$f'(x)$	$\dfrac{dy}{dx}$	$\dfrac{d}{dx}(f(x))$	$D_x y$
Second derivative:	y''	$f''(x)$	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^2}{dx^2}(f(x))$	$D_x^2 y$
Third derivative:	y'''	$f'''(x)$	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^3}{dx^3}(f(x))$	$D_x^3 y$
Fourth derivative:	$y^{(4)}$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$	$\dfrac{d^4}{dx^4}(f(x))$	$D_x^4 y$

CAUTION!

The symbol d^2y/dx^2 represents the second derivative of y. It is not the same as $(dy/dx)^2$, the square of the first derivative of y.

EXAMPLE 1 Finding Higher-Order Derivatives

a. If $f(x) = 6x^3 - 12x^2 + 6x - 2$, find all higher-order derivatives.

Solution: Differentiating $f(x)$ gives

$$f'(x) = 18x^2 - 24x + 6$$

Differentiating $f'(x)$ yields

$$f''(x) = 36x - 24$$

Similarly,

$$f'''(x) = 36$$
$$f^{(4)}(x) = 0$$

All successive derivatives are also 0: $f^{(5)}(x) = 0$, and so on.

b. If $f(x) = 7$, find $f''(x)$.

Solution:

$$f'(x) = 0$$
$$f''(x) = 0$$

Now Work Problem 1 ◁

EXAMPLE 2 Finding a Second-Order Derivative

APPLY IT ▷

8. The height $h(t)$ of a rock dropped off of a 200-foot building is given by $h(t) = 200 - 16t^2$, where t is the time measured in seconds. Find $\frac{d^2h}{dt^2}$, the acceleration of the rock at time t.

If $y = e^{x^2}$, find $\frac{d^2y}{dx^2}$.

Solution:

$$\frac{dy}{dx} = e^{x^2}(2x) = 2xe^{x^2}$$

By the product rule,

$$\frac{d^2y}{dx^2} = 2(x(e^{x^2})(2x) + e^{x^2}(1)) = 2e^{x^2}(2x^2 + 1)$$

Now Work Problem 5 ◁

EXAMPLE 3 Evaluating a Second-Order Derivative

APPLY IT ▷

9. If the cost to produce q units of a product is

$$c(q) = 7q^2 + 11q + 19$$

and the marginal-cost function is $c'(q)$, find the rate of change of the marginal cost function with respect to q when $q = 3$.

If $y = f(x) = \frac{16}{x+4}$, find $\frac{d^2y}{dx^2}$ and evaluate it when $x = 4$.

Solution: Since $y = 16(x+4)^{-1}$, the power rule gives

$$\frac{dy}{dx} = -16(x+4)^{-2}$$
$$\frac{d^2y}{dx^2} = 32(x+4)^{-3} = \frac{32}{(x+4)^3}$$

Evaluating when $x = 4$, we obtain

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{32}{8^3} = \frac{1}{16}$$

The second derivative evaluated at $x = 4$ is also denoted $f''(4)$ or $y''(4)$.

Now Work Problem 21 ◁

EXAMPLE 4 Finding the Rate of Change of $f''(x)$

If $f(x) = x \ln x$, find the rate of change of $f''(x)$.

The rate of change of $f''(x)$ is $f'''(x)$.

Solution: To find the rate of change of any function, we must find its derivative. Thus, we want the derivative of $f''(x)$, which is $f'''(x)$. Accordingly,

$$f'(x) = x\left(\frac{1}{x}\right) + (\ln x)(1) = 1 + \ln x$$
$$f''(x) = 0 + \frac{1}{x} = \frac{1}{x}$$
$$f'''(x) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

Now Work Problem 17 ◁

Higher-Order Implicit Differentiation

We will now find a higher-order derivative by means of implicit differentiation. Keep in mind that we will assume y to be a function of x.

EXAMPLE 5 **Higher-Order Implicit Differentiation**

Find $\dfrac{d^2y}{dx^2}$ if $x^2 + 4y^2 = 4$.

Solution: Differentiating both sides with respect to x, we obtain

$$2x + 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{4y} \tag{1}$$

$$\frac{d^2y}{dx^2} = \frac{4y\dfrac{d}{dx}(-x) - (-x)\dfrac{d}{dx}(4y)}{(4y)^2}$$

$$= \frac{4y(-1) - (-x)\left(4\dfrac{dy}{dx}\right)}{16y^2}$$

$$= \frac{-4y + 4x\dfrac{dy}{dx}}{16y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y + x\dfrac{dy}{dx}}{4y^2} \tag{2}$$

Although we have found an expression for d^2y/dx^2, our answer involves the derivative dy/dx. It is customary to express the answer without the derivative—that is, in terms of x and y only. This is easy to do. From Equation (1), $\dfrac{dy}{dx} = \dfrac{-x}{4y}$, so by substituting into Equation (2), we have

$$\frac{d^2y}{dx^2} = \frac{-y + x\left(\dfrac{-x}{4y}\right)}{4y^2} = \frac{-4y^2 - x^2}{16y^3} = -\frac{4y^2 + x^2}{16y^3}$$

In Example 5, the simplification of d^2y/dx^2 by making use of the original equation is not unusual.

We can further simplify the answer. Since $x^2 + 4y^2 = 4$ (the original equation),

$$\frac{d^2y}{dx^2} = -\frac{4}{16y^3} = -\frac{1}{4y^3}$$

Now Work Problem 23 ◁

EXAMPLE 6 **Higher-Order Implicit Differentiation**

Find $\dfrac{d^2y}{dx^2}$ if $y^2 = e^{x+y}$.

Solution: Differentiating both sides with respect to x gives

$$2y\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right)$$

Solving for dy/dx, we obtain

$$2y\frac{dy}{dx} = e^{x+y} + e^{x+y}\frac{dy}{dx}$$

$$2y\frac{dy}{dx} - e^{x+y}\frac{dy}{dx} = e^{x+y}$$

$$(2y - e^{x+y})\frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = \frac{e^{x+y}}{2y - e^{x+y}}$$

Since $y^2 = e^{x+y}$ (the original equation),

$$\frac{dy}{dx} = \frac{y^2}{2y - y^2} = \frac{y}{2 - y}$$

$$\frac{d^2y}{dx^2} = \frac{(2-y)\frac{dy}{dx} - y\left(-\frac{dy}{dx}\right)}{(2-y)^2} = \frac{2\frac{dy}{dx}}{(2-y)^2}$$

Now we express our answer without dy/dx. Since $\frac{dy}{dx} = \frac{y}{2-y}$,

$$\frac{d^2y}{dx^2} = \frac{2\left(\frac{y}{2-y}\right)}{(2-y)^2} = \frac{2y}{(2-y)^3}$$

Now Work Problem 31 ◁

PROBLEMS 12.7

In Problems 1–20, find the indicated derivatives.

1. $y = 4x^3 - 12x^2 + 6x + 2, y'''$

2. $y = x^5 + x^4 + x^3 + x^2 + x + 1, y'''$

3. $y = 8 - x, \dfrac{d^2y}{dx^2}$

4. $y = -x - x^2, \dfrac{d^2y}{dx^2}$

5. $y = x^3 + e^x, y^{(4)}$

6. $F(q) = \ln(q+1), \dfrac{d^3F}{dq^3}$

7. $f(x) = x^3 \ln x, f'''(x)$

8. $y = \dfrac{1}{x}, y'''$

9. $f(q) = \dfrac{1}{2q^4}, f'''(q)$

10. $f(x) = \sqrt{x}, f''(x)$

11. $f(r) = \sqrt{9-r}, f''(r)$

12. $y = e^{ax^2}, y''$

13. $y = \dfrac{1}{2x+3}, \dfrac{d^2y}{dx^2}$

14. $y = (3x+7)^5, y''$

15. $y = \dfrac{x+1}{x-1}, y''$

16. $y = 2x^{1/2} + (2x)^{1/2}, y''$

17. $y = \ln[x(x+a)], y''$

18. $y = \ln\dfrac{(2x+5)(5x-2)}{x+1}, y''$

19. $f(z) = z^2e^z, f''(z)$

20. $y = \dfrac{x}{e^x}, \dfrac{d^2y}{dx^2}$

21. If $y = e^{2x} + e^{3x}$, find $\left.\dfrac{d^5y}{dx^5}\right|_{x=0}$.

22. If $y = e^{2\ln(x^2+1)}$, find y'' when $x = 1$.

In Problems 23–32, find y''.

23. $x^2 + 4y^2 - 16 = 0$

24. $x^2 - y^2 = 16$

25. $y^2 = 4x$

26. $9x^2 + 16y^2 = 25$

27. $a\sqrt{x} + b\sqrt{y} = c$

28. $y^2 - 6xy = 4$

29. $xy + y - x = 4$

30. $x^2 + 2xy + y^2 = 1$

31. $y = e^{x+y}$

32. $e^x + e^y = x^2 + y^2$

33. If $x^2 + 3x + y^2 = 4y$, find d^2y/dx^2 when $x = 0$ and $y = 0$.

34. Show that the equation

$$f''(x) + 4f'(x) + 4f(x) = 0$$

is satisfied if $f(x) = (3x - 5)e^{-2x}$.

35. Find the rate of change of $f'(x)$ if $f(x) = (5x-3)^4$.

36. Find the rate of change of $f''(x)$ if

$$f(x) = 6\sqrt{x} + \frac{1}{6\sqrt{x}}$$

37. Marginal Cost If $c = 0.2q^2 + 2q + 500$ is a cost function, how fast is marginal cost changing when $q = 97.357$?

38. Marginal Revenue If $p = 400 - 40q - q^2$ is a demand equation, how fast is marginal revenue changing when $q = 4$?

39. If $f(x) = x^4 - 6x^2 + 5x - 6$, determine the values of x for which $f''(x) = 0$.

40. Suppose that $e^y = y^2e^x$. (a) Determine dy/dx, and express your answer in terms of y only. (b) Determine d^2y/dx^2, and express your answer in terms of y only.

In Problems 41 and 42, determine $f''(x)$. Then use your graphing calculator to find all real roots of $f''(x) = 0$. Round your answers to two decimal places.

41. $f(x) = 6e^x - x^3 - 15x^2$

42. $f(x) = \dfrac{x^5}{20} + \dfrac{x^4}{12} + \dfrac{5x^3}{6} + \dfrac{x^2}{2}$

Chapter 12 Review

Important Terms and Symbols

		Examples
Section 12.1	**Derivatives of Logarithmic Functions**	
	derivative of $\ln x$ and of $\log_b u$	Ex. 5, p. 544
Section 12.2	**Derivatives of Exponential Functions**	
	derivative of e^x and of b^u	Ex. 4, p. 548
Section 12.3	**Elasticity of Demand**	
	point elasticity of demand, η elastic unit elasticity inelastic	Ex. 2, p. 553
Section 12.4	**Implicit Differentiation**	
	implicit function	Ex. 1, p. 557
Section 12.5	**Logarithmic Differentiation**	
	logarithmic differentiation relative rate of change of revenue	Ex. 3, p. 562
Section 12.6	**Newton's Method**	
	recursion formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$	Ex. 1, p. 566
Section 12.7	**Higher-Order Derivatives**	
	higher-order derivatives, $f''(x)$, $\dfrac{d^3y}{dx^3}$, $\dfrac{d^4}{dx^4}[f(x)], \ldots$	Ex. 1, p. 569

Summary

The derivative formulas for natural logarithmic and exponential functions are

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$$

and

$$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$$

To differentiate logarithmic and exponential functions in bases other than e, first transform the function to base e and then differentiate the result. Alternatively, differentiation formulas can be applied:

$$\frac{d}{dx}(\log_b u) = \frac{1}{(\ln b)u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}(b^u) = b^u(\ln b)\cdot\frac{du}{dx}$$

Point elasticity of demand is a function that measures how consumer demand is affected by a change in price. It is given by

$$\eta = \frac{p}{q}\frac{dq}{dp}$$

where p is the price per unit at which q units are demanded. The three categories of elasticity are as follows:

$|\eta(p)| > 1$ demand is elastic

$|\eta(p)| = 1$ unit elasticity

$|\eta(p)| < 1$ demand is inelastic

For a given percentage change in price, if there is a greater (respectively, lesser) percentage change in quantity demanded, then demand is elastic (respectively, inelastic) and conversely.

Two relationships between elasticity and the rate of change of revenue are given by

$$\frac{dr}{dq} = p\left(1+\frac{1}{\eta}\right) \qquad \frac{dr}{dp} = q(1+\eta)$$

If an equation implicitly defines y as a function of x [rather than defining it explicitly in the form $y = f(x)$], then dy/dx can be found by implicit differentiation. With this method, we treat y as a function of x and differentiate both sides of the equation with respect to x. When doing this, remember that

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx}$$

and, more generally, that

$$\frac{d}{dx}(f(y)) = f'(y)\frac{dy}{dx}$$

Finally, we solve the resulting equation for dy/dx.

Suppose that $f(x)$ consists of products, quotients, or powers. To differentiate $y = \log_b(f(x))$, it may be helpful to use properties of logarithms to rewrite $\log_b(f(x))$ in terms of simpler logarithms and then differentiate that form. To differentiate $y = f(x)$, where $f(x)$ consists of products, quotients, or powers, the method of logarithmic differentiation may be used. In that method, we take the natural logarithm of both sides of $y = f(x)$ to obtain $\ln y = \ln(f(x))$. After simplifying $\ln(f(x))$ by using properties of logarithms, we differentiate both sides of $\ln y = \ln(f(x))$ with respect to x

and then solve for y'. Logarithmic differentiation can also be used to differentiate $y = u^v$, where both u and v are functions of x.

Newton's method is the name given to the following formula, which is used to approximate the roots of the equation $f(x) = 0$, provided that f is differentiable:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots$$

In many cases encountered, the approximation improves as n increases.

Because the derivative $f'(x)$ of a function $y = f(x)$ is itself a function, it can be successively differentiated to obtain the second derivative $f''(x)$, the third derivative $f'''(x)$, and other higher-order derivatives.

Review Problems

In Problems 1–30, differentiate.

1. $y = 3e^x + e^2 + e^{x^2} + x^{e^2}$

2. $f(w) = we^w + w^2$

3. $f(r) = \ln(7r^2 + 4r + 5)$

4. $y = e^{\ln x}$

5. $y = e^{x^2+4x+5}$

6. $f(t) = \log_6 \sqrt{t^2 + 1}$

7. $y = e^x(x^2 + 2)$

8. $y = 2^{3x^2}$

9. $y = \sqrt{(x-6)(x+5)(9-x)}$

10. $f(t) = e^{1/t}$

11. $y = \dfrac{\ln x}{e^x}$

12. $y = \dfrac{e^x + e^{-x}}{x^2}$

13. $f(q) = \ln[(q+a)^m(q+b)^n]$

14. $y = (x+2)^3(x+1)^4(x-2)^2$

15. $y = 2^{2x^2+2x-5}$

16. $y = (e + e^2)^0$

17. $y = \dfrac{4e^{3x}}{xe^{x-1}}$

18. $y = \dfrac{\ln x}{e^x}$

19. $y = \log_2(8x+5)^2$

20. $y = \ln\left(\dfrac{5}{x^2}\right)$

21. $f(l) = \ln(1 + l + l^2 + l^3)$

22. $y = (x^2)^{x^2}$

23. $y = (x^2+1)^{x+1}$

24. $y = \dfrac{1+e^x}{1-e^x}$

25. $\phi(t) = \ln\left(t\sqrt{4-t^2}\right)$

26. $y = (x+3)^{\ln x}$

27. $y = \dfrac{(x^2+1)^{1/2}(x^2+2)^{1/3}}{(2x^3+6x)^{2/5}}$

28. $y = (\ln x)\sqrt{x}$

29. $y = (x^x)^x$

30. $y = x^{(x^x)}$

In Problems 31–34, evaluate y' at the given value of x.

31. $y = (x+1)\ln x^2, x = 1$

32. $y = \dfrac{e^{x^2+1}}{\sqrt{x^2+1}}, x = 1$

33. $y = (1/x)^x, x = e$

34. $y = \left[\dfrac{2^{5x}(x^2-3x+5)^{1/3}}{(x^2-3x+7)^3}\right]^{-1}, x = 0$

In Problems 35 and 36, find an equation of the tangent line to the curve at the point corresponding to the given value of x.

35. $y = 3e^x, x = \ln 2$

36. $y = x + x^2 \ln x, x = 1$

37. Find the y-intercept of the tangent line to the graph of $y = x(2^{2-x^2})$ at the point where $x = 1$.

38. If $w = 2^x + \ln(1+x^2)$ and $x = \ln(1+t^2)$, find w and dw/dt when $t = 0$.

In Problems 39–42, find the indicated derivative at the given point.

39. $y = e^{x^2-2x+1}, y'', (1, 1)$

40. $y = x^2e^x, y''', (1, e)$

41. $y = \ln(2x), y''', (1, \ln 2)$

42. $y = x\ln x, y'', (1, 0)$

In Problems 43–46, find dy/dx.

43. $x^2 + 2xy + y^2 = 4$

44. $x^3y^3 = 3$

45. $\ln(xy^2) = xy$

46. $y^2e^{y\ln x} = e^2$

In Problems 47 and 48, find d^2y/dx^2 at the given point.

47. $x + xy + y = 5, (2, 1)$

48. $x^2 + xy + y^2 = 1, (0, -1)$

49. If y is defined implicitly by $e^y = (y+1)e^x$, determine both dy/dx and d^2y/dx^2 as explicit functions of y only.

50. If $\sqrt{x} + \sqrt{y} = 1$, find $\dfrac{d^2y}{dx^2}$.

51. Schizophrenia Several models have been used to analyze the length of stay in a hospital. For a particular group of schizophrenics, one such model is[9]

$$f(t) = 1 - (0.8e^{-0.01t} + 0.2e^{-0.0002t})$$

where $f(t)$ is the proportion of the group that was discharged at the end of t days of hospitalization. Determine the discharge rate (proportion discharged per day) at the end of t days.

52. Earthquakes According to Richter,[10] the number N of earthquakes of magnitude M or greater per unit of time is given by $\log N = A - bM$, where A and b are constants. Richter claims that

$$\log\left(-\frac{dN}{dM}\right) = A + \log\left(\frac{b}{q}\right) - bM$$

where $q = \log e$. Verify this statement.

53. If $f(x) = e^{x^4-10x^3+36x^2-2x}$, find all real roots of $f'(x) = 0$. Round your answers to two decimal places.

54. If $f(x) = \dfrac{x^5}{10} + \dfrac{x^4}{6} + \dfrac{2x^3}{3} + x^2 + 1$, find all roots of $f''(x) = 0$. Round your answers to two decimal places.

---

[9] Adapted from W. W. Eaton and G. A. Whitmore, "Length of Stay as a Stochastic Process: A General Approach and Application to Hospitalization for Schizophrenia," *Journal of Mathematical Sociology*, 5 (1977) 273–92.

[10] C. F. Richter, *Elementary Seismology* (San Francisco: W. H. Freeman and Company, Publishers, 1958).

13 Curve Sketching

13.1 Relative Extrema

13.2 Absolute Extrema on a Closed Interval

13.3 Concavity

13.4 The Second-Derivative Test

13.5 Asymptotes

13.6 Applied Maxima and Minima

Chapter 13 Review

EXPLORE & EXTEND
Population Change over Time

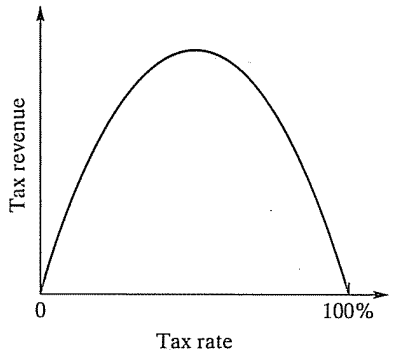
In the mid-1970s, economist Arthur Laffer was explaining his views on taxes to a politician. To illustrate his argument, Laffer grabbed a paper napkin and sketched the graph that now bears his name: the Laffer curve.

The Laffer curve describes total government tax revenue as a function of the tax rate. Obviously, if the tax rate is zero, the government gets nothing. But if the tax rate is 100%, revenue would again equal zero, because there is no incentive to earn money if it will all be taken away. Since tax rates between 0% and 100% do generate revenue, Laffer reasoned, the curve relating revenue to tax rate must look, qualitatively, more or less as shown in the figure below.

Laffer's argument was not meant to show that the optimal tax rate was 50%. It was meant to show that under some circumstances, namely when the tax rate is to the right of the peak of the curve, it is possible to *raise government revenue by lowering taxes*. This was a key argument made for the tax cuts passed by Congress during the first term of the Reagan presidency.

Because the Laffer curve is only a qualitative picture, it does not actually give an optimal tax rate. Revenue-based arguments for tax cuts involve the claim that the point of peak revenue lies to the left of the current taxation scheme on the horizontal axis. By the same token, those who urge raising taxes to raise government income are assuming either a different relationship between rates and revenues or a different location of the curve's peak.

By itself, then, the Laffer curve is too abstract to be of much help in determining the optimal tax rate. But even very simple sketched curves, like supply and demand curves and the Laffer curve, can help economists describe the causal factors that drive an economy. In this chapter, we will discuss techniques for sketching and interpreting curves.



Objective

To find when a function is increasing or decreasing, to find critical values, to locate relative maxima and relative minima, and to state the first-derivative test. Also, to sketch the graph of a function by using the information obtained from the first derivative.

13.1 Relative Extrema

Increasing or Decreasing Nature of a Function

Examining the graphical behavior of functions is a basic part of mathematics and has applications to many areas of study. When we sketch a curve, just plotting points may not give enough information about its shape. For example, the points $(-1, 0), (0, -1)$, and $(1, 0)$ satisfy the equation given by $y = (x+1)^3(x-1)$. On the basis of these points, we might hastily conclude that the graph should appear as in Figure 13.1(a), but in fact the true shape is given in Figure 13.1(b). In this chapter we will explore the powerful role that differentiation plays in analyzing a function so that we can determine the true shape and behavior of its graph.

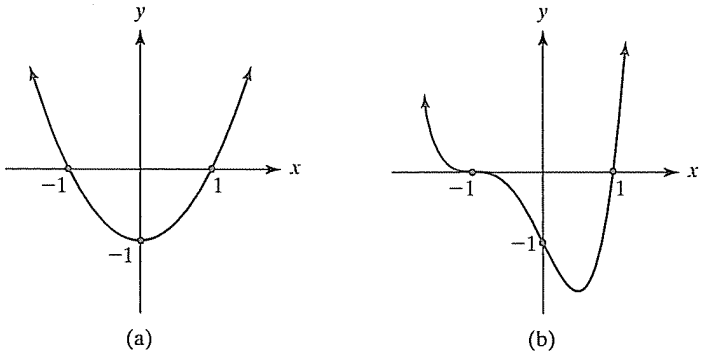


FIGURE 13.1 Curves passing through $(-1, 0), (0, -1)$, and $(1, 0)$.

We begin by analyzing the graph of the function $y = f(x)$ in Figure 13.2. Notice that as x increases (goes from left to right) on the interval I_1, between a and b, the values of $f(x)$ increase and the curve is rising. Mathematically, this observation means that if x_1 and x_2 are any two points in I_1 such that $x_1 < x_2$, then $f(x_1) < f(x_2)$. Here f is said to be an *increasing function* on I_1. On the other hand, as x increases on the interval I_2 between c and d, the curve is falling. On this interval, $x_3 < x_4$ implies that $f(x_3) > f(x_4)$, and f is said to be a *decreasing function* on I_2. We summarize these observations in the following definition.

Definition

A function f is said to be ***increasing*** on an interval I when, for any two numbers x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) < f(x_2)$. A function f is ***decreasing*** on an interval I when, for any two numbers x_1, x_2 in I, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

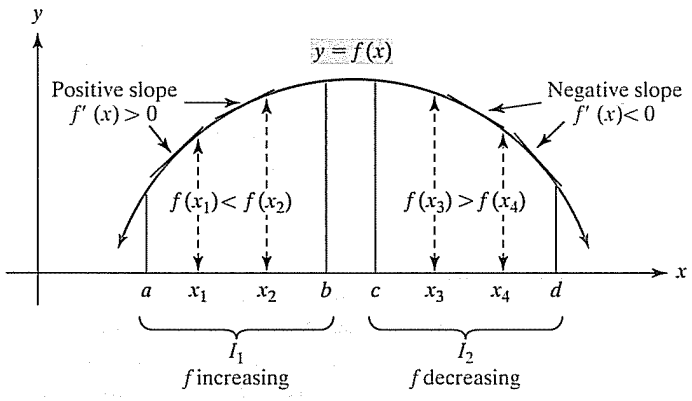


FIGURE 13.2 Increasing or decreasing nature of function.

In terms of the graph of the function, f is increasing on I if the curve rises to the right and f is decreasing on I if the curve falls to the right. Recall that a straight line with positive slope rises to the right while a straight line with negative slope falls to the right.

Turning again to Figure 13.2, we note that over the interval I_1, tangent lines to the curve have positive slopes, so $f'(x)$ must be positive for all x in I_1. A positive derivative implies that the curve is rising. Over the interval I_2, the tangent lines have negative slopes, so $f'(x) < 0$ for all x in I_2. The curve is falling where the derivative is negative. We thus have the following rule, which allows us to use the derivative to determine when a function is increasing or decreasing:

Rule 1 Criteria for Increasing or Decreasing Function

Let f be differentiable on the interval (a, b). If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b). If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b).

To illustrate these ideas, we will use Rule 1 to find the intervals on which $y = 18x - \frac{2}{3}x^3$ is increasing and the intervals on which y is decreasing. Letting $y = f(x)$, we must determine when $f'(x)$ is positive and when $f'(x)$ is negative. We have

$$f'(x) = 18 - 2x^2 = 2(9 - x^2) = 2(3 + x)(3 - x)$$

Using the technique of Section 10.4, we can find the sign of $f'(x)$ by testing the intervals determined by the roots of $2(3 + x)(3 - x) = 0$, namely, -3 and 3. These should be arranged in increasing order on the top of a sign chart for f' so as to divide the domain of f into intervals. (See Figure 13.3.) In each interval, the sign of $f'(x)$ is determined by the signs of its factors:

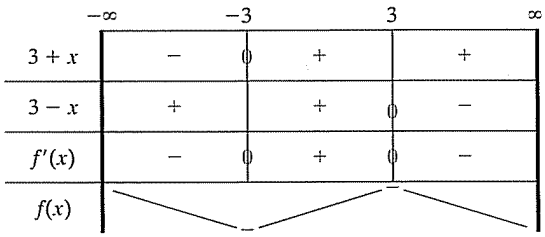


FIGURE 13.3 Sign chart for $f'(x) = 18 - 9x^2$ and its interpretation for $f(x)$.

If $x < -3$, then $\text{sign}(f'(x)) = 2(-)(+) = -$, so f is *decreasing*.

If $-3 < x < 3$, then $\text{sign}(f'(x)) = 2(+)(+) = +$, so f is *increasing*.

If $x > 3$, then $\text{sign}(f'(x)) = 2(+)(-) = -$, so f is *decreasing*.

These results are indicated in the sign chart given by Figure 13.3, where the bottom line is a schematic version of what the signs of f' say about f itself. Notice that the horizontal line segments in the bottom row indicate horizontal tangents for f at -3 and at 3. Thus, f is decreasing on $(-\infty, -3)$ and $(3, \infty)$ and is increasing on $(-3, 3)$. This corresponds to the rising and falling nature of the graph of f shown in Figure 13.4. Indeed, the point of a well-constructed sign chart is to provide a schematic for subsequent construction of the graph itself.

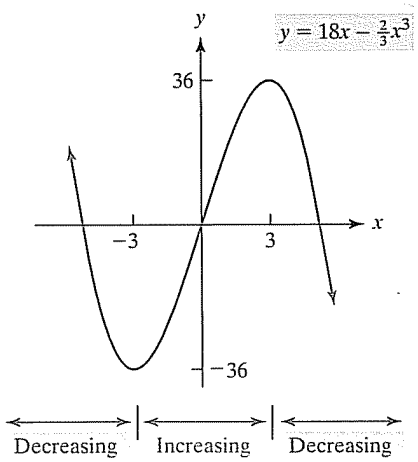


FIGURE 13.4 Increasing/decreasing for $y = 18x - \frac{2}{3}x^3$.

Extrema

Look now at the graph of $y = f(x)$ in Figure 13.5. Some observations can be made. First, there is something special about the points P, Q, and R. Notice that P is *higher* than any other "nearby" point on the curve—and likewise for R. The point Q is *lower* than any other "nearby" point on the curve. Since P, Q, and R may not necessarily be the highest or lowest points on the *entire* curve, we say that the graph

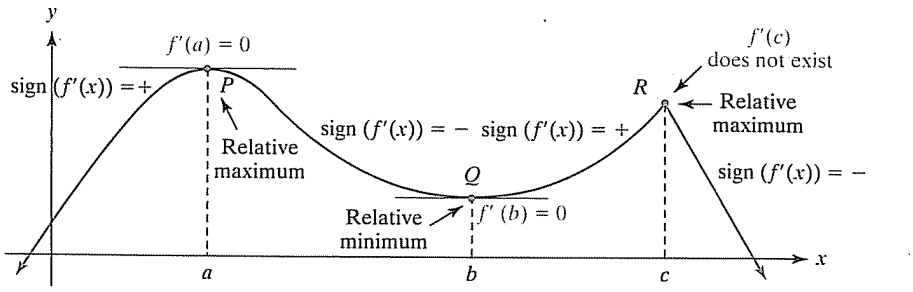


FIGURE 13.5 Relative maxima and relative minima.

CAUTION!

Be sure to note the difference between relative extreme *values* and *where* they occur.

of f *has relative maxima at* a *and at* c*; and has a relative minimum at* b. The function f *has relative maximum values of* $f(a)$ *at* a *and* $f(c)$ *at* c*; and has a relative minimum value of* $f(b)$ *at* b. We also say that $(a,f(a))$ *and* $(c,f(c))$ *are relative maximum points and* $(b,f(b))$ *is a relative minimum point on the graph of* f.

Turning back to the graph, we see that there is an *absolute maximum* (highest point on the entire curve) at a, but there is no *absolute minimum* (lowest point on the entire curve) because the curve is assumed to extend downward indefinitely. More precisely, we define these new terms as follows:

Definition

A function f has a ***relative maximum*** at a if there is an open interval containing a on which $f(a) \geq f(x)$ for all x in the interval. The relative maximum value is $f(a)$. A function f has a ***relative minimum*** at a if there is an open interval containing a on which $f(a) \leq f(x)$ for all x in the interval. The relative minimum value is $f(a)$.

If it exists, an absolute maximum value is unique; however, it may occur at more than one value of x. A similar statement is true for an absolute minimum.

Definition

A function f has an ***absolute maximum*** at a if $f(a) \geq f(x)$ for all x in the domain of f. The absolute maximum value is $f(a)$. A function f has an ***absolute minimum*** at a if $f(a) \leq f(x)$ for all x in the domain of f. The absolute minimum value is $f(a)$.

We refer to either a relative maximum or a relative minimum as a **relative extremum** (plural: *relative extrema*). Similarly, we speak of **absolute extrema.**

When dealing with relative extrema, we compare the function value at a point with values of nearby points; however, when dealing with absolute extrema, we compare the function value at a point with all other values determined by the domain. Thus, relative extrema are *local* in nature, whereas absolute extrema are *global* in nature.

Referring to Figure 13.5, we notice that at a relative extremum the derivative may not be defined (as when $x = c$). But whenever it is defined at a relative extremum, it is 0 (as when $x = a$ and when $x = b$), and hence the tangent line is horizontal. We can state the following:

Rule 2 A Necessary Condition for Relative Extrema

If f has a relative extremum at a, then $f'(a) = 0$ or $f'(a)$ does not exist.

The implication in Rule 2 goes in only one direction:

$$\left.\begin{array}{c}\text{relative extremum}\\ \text{at } a\end{array}\right\} \text{ implies } \begin{cases} f'(a) = 0 \\ \text{or} \\ f'(a) \text{ does not exist} \end{cases}$$

Rule 2 does *not* say that if $f'(a)$ is 0 or $f'(a)$ does not exist, then there must be a relative extremum at a. In fact, there may not be one at all. For example, in Figure 13.6(a), $f'(a)$

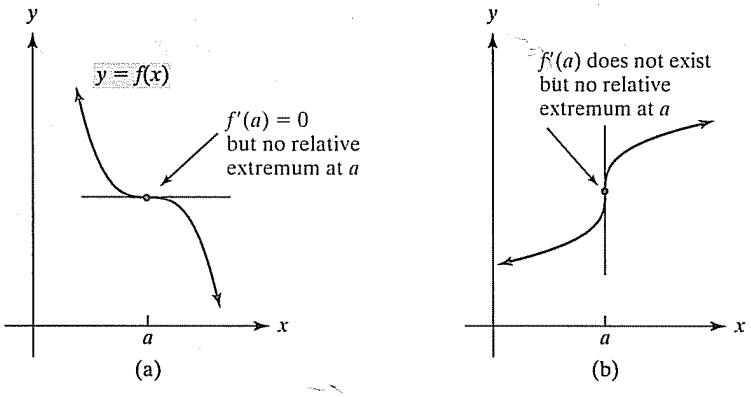


FIGURE 13.6 No relative extremum at a.

is 0 because the tangent line is horizontal at a, but there is no relative extremum there. In Figure 13.6(b), $f'(a)$ does not exist because the tangent line is vertical at a, but again there is no relative extremum there.

But if we want to find all relative extrema of a function—and this is an important task—what Rule 2 *does* tell us is that we can limit our search to those values of x in the domain of f for which *either* $f'(x) = 0$ *or* $f'(x)$ does not exist. Typically, in applications, this cuts down our search for relative extrema from the infinitely many x for which f is defined to a small finite number of *possibilities*. Because these values of x are so important for locating the relative extrema of f, they are called the *critical values* for f, and if a is a critical value for f, then we also say that $(a, f(a))$ is a *critical point* on the graph of f. Thus, in Figure 13.5, the numbers a, b, and c are critical values, and P, Q, and R are critical points.

Definition

For a in the domain of f, if either $f'(a) = 0$ or $f'(a)$ does not exist, then a is called a ***critical value*** for f. If a is a critical value, then the point $(a, f(a))$ is called a ***critical point*** for f.

At a critical point, there may be a relative maximum, a relative minimum, or neither. Moreover, from Figure 13.5, we observe that each relative extremum occurs at a point around which the sign of $f'(x)$ is changing. For the relative maximum at a, the sign of $f'(x)$ goes from $+$ for $x < a$ to $-$ for $x > a$, *as long as x is near a.* For the relative minimum at b, the sign of $f'(x)$ goes from $-$ to $+$, and for the relative maximum at c, it again goes from $+$ to $-$. Thus, *around relative maxima, f is increasing and then decreasing, and the reverse holds for relative minima.* More precisely, we have the following rule:

Rule 3 Criteria for Relative Extrema

Suppose f is continuous on an open interval I that contains the critical value a and f is differentiable on I, except possibly at a.

1. If $f'(x)$ changes from positive to negative as x increases through a, then f has a relative maximum at a.

2. If $f'(x)$ changes from negative to positive as x increases through a, then f has a relative minimum at a.

To illustrate Rule 3 with a concrete example, refer again to Figure 13.3, the sign chart for $f'(x) = 18 - 2x^2$. The row labeled by $f'(x)$ shows clearly that $f(x) = 18x - \frac{2}{3}x^2$ has a relative minimum at -3 and a relative maximum at 3. The row providing the interpretation of the chart for f, labeled $f(x)$, is immediately deduced

CAUTION!

We point out again that not every critical value corresponds to a relative extremum. For example, if $y = f(x) = x^3$, then $f'(x) = 3x^2$. Since $f'(0) = 0$, 0 is a critical value. But if $x < 0$, then $3x^2 > 0$, and if $x > 0$, then $3x^2 > 0$. Since $f'(x)$ does not change sign at 0, there is no relative extremum at 0. Indeed, since $f'(x) \geq 0$ for all x, the graph of f never falls, and f is said to be *nondecreasing*. (See Figure 13.8.)

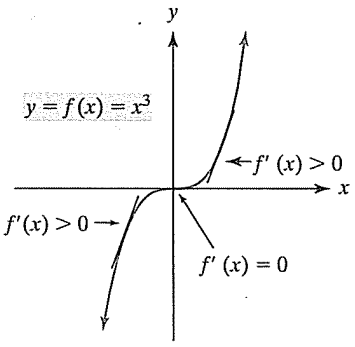


FIGURE 13.8 Zero is a critical value, but does not give a relative extremum.

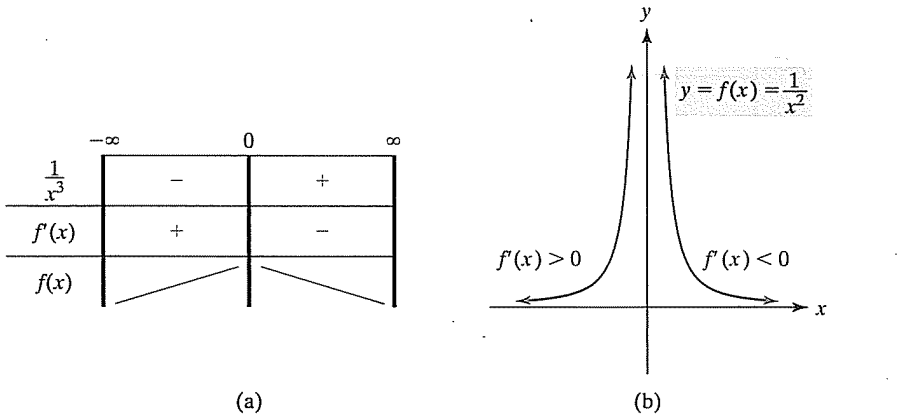


FIGURE 13.7 $f'(0)$ is not defined, but 0 is not a critical value because 0 is not in the domain of f.

from the row above it. The significance of the $f(x)$ row is that it provides an intermediate step in actually sketching the graph of f. In this row it stands out, visually, that f has a relative minimum at -3 and a relative maximum at 3.

When searching for extrema of a function f, care must be paid to those a that are not in the domain of f but that are near values in the domain of f. Consider the following example. If

$$y = f(x) = \frac{1}{x^2}, \quad \text{then} \quad f'(x) = -\frac{2}{x^3}$$

Although $f'(x)$ does not exist at 0, 0 is not a critical value, because 0 is not in the domain of f. Thus, a relative extremum cannot occur at 0. Nevertheless, the derivative may change sign around any x-value where $f'(x)$ is not defined, so such values are important in determining intervals over which f is increasing or decreasing. In particular, such values should be included in a sign chart for f'. See Figure 13.7(a) and the accompanying graph in Figure 13.7(b).

Observe that the thick vertical rule at 0 on the chart serves to indicate that 0 is not in the domain of f. Here there are no extrema of any kind.

In Rule 3 the hypotheses must be satisfied, or the conclusion need not hold. For example, consider the case-defined function

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Here, 0 is explicitly in the domain of f but f is not continuous at 0. We recall from Section 11.1 that if a function f is not continuous at a, then f is not differentiable at a, meaning that $f'(a)$ does not exist. Thus $f'(0)$ does not exist and 0 is a critical value that must be included in the sign chart for f' shown in Figure 13.9(a). We extend our sign

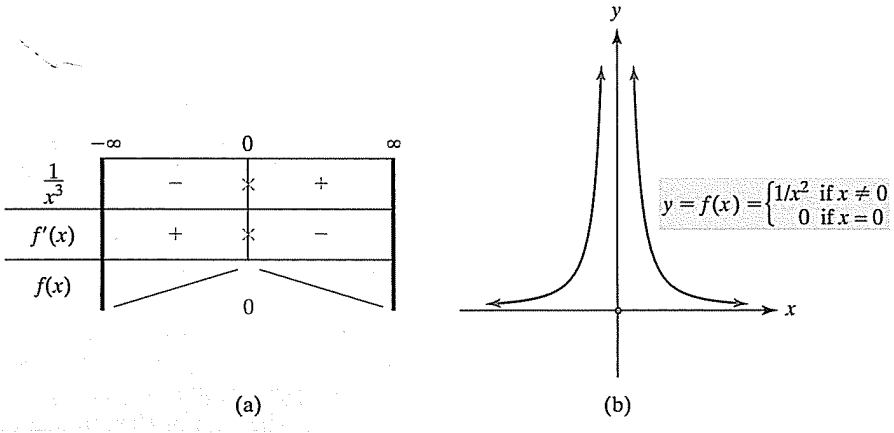


FIGURE 13.9 Zero is a critical value, but Rule 3 does not apply.

chart conventions by indicating with a $\times$ symbol those values for which f' does not exist. We see in this example that $f'(x)$ changes from positive to negative as x increases through 0 but f does *not* have a relative maximum at 0. Here Rule 3 does not apply because its continuity hypothesis is not met. In Figure 13.9(b), 0 is displayed in the domain of f. It is clear that f has an absolute *minimum* at 0 because $f(0) = 0$ and, for all $x \neq 0, f(x) > 0$.

Summarizing the results of this section, we have the *first-derivative test* for the relative extrema of $y = f(x)$:

First-Derivative Test for Relative Extrema

Step 1. Find $f'(x)$.

Step 2. Determine all critical values of f [those a where $f'(a) = 0$ or $f'(a)$ does not exist] and any a that are not in the domain of f but that are near values in the domain of f, and construct a sign chart that shows for each of the intervals determined by these values whether f is increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$).

Step 3. For each critical value a at which f is continuous, determine whether $f'(x)$ changes sign as x increases through a. There is a relative maximum at a if $f'(x)$ changes from $+$ to $-$ going from left to right and a relative minimum if $f'(x)$ changes from $-$ to $+$ going from left to right. If $f'(x)$ does not change sign, there is no relative extremum at a.

Step 4. For critical values a at which f is not continuous, analyze the situation by using the definitions of extrema directly.

APPLY IT ▶

1. The cost equation for a hot dog stand is given by $c(q) = 2q^3 - 21q^2 + 60q + 500$, where q is the number of hot dogs sold, and $c(q)$ is the cost in dollars. Use the first-derivative test to find where relative extrema occur.

EXAMPLE 1 First-Derivative Test

If $y = f(x) = x + \dfrac{4}{x+1}$, for $x \neq -1$ use the first-derivative test to find where relative extrema occur.

Solution:

Step 1. $f(x) = x + 4(x+1)^{-1}$, so

$$f'(x) = 1 + 4(-1)(x+1)^{-2} = 1 - \frac{4}{(x+1)^2}$$

$$= \frac{(x+1)^2 - 4}{(x+1)^2} = \frac{x^2 + 2x - 3}{(x+1)^2}$$

$$= \frac{(x+3)(x-1)}{(x+1)^2} \quad \text{for } x \neq -1$$

Note that we expressed $f'(x)$ as a quotient with numerator and denominator fully factored. This enables us in Step 2 to determine easily where $f'(x)$ is 0 or does not exist and the signs of f'.

Step 2. Setting $f'(x) = 0$ gives $x = -3, 1$. The denominator of $f'(x)$ is 0 when x is -1. We note that -1 is not in the domain of f but all values near -1 are in the domain of f. We construct a sign chart, headed by the values -3, -1, and 1 (which we have placed in increasing order). See Figure 13.10.

The three values lead us to test four intervals as shown in our sign chart. On each of these intervals, f is differentiable and is not zero. We determine the sign of f' on each interval by first determining the sign of each of its factors on each iterval. For example, considering first the interval $(-\infty, -3)$, it is not easy to see immediately that $f'(x) > 0$ there; but it is easy to see that $x + 3 < 0$ for $x < -3$, while $(x+1)^{-2} > 0$ for all $x \neq -1$, and $x - 1 < 0$ for $x < 1$. These observations account for the signs of the factors in the $(-\infty, -3)$ column of the chart. The sign of $f'(x)$ in that column is obtained by "multiplying signs" (downward): $(-)(+)(-) = +$. We repeat these considerations for the other three intervals. Note that the thick vertical line at -1 in the chart indicates that

	$-\infty$		-3		-1		1		∞
$x+3$	$-$		0	$+$		$+$		$+$	
$(x+1)^{-2}$	$+$			$+$		$+$		$+$	
$x-1$	$-$			$-$		$-$	0	$+$	
$f'(x)$	$+$		0	$-$		$-$	0	$+$	
$f(x)$		↗		↘		↘		↗	

FIGURE 13.10 Sign chart for $f'(x) = \dfrac{(x+3)(x-1)}{(x+1)^2}$.

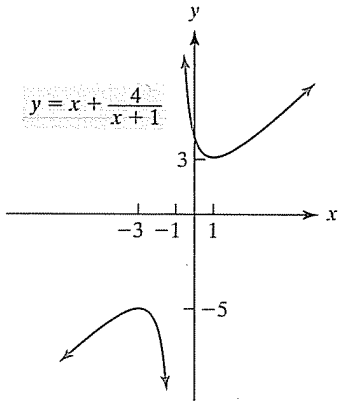


FIGURE 13.11 Graph of $y = x + \dfrac{4}{x+1}$.

-1 is not in the domain of f and hence cannot give rise to any extrema. In the bottom row of the sign chart we record, graphically, the nature of tangent lines to $f(x)$ in each interval and at the values where f' is 0.

Step 3. From the sign chart alone we conclude that at -3 there is a relative maximum (since $f'(x)$ changes from $+$ to $-$ at -3). Going beyond the chart, we compute $f(-3) = -3 + (4/-2) = -5$, and this gives the relative maximum value of -5 at -3. We also conclude from the chart that there is a relative minimum at 1 [because $f'(x)$ changes from $-$ to $+$ at 1]. From $f(1) = 1 + 4/2 = 3$ we see that at 1 the relative minimum value is 3.

Step 4. There are no critical values at which f is not continuous, so our considerations above provide the whole story about the relative extrema of $f(x)$, whose graph is given in Figure 13.11. Note that the general shape of the graph was indeed forecast by the bottom row of the sign chart (Figure 13.10).

Now Work Problem 37 ◁

	$-\infty$		0		∞
$(x)^{-1/3}$	$-$		$\times$	$+$	
$f'(x)$	$-$		$\times$	$+$	
$f(x)$		↘		↗	

FIGURE 13.12 Sign chart for $f'(x) = \dfrac{2}{3\sqrt[3]{x}}$.

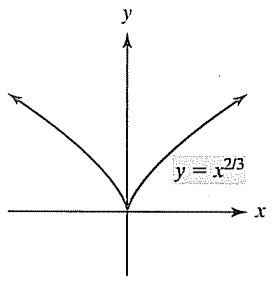


FIGURE 13.13 Derivative does not exist at 0 and there is a minimum at 0.

EXAMPLE 2 A Relative Extremum where $f'(x)$ Does Not Exist

Test $y = f(x) = x^{2/3}$ for relative extrema.

Solution: We have

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

and the sign chart is given in Figure 13.12. Again, we use the symbol $\times$ on the vertical line at 0 to indicate that the factor $x^{-1/3}$ does not exist at 0. Hence $f'(0)$ does not exist. Since f is continuous at 0, we conclude from Rule 3 that f has a relative minimum at 0 of $f(0) = 0$, and there are no other relative extrema. We note further, by inspection of the sign chart, that f has an *absolute* minimum at 0. The graph of f follows as Figure 13.13. Note that we could have predicted its shape from the bottom line of the sign chart in Figure 13.12, which shows there can be no tangent with a slope at 0. (Of course, the tangent does exist at 0 but it is a vertical line.)

Now Work Problem 41 ◁

EXAMPLE 3 Finding Relative Extrema

Test $y = f(x) = x^2e^x$ for relative extrema.

Solution: By the product rule,

$$f'(x) = x^2e^x + e^x(2x) = xe^x(x+2)$$

	$-\infty$		-2		0		∞
$x+2$	$-$	0	$+$		$+$		
x	$-$		$-$	0	$+$		
e^x	$+$		$+$		$+$		
$f'(x)$	$+$	0	$-$	0	$+$		
$f(x)$		↗		↘		↗	

FIGURE 13.14 Sign chart for $f'(x) = x(x+2)e^x$.

APPLY IT ▷

2. A drug is injected into a patient's bloodstream. The concentration of the drug in the bloodstream t hours after the injection is approximated by $C(t) = \frac{0.14t}{t^2+4t+4}$. Find the relative extrema for $t > 0$, and use them to determine when the drug is at its greatest concentration.

Noting that e^x is always positive, we obtain the critical values 0 and -2. From the sign chart of $f'(x)$ given in Figure 13.14, we conclude that there is a relative maximum when $x = -2$ and a relative minimum when $x = 0$.

Now Work Problem 49 ◁

Curve Sketching

In the next example we show how the first-derivative test, in conjunction with the notions of intercepts and symmetry, can be used as an aid in sketching the graph of a function.

EXAMPLE 4 **Curve Sketching**

Sketch the graph of $y = f(x) = 2x^2 - x^4$ with the aid of intercepts, symmetry, and the first-derivative test.

Solution:

Intercepts If $x = 0$, then $f(x) = 0$ so that the y-intercept is $(0, 0)$. Next note that

$$f(x) = 2x^2 - x^4 = x^2(2 - x^2) = x^2\left(\sqrt{2} + x\right)\left(\sqrt{2} - x\right)$$

So if $y = 0$, then $x = 0, \pm\sqrt{2}$ and the x-intercepts are $(-\sqrt{2}, 0)$, $(0, 0)$, and $(\sqrt{2}, 0)$. We have the sign chart *for f itself* (Figure 13.15), which shows the intervals over which the graph of $y = f(x)$ is above the x-axis $(+)$ and the intervals over which the graph of $y = f(x)$ is below the x-axis $(-)$.

	$-\infty$		$-\sqrt{2}$		0		$\sqrt{2}$		∞
$\sqrt{2}+x$	$-$	0	$+$		$+$		$+$		
x^2	$+$		$+$	0	$+$		$+$		
$\sqrt{2}-x$	$+$		$+$		$+$	0	$-$		
$f(x)$	$-$	0	$+$	0	$+$	0	$-$		

FIGURE 13.15 Sign chart for $f(x) = (\sqrt{2} + x)x^2(\sqrt{2} - x)$.

Symmetry Testing for y-axis symmetry, we have

$$f(-x) = 2(-x)^2 - (-x)^4 = 2x^2 - x^4 = f(x)$$

So the graph is symmetric with respect to the y-axis. Because y is a function (and not the zero function), there is no x-axis symmetry and hence no symmetry about the origin.

First-Derivative Test

Step 1. $y' = 4x - 4x^3 = 4x(1 - x^2) = 4x(1 + x)(1 - x)$.

Step 2. Setting $y' = 0$ gives the critical values $x = 0, \pm 1$. Since f is a polynomial, it is defined and differentiable for all x. Thus the only values to head the sign chart for f' are -1, 0, 1 (in increasing order) and the sign chart is given in Figure 13.16. Since we are interested in the graph, the critical *points* are important to us. By substituting the critical values into the *original* equation, $y = 2x^2 - x^4$,

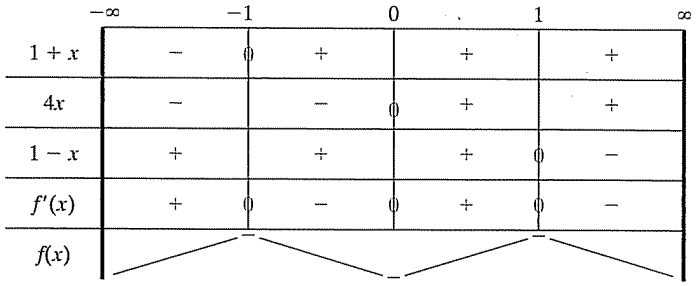


FIGURE 13.16 Sign chart of $y' = (1 + x)4x(1 - x)$.

we obtain the y-coordinates of these points. We find the critical points to be $(-1, 1)$, $(0, 0)$, and $(1, 1)$.

Step 3. From the sign chart and evaluations in step 2, it is clear that f has relative maxima $(-1, 1)$ and $(1, 1)$ and relative minimum $(0, 0)$. (Step 4 does not apply here.)

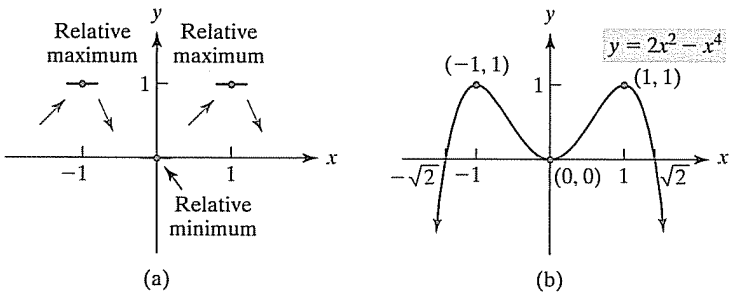


FIGURE 13.17 Putting together the graph of $y = 2x^2 - x^4$.

Discussion In Figure 13.17(a), we have indicated the horizontal tangents at the relative maximum and minimum points. We know the curve rises from the left, has a relative maximum, then falls, has a relative minimum, then rises to a relative maximum, and falls thereafter. By symmetry, it suffices to sketch the graph on one side of the y-axis and construct a mirror image on the other side. We also know, from the sign chart for f, where the graph crosses and touches the x-axis, and this adds further precision to our sketch, which is shown in Figure 13.17(b).

As a passing comment, we note that *absolute* maxima occur at $x = \pm 1$. See Figure 13.17(b). There is no absolute minimum.

Now Work Problem 59 ◁

TECHNOLOGY

A graphing calculator is a powerful tool for investigating relative extrema. For example, consider the function

$$f(x) = 3x^4 - 4x^3 + 4$$

whose graph is shown in Figure 13.18. It appears that there is a relative minimum near $x = 1$. We can locate this minimum by either using "trace and zoom" or (on a TI-83 Plus) using the "minimum" feature. Figure 13.19 shows the latter approach. The relative minimum point is estimated to be (1.00, 3).

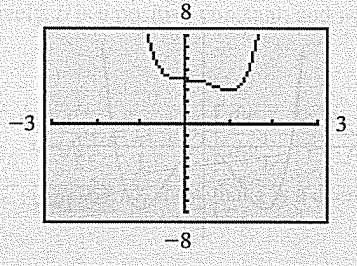


FIGURE 13.18 Graph of $f(x) = 3x^4 - 4x^3 + 4$.

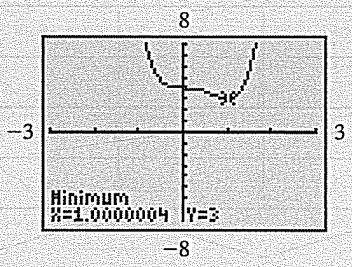


FIGURE 13.19 Relative minimum at (1.00, 3).

Now let us see how the graph of f' indicates when extrema occur. We have

$$f'(x) = 12x^3 - 12x^2$$

whose graph is shown in Figure 13.20. It appears that $f'(x)$ is 0 at two points. Using "trace and zoom" or the "zero" feature, we estimate the roots of $f' = 0$ (the critical values of f) to be 1 and 0. Around $x = 1$, we see that $f'(x)$ goes from negative values to positive values. (That is, the graph of f' goes from below the x-axis to above it.) Thus, we conclude that f has a relative minimum at $x = 1$, which confirms our previous result.

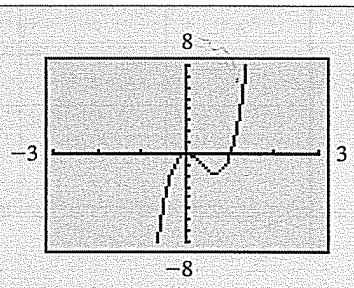


FIGURE 13.20 Graph of $f'(x) = 12x^3 - 12x^2$.

Around the critical value $x = 0$, the values of $f'(x)$ are negative. Since $f'(x)$ does not change sign, we conclude that there is no relative extremum at $x = 0$. This is also apparent from the graph in Figure 13.18.

It is worthwhile to note that we can approximate the graph of f' without determining $f'(x)$ itself. We make use of the "nDeriv" feature. First we enter the function f as Y_1. Then we set

$$Y_2 = \text{nDeriv}(Y_1, X, X)$$

The graph of Y_2 approximates the graph of $f'(x)$.

PROBLEMS 13.1

In Problems 1–4, the graph of a function is given (Figures 13.21–13.24). Find the open intervals on which the function is increasing, the open intervals on which the function is decreasing, and the coordinates of all relative extrema.

1.

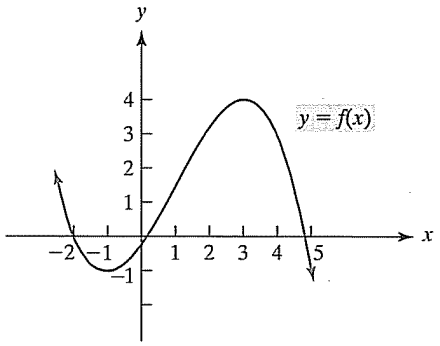


FIGURE 13.21

2.

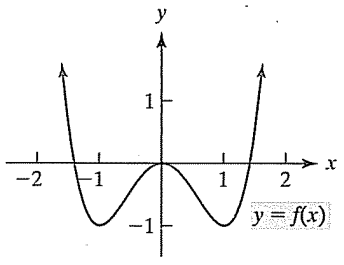


FIGURE 13.22

3.

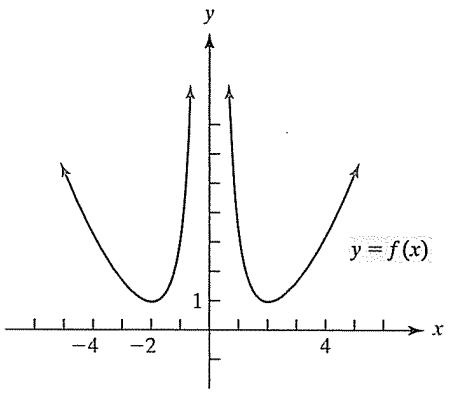


FIGURE 13.23

4.

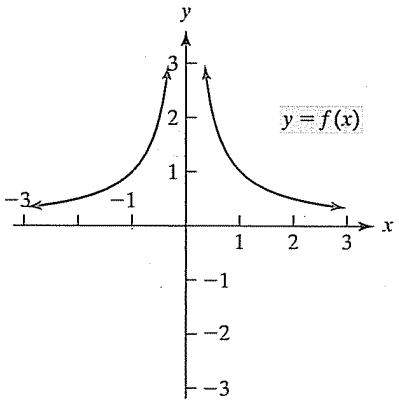


FIGURE 13.24

In Problems 5–8, the derivative of a continuous function f is given. Find the open intervals on which f is **(a)** *increasing;* **(b)** *decreasing; and* **(c)** *find the x-values of all relative extrema.*

5. $f'(x) = (x+3)(x-1)(x-2)$

6. $f'(x) = 2x(x-1)^3$

7. $f'(x) = (x+1)(x-3)^2$ **8.** $f'(x) = \dfrac{x(x+2)}{x^2+1}$

In Problems 9–52, determine where the function is **(a)** *increasing;* **(b)** *decreasing; and* **(c)** *determine where relative extrema occur. Do not sketch the graph.*

9. $y = -x^3 - 1$

10. $y = x^2 + 4x + 3$

11. $y = x - x^2 + 2$

12. $y = x^3 - \dfrac{5}{2}x^2 - 2x + 6$

13. $y = -\dfrac{x^3}{3} - 2x^2 + 5x - 2$

14. $y = -\dfrac{x^4}{4} - x^3$

15. $y = x^4 - 2x^2$

16. $y = -3 + 12x - x^3$

17. $y = x^3 - \dfrac{7}{2}x^2 + 2x - 5$

18. $y = x^3 - 6x^2 + 12x - 6$

19. $y = 2x^3 - \dfrac{19}{2}x^2 + 10x + 2$

20. $y = -5x^3 + x^2 + x - 7$

21. $y = \dfrac{x^3}{3} - 5x^2 + 22x + 1$

22. $y = \dfrac{9}{5}x^5 - \dfrac{47}{3}x^3 + 10x$

23. $y = 3x^5 - 5x^3$

24. $y = 3x - \dfrac{x^6}{2}$ (Remark: $x^4 + x^3 + x^2 + x + 1 = 0$ has no real roots.)

25. $y = -x^5 - 5x^4 + 200$

26. $y = \dfrac{3x^4}{2} - 4x^3 + 17$

27. $y = 8x^4 - x^8$

28. $y = \dfrac{4}{5}x^5 - \dfrac{13}{3}x^3 + 3x + 4$

29. $y = (x^2 - 4)^4$

30. $y = \sqrt[3]{x}(x-2)$

31. $y = \dfrac{5}{x-1}$

32. $y = \dfrac{3}{x}$

33. $y = \dfrac{10}{\sqrt{x}}$

34. $y = \dfrac{ax+b}{cx+d}$
(a) for $ad - bc > 0$
(b) for $ad - bc < 0$

35. $y = \dfrac{x^2}{2-x}$

36. $y = 4x^2 + \dfrac{1}{x}$

37. $y = \dfrac{x^2-3}{x+2}$

38. $y = \dfrac{2x^2}{4x^2-25}$

39. $y = \dfrac{ax^2+b}{cx^2+d}$ for $d/c < 0$
(a) for $ad - bc > 0$
(b) for $ad - bc < 0$

40. $y = \sqrt[3]{x^3 - 9x}$

41. $y = (x-1)^{2/3}$

42. $y = x^2(x+3)^4$

43. $y = x^3(x-6)^4$

44. $y = (1-x)^{2/3}$

45. $y = e^{-\pi x} + \pi$

46. $y = x \ln x$

47. $y = x^2 - 9 \ln x$

48. $y = x^{-1}e^x$

49. $y = e^x - e^{-x}$

50. $y = e^{-x^2/2}$

51. $y = x \ln x - x$

52. $y = (x^2+1)e^{-x}$

In Problems 53–64, determine intervals on which the function is increasing; intervals on which the function is decreasing; relative extrema; symmetry; and those intercepts that can be obtained conveniently. Then sketch the graph.

53. $y = x^2 - 3x - 10$

54. $y = 2x^2 + x - 10$

55. $y = 3x - x^3$

56. $y = x^4 - 16$

57. $y = 2x^3 - 9x^2 + 12x$

58. $y = 2x^3 - x^2 - 4x + 4$

59. $y = x^4 - 2x^2$

60. $y = x^6 - \dfrac{6}{5}x^5$

61. $y = (x-1)^2(x+2)^2$

62. $y = \sqrt{x}(x^2 - x - 2)$

63. $y = 2\sqrt{x} - x$

64. $y = x^{5/3} - 2x^{2/3}$

65. Sketch the graph of a continuous function f such that $f(2) = 2$, $f(4) = 6$, $f'(2) = f'(4) = 0$, $f'(x) < 0$ for $x < 2$, $f'(x) > 0$ for $2 < x < 4$, f has a relative maximum at 4, and $\lim_{x\to\infty} f(x) = 0$.

66. Sketch the graph of a continuous function f such that $f(1) = 2$, $f(4) = 5$, $f'(1) = 0$, $f'(x) \geq 0$ for $x < 4$, f has a relative maximum when $x = 4$, and there is a vertical tangent line when $x = 4$.

67. Average Cost If $c_f = 25{,}000$ is a fixed-cost function, show that the average fixed-cost function $\overline{cf} = c_f/q$ is a decreasing function for $q > 0$. Thus, as output q increases, each unit's portion of fixed cost declines.

68. Marginal Cost If $c = 3q - 3q^2 + q^3$ is a cost function, when is marginal cost increasing?

69. Marginal Revenue Given the demand function

$$p = 500 - 5q$$

find when marginal revenue is increasing.

70. Cost Function For the cost function $c = \sqrt{q}$, show that marginal and average costs are always decreasing for $q > 0$.

71. Revenue For a manufacturer's product, the revenue function is given by $r = 240q + 57q^2 - q^3$. Determine the output for maximum revenue.

72. Labor Markets Eswaran and Kotwal[1] consider agrarian economies in which there are two types of workers, permanent and casual. Permanent workers are employed on long-term contracts and may receive benefits such as holiday gifts and emergency aid. Casual workers are hired on a daily basis and perform routine and menial tasks such as weeding, harvesting, and threshing. The difference z in the present-value cost of hiring a permanent worker over that of hiring a casual worker is given by

$$z = (1+b)w_p - bw_c$$

where w_p and w_c are wage rates for permanent labor and casual labor, respectively, b is a positive constant, and w_p is a function of w_c.

(a) Show that

$$\frac{dz}{dw_c} = (1+b)\left[\frac{dw_p}{dw_c} - \frac{b}{1+b}\right]$$

(b) If $dw_p/dw_c < b/(1+b)$, show that z is a decreasing function of w_c.

[1] M. Eswaran and A. Kotwal, "A Theory of Two-Tier Labor Markets in Agrarian Economics," *The American Economic Review*, 75, no. 1 (1985), 162–77.

73. Thermal Pollution In Shonle's discussion of thermal pollution,[2] the efficiency of a power plant is given by

$$E = 0.71\left(1 - \frac{T_c}{T_h}\right)$$

where T_h and T_c are the respective absolute temperatures of the hotter and colder reservoirs. Assume that T_c is a positive constant and that T_h is positive. Using calculus, show that as T_h increases, the efficiency increases.

74. Telephone Service In a discussion of the pricing of local telephone service, Renshaw[3] determines that total revenue r is given by

$$r = 2F + \left(1 - \frac{a}{b}\right)p - p^2 + \frac{a^2}{b}$$

where p is an indexed price per call, and a, b, and F are constants. Determine the value of p that maximizes revenue.

75. Storage and Shipping Costs In his model for storage and shipping costs of materials for a manufacturing process, Lancaster[4] derives the cost function

$$C(k) = 100\left(100 + 9k + \frac{144}{k}\right) \quad 1 \le k \le 100$$

where $C(k)$ is the total cost (in dollars) of storage and transportation for 100 days of operation if a load of k tons of material is moved every k days.

(a) Find $C(1)$.
(b) For what value of k does $C(k)$ have a minimum?
(c) What is the minimum value?

76. Physiology—The Bends When a deep-sea diver undergoes decompression or a pilot climbs to a high altitude, nitrogen may bubble out of the blood, causing what is commonly called *the bends*. Suppose the percentage P of people who suffer effects of the bends at an altitude of h thousand feet is given by[5]

$$P = \frac{100}{1 + 100{,}000e^{-0.36h}}$$

Is P an increasing function of h?

In Problems 77–80, from the graph of the function, find the coordinates of all relative extrema. Round your answers to two decimal places.

77. $y = 0.3x^2 + 2.3x + 5.1$ **78.** $y = 3x^4 - 4x^3 - 5x + 1$

79. $y = \dfrac{8.2x}{0.4x^2 + 3}$ **80.** $y = \dfrac{e^x(3 - x)}{7x^2 + 1}$

81. Graph the function

$$f(x) = [x(x - 2)(2x - 3)]^2$$

in the window $-1 \le x \le 3$, $-1 \le y \le 3$. Upon first glance, it may appear that this function has two relative minimum points and one relative maximum point. However, in reality, it has three relative minimum points and two relative maximum points. Determine the x-values of all these points. Round answers to two decimal places.

82. If $f(x) = 3x^3 - 7x^2 + 4x + 2$, display the graphs of f and f' on the same screen. Notice that $f'(x) = 0$ where relative extrema of f occur.

83. Let $f(x) = 6 + 4x - 3x^2 - x^3$. (a) Find $f'(x)$. (b) Graph $f'(x)$. (c) Observe where $f'(x)$ is positive and where it is negative. Give the intervals (rounded to two decimal places) where f is increasing and where f is decreasing. (d) Graph f and f' on the same screen, and verify your results to part (c).

84. If $f(x) = x^4 - x^2 - (x + 2)^2$, find $f'(x)$. Determine the critical values of f. Round your answers to two decimal places.

13.2 Absolute Extrema on a Closed Interval

Objective

To find extreme values on a closed interval.

If a function f is *continuous* on a *closed* interval $[a, b]$, it can be shown that of *all* the function values $f(x)$ for x in $[a, b]$, there must be an absolute maximum value and an absolute minimum value. These two values are called **extreme values** of f on that interval. This important property of continuous functions is called the *extreme-value theorem.*

Extreme-Value Theorem

If a function is continuous on a closed interval, then the function has *both* a maximum value *and* a minimum value on that interval.

For example, each function in Figure 13.25 is continuous on the closed interval [1, 3]. Geometrically, the extreme-value theorem assures us that over this interval each graph has a highest point and a lowest point.

In the extreme-value theorem, it is important that we are dealing with

1. a closed interval and
2. a function continuous on that interval

[2] J. I. Shonle, *Environmental Applications of General Physics* (Reading, MA: Addison-Wesley Publishing Company, Inc., 1975).

[3] E. Renshaw, "A Note on Equity and Efficiency in the Pricing of Local Telephone Services," *The American Economic Review*, 75, no. 3 (1985), 515–18.

[4] P. Lancaster, *Mathematics: Models of the Real World* (Englewood Cliffs, NJ: Prentice-Hall, Inc., 1976).

[5] Adapted from G. E. Folk, Jr., *Textbook of Environmental Physiology*, 2nd ed. (Philadelphia: Lea & Febiger, 1974).

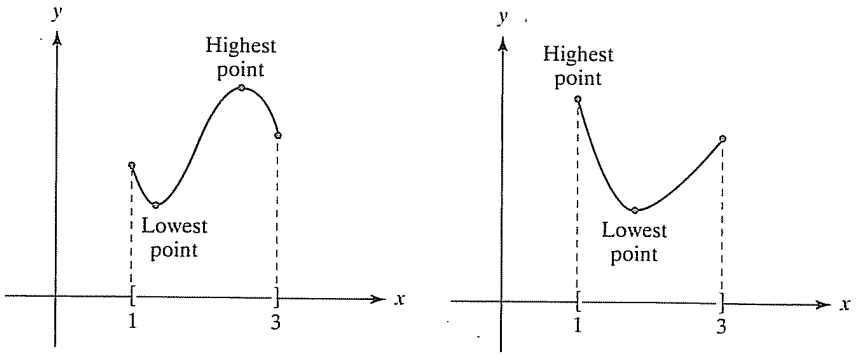
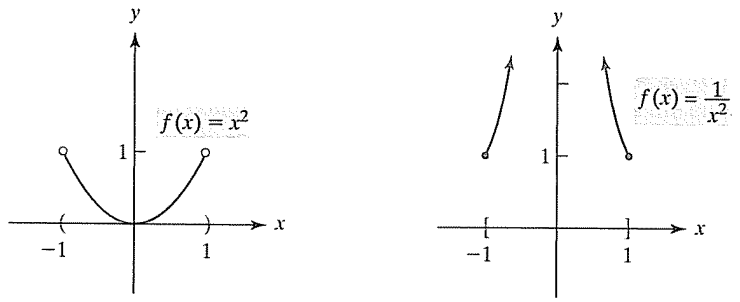


FIGURE 13.25 Illustrating the extreme-value theorem.

If either condition (1) or condition (2) is not met, then extreme values are not guaranteed. For example, Figure 13.26(a) shows the graph of the continuous function $f(x) = x^2$ on the *open* interval $(-1, 1)$. You can see that f has no maximum value on the interval (although f has a minimum value there). Now consider the function $f(x) = 1/x^2$ on the closed interval $[-1, 1]$. Here f is *not continuous* at 0. From the graph of f in Figure 13.26(b), you can see that f has no maximum value (although there is a minimum value).



Open interval $(-1, 1)$
No maximum, minimum = 0
(a)

Not continuous at 0
No maximum, minimum = 1
(b)

FIGURE 13.26 Extreme-value theorem does not apply.

In the previous section, our emphasis was on relative extrema. Now we will focus our attention on absolute extrema and make use of the extreme-value theorem where possible. If the domain of a function is a closed interval, to determine *absolute* extrema we must examine the function not only at critical values, but also at the endpoints. For example, Figure 13.27 shows the graph of the continuous function $y = f(x)$ over $[a, b]$. The extreme-value theorem guarantees absolute extrema over the interval. Clearly, the important points on the graph occur at $x = a, b, c$, and d, which correspond to endpoints

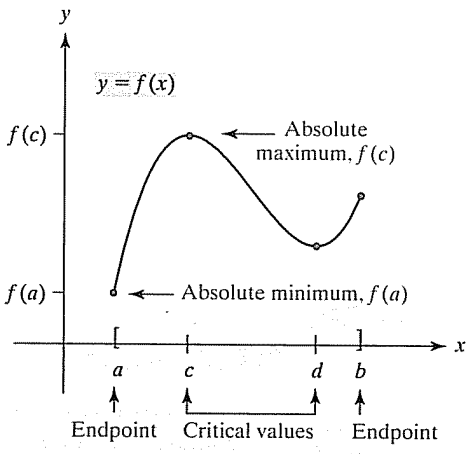


FIGURE 13.27 Absolute extrema.

or critical values. Notice that the absolute maximum occurs at the critical value c and the absolute minimum occurs at the endpoint a. These results suggest the following procedure:

Procedure to Find Absolute Extrema for a Function f That Is Continuous on $[a, b]$

Step 1. Find the critical values of f.

Step 2. Evaluate $f(x)$ at the endpoints a and b and at the critical values in (a, b).

Step 3. The maximum value of f is the greatest of the values found in step 2. The minimum value of f is the least of the values found in step 2.

EXAMPLE 1 Finding Extreme Values on a Closed Interval

Find absolute extrema for $f(x) = x^2 - 4x + 5$ over the closed interval $[1, 4]$.

Solution: Since f is continuous on $[1, 4]$, the foregoing procedure applies.

Step 1. To find the critical values of f, we first find f':

$$f'(x) = 2x - 4 = 2(x - 2)$$

This gives the critical value $x = 2$.

Step 2. Evaluating $f(x)$ at the endpoints 1 and 4 and at the critical value 2, we have

$f(1) = 2$
$f(4) = 5$ values of f at endpoints

and

$f(2) = 1$ value of f at critical value 2 in $(1, 4)$

Step 3. From the function values in Step 2, we conclude that the maximum is $f(4) = 5$ and the minimum is $f(2) = 1$. (See Figure 13.28.)

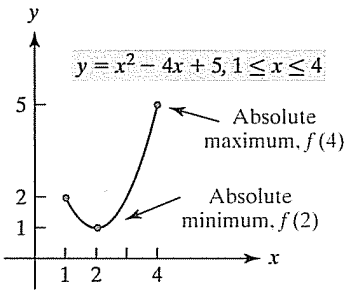


FIGURE 13.28 Extreme values for Example 1.

Now Work Problem 1 ◁

PROBLEMS 13.2

In Problems 1–14, find the absolute extrema of the given function on the given interval.

1. $f(x) = x^2 - 2x + 3$, $[0, 3]$
2. $f(x) = -2x^2 - 6x + 5$, $[-3, 2]$
3. $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + 1$, $[-1, 0]$
4. $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^2$, $[0, 1]$
5. $f(x) = x^3 - 5x^2 - 8x + 50$, $[0, 5]$
6. $f(x) = x^{2/3}$, $[-8, 8]$
7. $f(x) = -3x^5 + 5x^3$, $[-2, 0]$
8. $f(x) = \frac{7}{3}x^3 + 2x^2 - 3x + 1$, $[0, 3]$
9. $f(x) = 3x^4 - x^6$, $[-1, 2]$
10. $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 2$, $[0, 4]$
11. $f(x) = x^4 - 9x^2 + 2$, $[-1, 3]$
12. $f(x) = \dfrac{x}{x^2 + 1}$, $[0, 2]$
13. $f(x) = (x - 1)^{2/3}$, $[-26, 28]$
14. $f(x) = 0.2x^3 - 3.6x^2 + 2x + 1$, $[-1, 2]$
15. Consider the function

$$f(x) = x^4 + 8x^3 + 21x^2 + 20x + 9$$

over the interval $[-4, 9]$.

(a) Determine the value(s) (rounded to two decimal places) of x at which f attains a minimum value.
(b) What is the minimum value (rounded to two decimal places) of f?
(c) Determine the value(s) of x at which f attains a maximum value.
(d) What is the maximum value of f?

13.3 Concavity

Objective

To test a function for concavity and inflection points. Also, to sketch curves with the aid of the information obtained from the first and second derivatives.

The first derivative provides much information for sketching curves. It is used to determine where a function is increasing, is decreasing, has relative maxima, and has relative minima. However, to be sure we know the true shape of a curve, we may need more

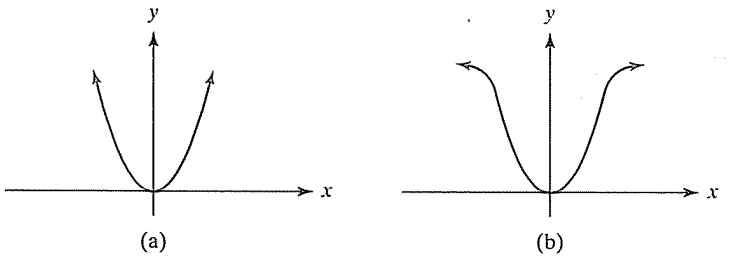


FIGURE 13.29 Two functions with $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

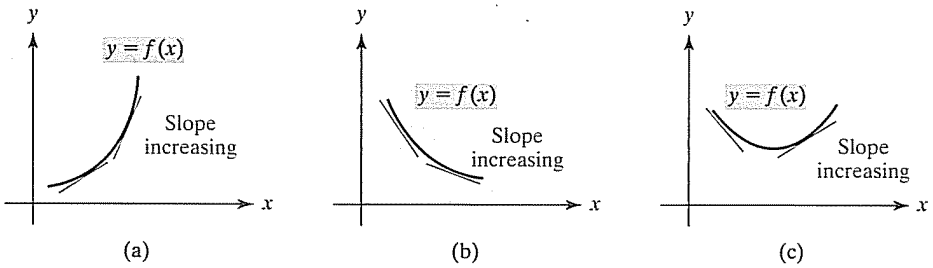


FIGURE 13.30 Each curve is concave up.

information. For example, consider the curve $y = f(x) = x^2$. Since $f'(x) = 2x, x = 0$ is a critical value. If $x < 0$, then $f'(x) < 0$, and f is decreasing; if $x > 0$, then $f'(x) > 0$, and f is increasing. Thus, there is a relative minimum when $x = 0$. In Figure 13.29, both curves meet the preceding conditions. But which one truly describes the curve $y = x^2$? This question will be settled easily by using the second derivative and the notion of *concavity*.

In Figure 13.30, note that each curve $y = f(x)$ "bends" (or opens) upward. This means that if tangent lines are drawn to each curve, the curves lie *above* them. Moreover, the slopes of the tangent lines *increase* in value as x increases: In part (a), the slopes go from small positive values to larger values; in part (b), they are negative and approaching zero (and thus increasing); in part (c), they pass from negative values to positive values. Since $f'(x)$ gives the slope at a point, an increasing slope means that f' must be an increasing function. To describe this property, each curve (or function f) in Figure 13.30 is said to be *concave up*.

In Figure 13.31, it can be seen that each curve lies *below* the tangent lines and the curves are bending downward. As x increases, the slopes of the tangent lines are *decreasing*. Thus, f' must be a decreasing function here, and we say that f is *concave down*.

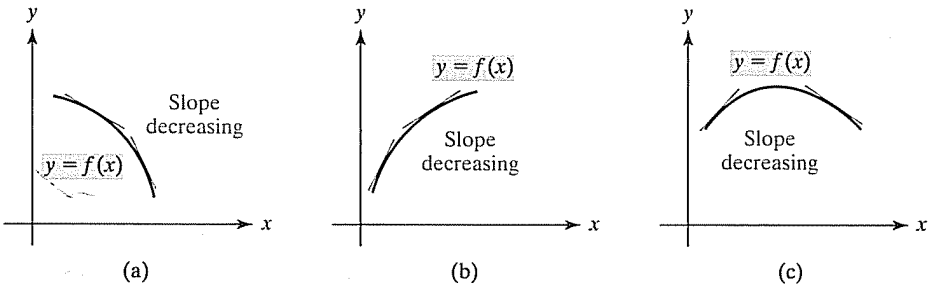


FIGURE 13.31 Each curve is concave down.

Definition

Let f be differentiable on the interval (a, b). Then f is said to be ***concave up*** **[*concave down*]** on (a, b) if f' is increasing [decreasing] on (a, b).

CAUTION!

Concavity relates to whether f', not f, is increasing or decreasing. In Figure 13.30(b), note that f is concave up and decreasing; however, in Figure 13.31(a), f is concave down and decreasing.

Remember: If f is concave up on an interval, then, geometrically, its graph is bending upward there. If f is concave down, then its graph is bending downward.

Since f' is increasing when its derivative $f''(x)$ is positive, and f' is decreasing when $f''(x)$ is negative, we can state the following rule:

Rule 1 Criteria for Concavity

Let f' be differentiable on the interval (a, b). If $f''(x) > 0$ for all x in (a, b), then f is concave up on (a, b). If $f''(x) < 0$ for all x in (a, b), then f is concave down on (a, b).

A function f is also said to be concave up at a point c if there exists an open interval around c on which f is concave up. In fact, for the functions that we will consider, if $f''(c) > 0$, then f is concave up at c. Similarly, f is concave down at c if $f''(c) < 0$.

EXAMPLE 1 Testing for Concavity

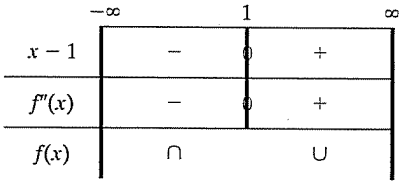
Determine where the given function is concave up and where it is concave down.

a. $y = f(x) = (x - 1)^3 + 1$.

Solution: To apply Rule 1, we must examine the signs of y''. Now, $y' = 3(x - 1)^2$, so

$$y'' = 6(x - 1)$$

Thus, f is concave up when $6(x - 1) > 0$; that is, when $x > 1$. And f is concave down when $6(x - 1) < 0$; that is, when $x < 1$. We now use a sign chart for f'' (together with an interpretation line for f) to organize our findings. (See Figure 13.32.)



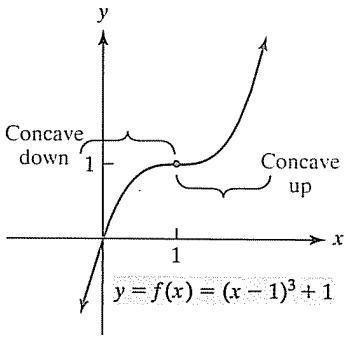


FIGURE 13.32 Sign chart for f'' and concavity for $f(x) = (x - 1)^3 + 1$.

b. $y = x^2$.

Solution: We have $y' = 2x$ and $y'' = 2$. Because y'' is always positive, the graph of $y = x^2$ must always be concave up, as in Figure 13.29(a). The graph cannot appear as in Figure 13.29(b), for that curve is sometimes concave down.

Now Work Problem 1 ◁

A point on a graph where concavity changes from concave down to concave up, or vice versa, such as (1, 1) in Figure 13.32, is called an *inflection point* or a *point of inflection*. Around such a point, the sign of $f''(x)$ goes from $-$ to $+$ or from $+$ to $-$. More precisely, we have the following definition:

Definition

A function f has an ***inflection point*** at a if and only if f is continuous at a and f changes concavity at a.

The definition of an inflection point implies that a is in the domain of f.

To test a function for concavity and inflection points, first find the values of x where $f''(x)$ is 0 or not defined. These values of x determine intervals. On each interval,

determine whether $f''(x) > 0$ (f is concave up) or $f''(x) < 0$ (f is concave down). If concavity changes around one of these x-values and f is continuous there, then f has an inflection point at this x-value. The continuity requirement implies that the x-value must be in the domain of the function. In brief, a *candidate* for an inflection point must satisfy two conditions:

1. f'' must be 0 or fail to exist at that point.
2. f must be continuous at that point.

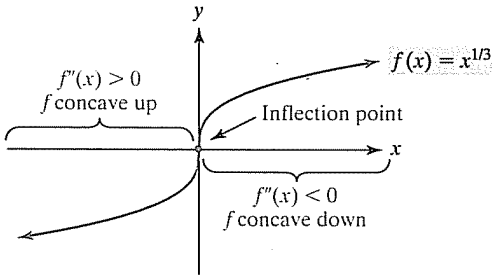


FIGURE 13.33 Inflection point for $f(x) = x^{1/3}$.

The candidate *will be* an inflection point if concavity changes around it. For example, if $f(x) = x^{1/3}$, then $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$f''(x) = -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}}$$

Because f'' does not exist at 0, but f is continuous at 0, there is a candidate for an inflection point at 0. If $x > 0$, then $f''(x) < 0$, so f is concave down for $x > 0$; if $x < 0$, then $f''(x) > 0$, so f is concave up for $x < 0$. Because concavity changes at 0, there is an inflection point there. (See Figure 13.33.)

EXAMPLE 2 Concavity and Inflection Points

Test $y = 6x^4 - 8x^3 + 1$ for concavity and inflection points.

Solution: We have

$$y' = 24x^3 - 24x^2$$
$$y'' = 72x^2 - 48x = 24x(3x - 2)$$

	$-\infty$		0		2/3		∞
x	−		0	+		+	
$3x - 2$	−			−	0	+	
y''	+		0	−	0	+	
y	∪			∩		∪	

FIGURE 13.34 Sign chart of $y'' = 24x(3x - 2)$ for $y = 6x^4 - 8x^3 + 1$.

To find where $y'' = 0$, we set each factor in y'' equal to 0. This gives $x = 0, \frac{2}{3}$. We also note that y'' is never undefined. Thus, there are three intervals to consider, as recorded on the top of the sign chart in Figure 13.34. Since y is continuous at 0 and $\frac{2}{3}$, these points are candidates for inflection points. Having completed the sign chart, we see that concavity changes at 0 and at $\frac{2}{3}$. Thus these candidates are indeed inflection points. (See Figure 13.35.) In summary, the curve is concave up on $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$ and is concave down on $(0, \frac{2}{3})$. Inflection points occur at 0 and at $\frac{2}{3}$. These points are $(0, y(0)) = (0, 1)$ and $(\frac{2}{3}, y(\frac{2}{3})) = (\frac{2}{3}, -\frac{5}{27})$.

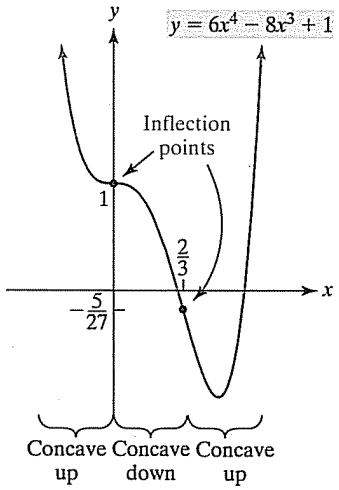


FIGURE 13.35 Graph of $y = 6x^4 - 8x^3 + 1$.

Now Work Problem 13 ◁

	$-\infty$		0		∞
$\frac{1}{x^3}$		−		+	
$f''(x)$		−		+	
$f(x)$		∩		∪	

FIGURE 13.36 Sign chart for $f''(x)$.

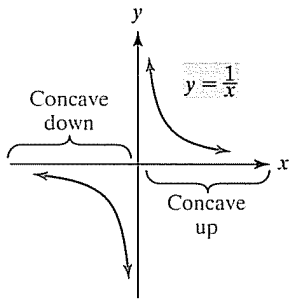


FIGURE 13.37 Graph of $y = \frac{1}{x}$.

CAUTION!

A candidate for an inflection point may not necessarily be an inflection point. For example, if $f(x) = x^4$, then $f''(x) = 12x^2$ and $f''(0) = 0$. But $f''(x) > 0$ both when $x < 0$ and when $x > 0$. Thus, concavity does not change, and there are no inflection points. (See Figure 13.38.)

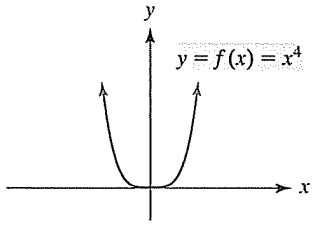


FIGURE 13.38 Graph of $f(x) = x^4$.

As we did in the analysis of increasing and decreasing, so we must in concavity analysis consider also those points a that are not in the domain of f but that are near points in the domain of f. The next example will illustrate.

EXAMPLE 3 A Change in Concavity with No Inflection Point

Discuss concavity and find all inflection points for $f(x) = \frac{1}{x}$.

Solution: Since $f(x) = x^{-1}$ for $x \neq 0$,

$$f'(x) = -x^{-2} \quad \text{for } x \neq 0$$
$$f''(x) = 2x^{-3} = \frac{2}{x^3} \quad \text{for } x \neq 0$$

We see that $f''(x)$ is never 0 but it is not defined when $x = 0$. Since f is not continuous at 0, we conclude that 0 is not a candidate for an inflection point. Thus, the given function has no inflection point. However, 0 must be considered in an analysis of concavity. See the sign chart in Figure 13.36; note that we have a thick vertical line at 0 to indicate that 0 is not in the domain of f and cannot correspond to an inflection point. If $x > 0$, then $f''(x) > 0$; if $x < 0$, then $f''(x) < 0$. Hence, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. (See Figure 13.37.) Although concavity changes around $x = 0$, there is no inflection point there because f is not continuous at 0 (nor is it even defined there).

Now Work Problem 23 ◁

Curve Sketching

EXAMPLE 4 Curve Sketching

Sketch the graph of $y = 2x^3 - 9x^2 + 12x$.

Solution:

Intercepts If $x = 0$, then $y = 0$. Setting $y = 0$ gives $0 = x(2x^2 - 9x + 12)$. Clearly, $x = 0$ is a solution, and using the quadratic formula on $2x^2 - 9x + 12 = 0$ gives no real roots. Thus, the only intercept is $(0, 0)$. In fact, since $2x^2 - 9x + 12$ is a continuous function whose value at 0 is $2 \cdot 0^2 - 9 \cdot 0 + 12 = 12 > 0$, we conclude that $2x^2 - 9x + 12 > 0$ for all x, which gives the sign chart in Figure 13.39 for y.

Note that this chart tells us the graph of $y = 2x^3 - 9x^2 + 12x$ is confined to the third and first quadrants of the xy-plane.

Symmetry None.

Maxima and Minima We have

$$y' = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

The critical values are $x = 1, 2$, so these and the factors $x - 1$ and $x - 2$ determine the sign chart of y' (Figure 13.40).

From the sign chart for y' we see that there is a relative maximum at 1 and a relative minimum at 2. Note too that the bottom line of Figure 13.40, together with that of Figure 13.39, comes close to determining a precise graph of $y = 2x^3 - 9x^2 + 12x$.

	$-\infty$		0		∞
x		−	0	+	
$2x^2 - 9x + 12$		+		+	
y		−	0	+	

FIGURE 13.39 Sign chart for y.

	$-\infty$		1		2		∞
$x - 1$		−	0	+		+	
$x - 2$		−		−	0	+	
y'		+	0	−	0	+	
y		↗		↘		↗	

FIGURE 13.40 Sign chart of $y' = 6(x - 1)(x - 2)$.

	$-\infty$	$3/2$	∞
$2x-3$	$-$	0	$+$
y''	$-$	0	$+$
y	$\cap$		$\cup$

FIGURE 13.41 Sign chart of y''.

Of course, it will help to know the relative maximum $y(1) = 5$, which occurs at 1, and the relative minimum $y(2) = 4$, which occurs at 2, so that in addition to the intercept $(0, 0)$ we will actually plot also $(1, 5)$ and $(2, 4)$.

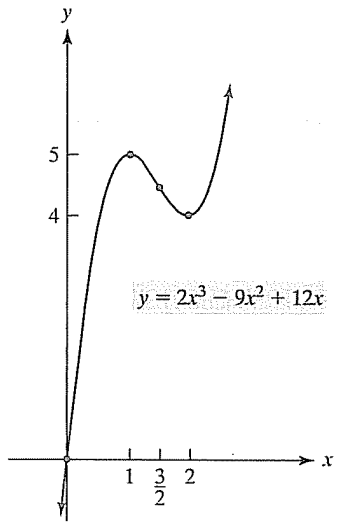


FIGURE 13.42 Graph of $y = 2x^3 - 9x^2 + 12x$.

Concavity

$$y'' = 12x - 18 = 6(2x - 3)$$

Setting $y'' = 0$ gives a possible inflection point at $x = \frac{3}{2}$ from which we construct the simple sign chart for y'' in Figure 13.41.

Since concavity changes at $x = \frac{3}{2}$, at which point f is certainly continuous, there is an inflection point at $\frac{3}{2}$.

Discussion We know the coordinates of three of the important points on the graph. The only other important point from our perspective is the inflection point, and since $y(3/2) = 2(3/2)^3 - 9(3/2)^2 + 12(3/2) = 9/2$ the inflection point is $(3/2, 9/2)$.

We plot the four points noted above and observe from all three sign charts jointly that the curve increases through the third quadrant and passes through $(0, 0)$, all the while concave down until a relative maximum is attained at $(1, 5)$. The curve then falls until it reaches a relative minimum at $(2, 4)$. However, along the way the concavity changes at $(3/2, 9/2)$ from concave down to concave up and remains so for the rest of the curve. After $(2, 4)$ the curve increases through the first quadrant. The curve is shown in Figure 13.42.

Now Work Problem 39 ◁

TECHNOLOGY

Suppose that you need to find the inflection points for

$$f(x) = \frac{1}{20}x^5 - \frac{17}{16}x^4 + \frac{273}{32}x^3 - \frac{4225}{128}x^2 + \frac{750}{4}$$

The second derivative of f is given by

$$f''(x) = x^3 - \frac{51}{4}x^2 + \frac{819}{16}x - \frac{4225}{64}$$

Here the roots of $f'' = 0$ are not obvious. Thus, we will graph f'' using a graphing calculator. (See Figure 13.43.) We find that the roots of $f'' = 0$ are approximately 3.25 and 6.25. Around $x = 6.25, f''(x)$ goes from negative to positive values. Therefore, at $x = 6.25$, there is an inflection point. Around $x = 3.25, f''(x)$ does not change sign, so no inflection point exists at $x = 3.25$. Comparing our results with the graph of f in Figure 13.44, we see that everything checks out.

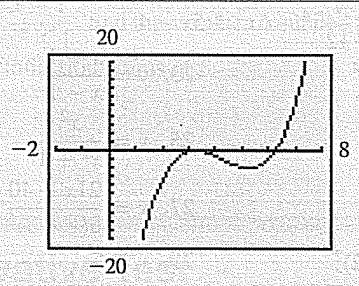


FIGURE 13.43 Graph of f''; roots of $f'' = 0$ are approximately 3.25 and 6.25.

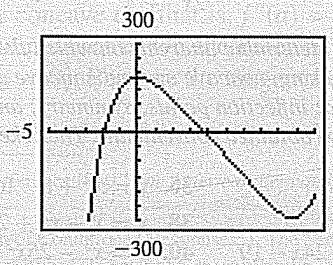


FIGURE 13.44 Graph of f; inflection point at $x = 6.25$, but not at $x = 3.25$.