BME2301 - Circuit Theory

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RC and RL Circuits

First Order Circuits

Objectives of Lecture

- Explain the operation of a RC circuit in dc circuits
  - As the capacitor stores energy when voltage is first applied to the circuit or the voltage applied across the capacitor is increased during the circuit operation.
  - As the capacitor releases energy when voltage is removed from the circuit or the voltage applied across the capacitor is decreased during the circuit operation.
- Explain the operation of a RL circuit in dc circuit
  - As the inductor stores energy when current begins to flow in the circuit or the current flowing through the inductor is increased during the circuit operation.
  - As the inductor releases energy when current stops flowing in the circuit or the current flowing through the inductor is decreased during the circuit operation.

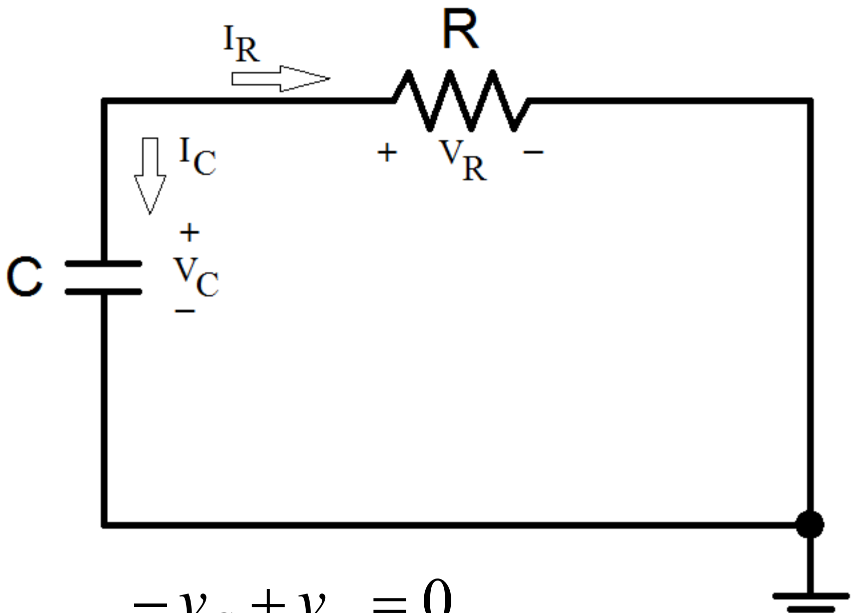
Natural Response (The Source-Free Response)

- The behavior of the circuit with no external sources of excitation.
  - There is stored energy in the capacitor or inductor at $time = 0$ s.
  - For $t > 0$ s, the stored energy is released
    - Current flows through the circuit and voltages exist across components in the circuit as the stored energy is released.
    - The stored energy will decay to zero as time approaches infinite, at which point the currents and voltages in the circuit become zero.

RC Circuit – Natural Response

- Suppose there is some charge on a capacitor at time $t = 0$ s.
  - This charge could have been stored because a voltage or current source had been in the circuit at $t < 0$ s, but was switched off at $t = 0$ s.
- We can use the equations relating voltage and current to determine how the charge on the capacitor is removed as a function of time.
  - The charge flows from one plate of the capacitor through the resistor R to the other plate to neutralize the charge on the opposite plate of the capacitor.

Equations for the Natural Response of RC Circuit



$$-v_C + v_R = 0$$

$$i_C = -i_R$$

$$i_C = C\frac{dv_C}{dt}$$

$$i_R = \frac{v_R}{R}$$

$$C\frac{dV_C}{dt} + \frac{V_R}{R} = 0$$

$$V_R = V_C$$

$$\frac{dV_C}{dt} + \frac{V_C}{RC} = 0$$

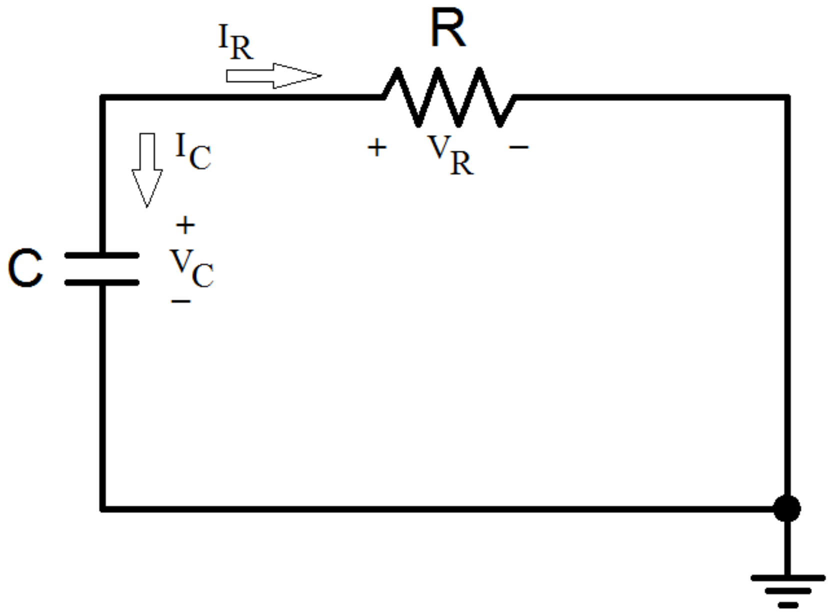
$$V_C = Ae^{st}$$

$$Ase^{st} + \frac{1}{RC}Ae^{st} = 0$$

$$Ae^{st}\left(s + \frac{1}{RC}\right) = 0$$

$$s = -\frac{1}{RC}$$

Equations for the Natural Response of RC Circuit



If $V_o = V_C\big|_{t=0s}$ and $\tau = RC$

$$V_C(t) = V_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$I_R(t) = -I_C(t) = \frac{V_o}{R} e^{-\frac{t}{\tau}}$$

$$p_R(t) = V_R I_R = \frac{V_o^2}{R} e^{-\frac{2t}{\tau}}$$

Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the capacitor.

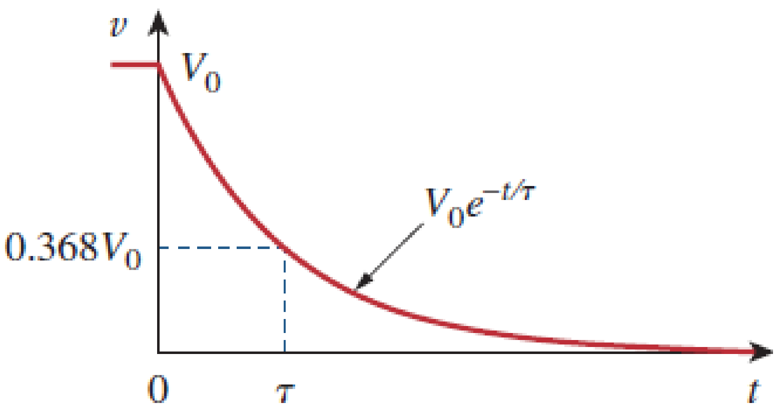
$$w(t) = \int_{0s}^{t} p_R(t) dt = \frac{CV_o^2}{2}\left[1 - e^{-\frac{2t}{\tau}}\right]$$

The Key to Working with a Source-Free RC Circuit Is Finding:

- The initial voltage $v(0) = V_0$ across the capacitor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least $t = -5\tau$) and then removing it at $t = 0$.
    - Capacitor
      - Open Circuit Voltage
- The time constant τ.
  - In finding the time constant $\tau = RC$, R is often the Thevenin equivalent resistance at the terminals of the capacitor;
    - that is, we take out the capacitor C and find $R = R_{Th}$ at its terminals

Time constant - τ

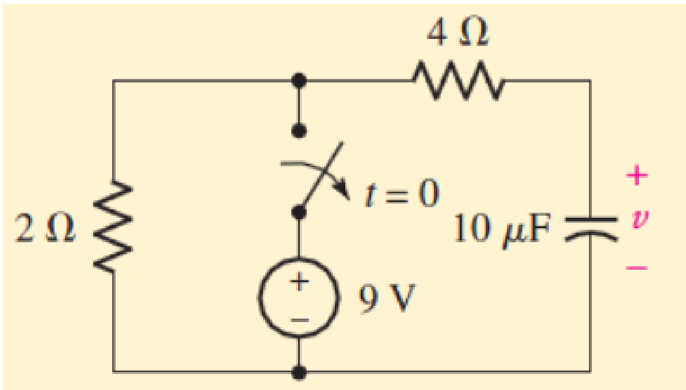
- The natural response of a capacitive circuit refers to the behavior (in terms of voltages) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the capacitor.
- The voltage response of the *RC* circuit

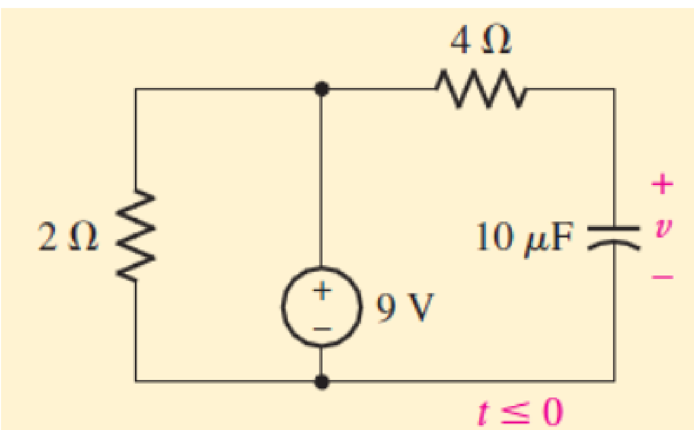


  - Time constant, $\tau = RC$
    - The time required for the voltage across the capacitor to decay by a factor of 1/e or 36.8% of its initial value.

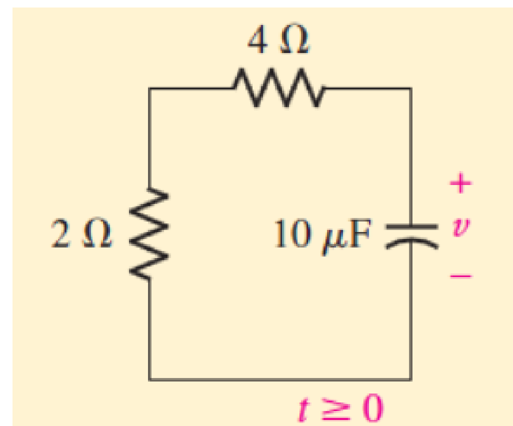
Example 1

For the circuit, find the voltage labeled *v* at t 200 μs



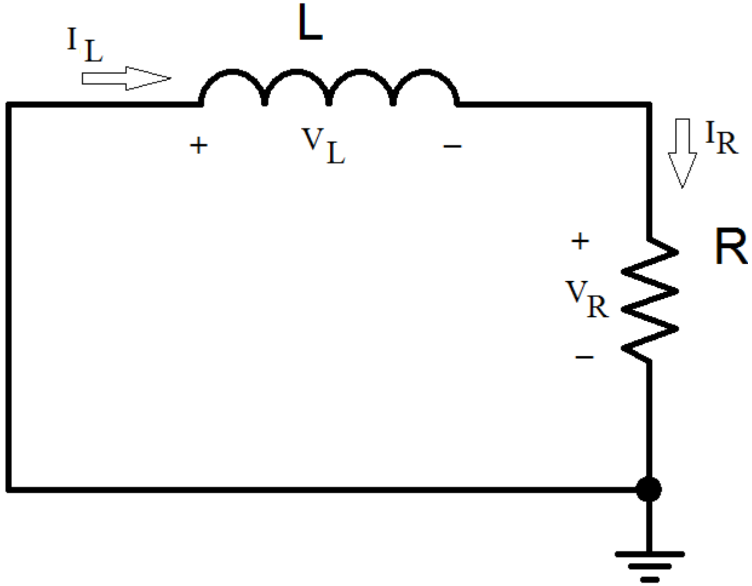


before the switch is thrown



after the switch is thrown

Equations for the Natural Response of RL Circuit



$$V_L + V_R = 0$$
$$I_L = I_R$$

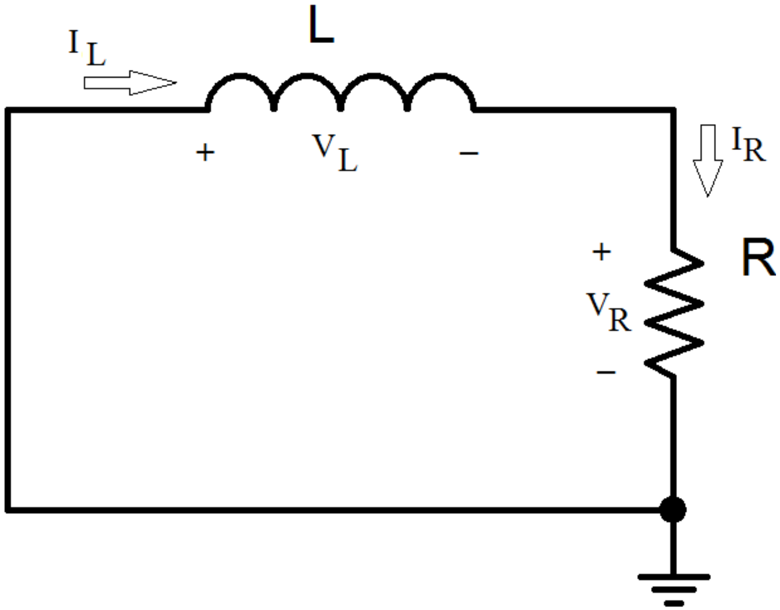
$$V_L = L\frac{dI_L}{dt}$$
$$I_R = V_R / R$$

$$L\frac{dI_L}{dt} + RI_R = 0$$

$$\frac{dI_L}{dt} + \frac{RI_L}{L} = 0$$

$$I_L = Ae^{st}$$
$$Ase^{st} + \frac{R}{L}Ae^{st} = 0$$
$$Ae^{st}\left(s + \frac{R}{L}\right) = 0$$
$$s = -\frac{R}{L}$$

Equations for the Natural Response of RL Circuit



If $I_o = I_L\big|_{t=0s}$ and $\tau = \dfrac{L}{R}$

$$I_L(t) = I_o e^{-\frac{t}{\tau}} \quad \text{when } t \geq 0s$$

$$V_R(t) = -V_L(t) = RI_o e^{-\frac{t}{\tau}}$$

$$p_R(t) = V_R I_R = RI_o^{\,2} e^{-\frac{2t}{\tau}}$$

$$w(t) = \int_{0s}^{t} p_R(t)dt = \frac{LI_o^{\,2}}{2}\left[1 - e^{-\frac{2t}{\tau}}\right]$$

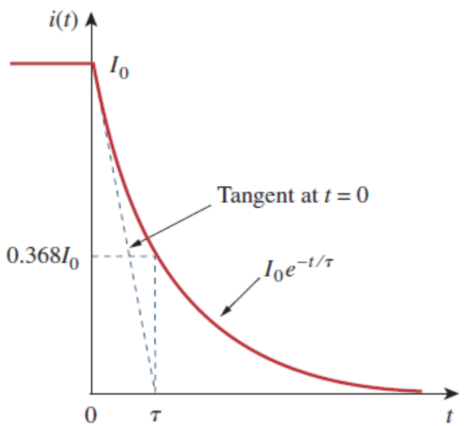
Since the voltages are equal and the currents have the opposite sign, the power that is dissipated by the resistor is the power that is being released by the inductor.

The Key to Working with a Source-Free RL Circuit Is Finding:

- The initial current $i(0) = I_0$ through the inductor.
  - Can be obtained by inserting a d.c. source to the circuit for a time much longer than τ (at least t = -5τ) and then removing it at $t = 0$.
    - Inductor
      - Short Circuit Current
- The time constant τ.
  - In finding the time constant $\tau = L/R$, R is often the Thevenin equivalent resistance at the terminals of the inductor;
    - that is, we take out the inductor L and find $R = R_{Th}$ at its terminals

Time constant - τ

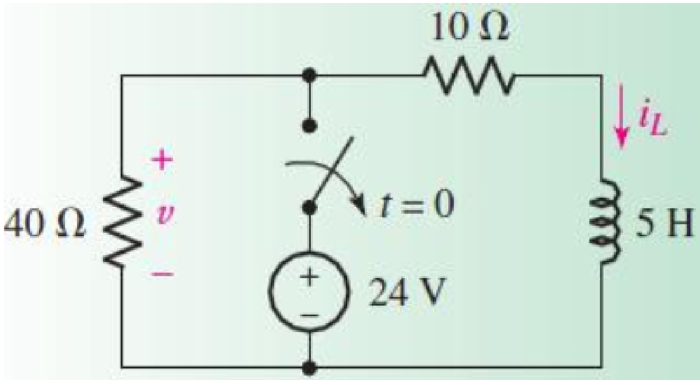
- The natural response of an inductive circuit refers to the behavior (in terms of currents) of the circuit itself, with no external sources of excitation.
  - The natural response depends on the nature of the circuit alone, with no external sources.
    - In fact, the circuit has a response only because of the energy initially stored in the inductor.
- The current response of the *RL* circuit

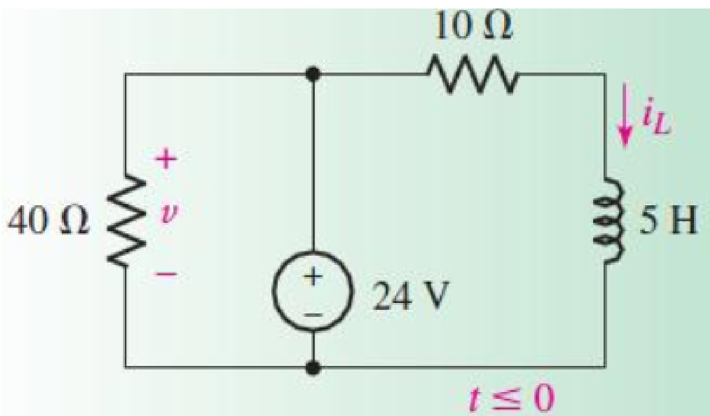


  - Time constant, $\tau = L/R$
    - The time required for the current in the inductor to decay by a factor of 1/e or 36.8% of its initial value.

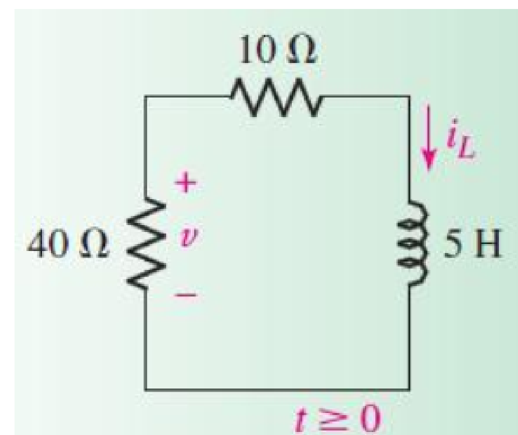
Example 2

For the circuit, *find the voltage labeled v at t 200 ms.*





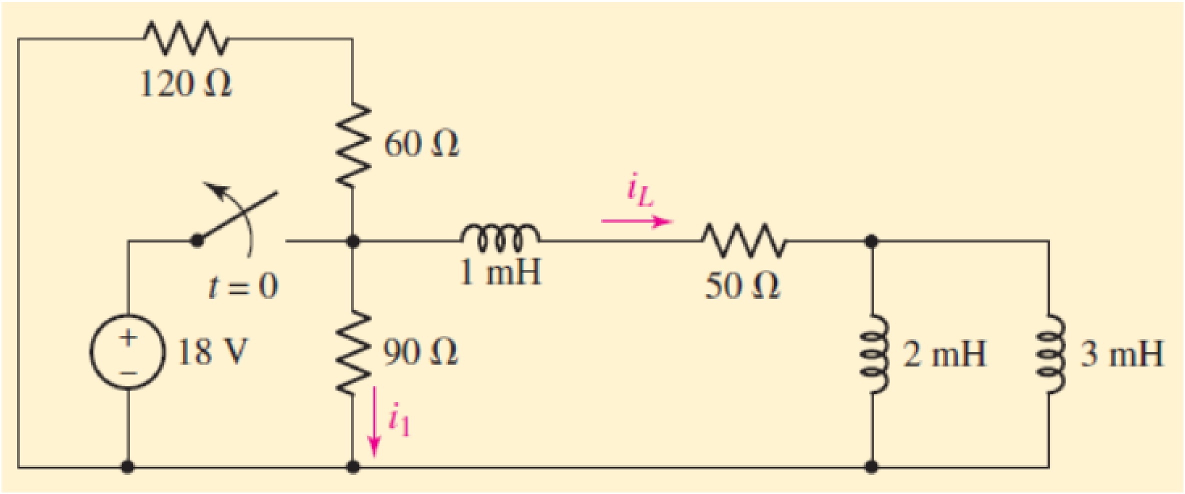
before the switch is thrown

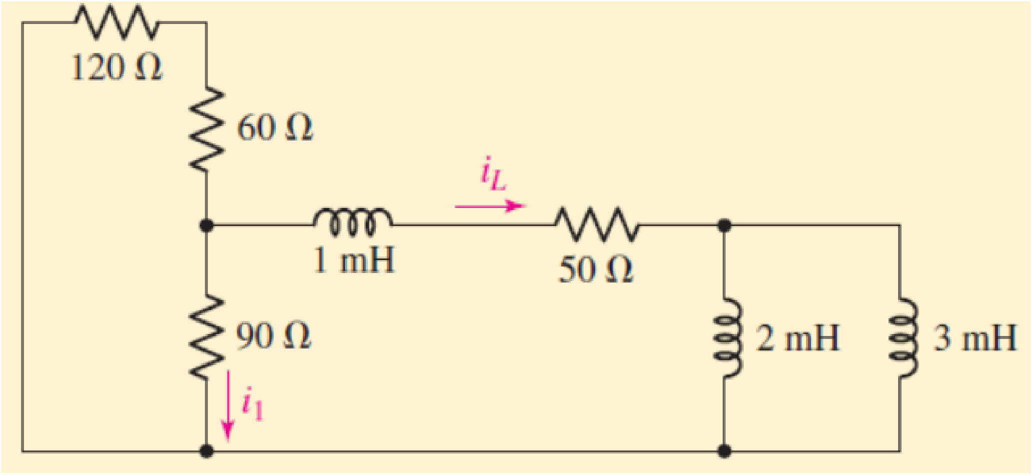


after the switch is thrown

Example 3

Determine both i_1 and i_L in the circuit for t > 0.

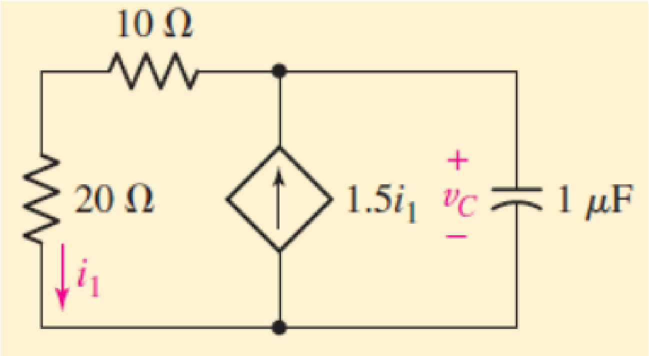


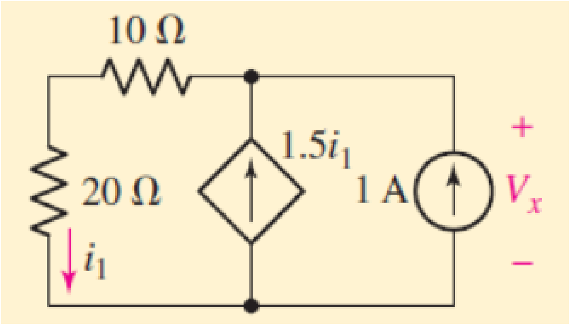


after the switch is thrown

Example 4

For the circuit, find the voltage labeled v_C for $t > 0$ if $v_C(0^-) = 2$ V.





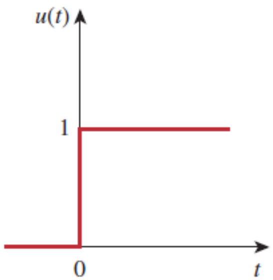
Circuit for finding the Thévenin equivalent of the network connected to the capacitor

Singularity Functions

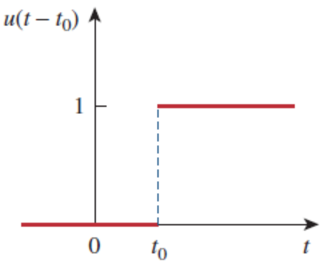
- Singularity functions (also called switching functions) are very useful in circuit analysis.
- They serve as good approximations to the switching signals that arise in circuits with switching operations.
- They are helpful in the neat, compact description of some circuit phenomena,
  - especially the step response of *RC* or *RL* circuits
- Singularity functions are functions that either are discontinuous or have discontinuous derivatives.

Unit Step Function

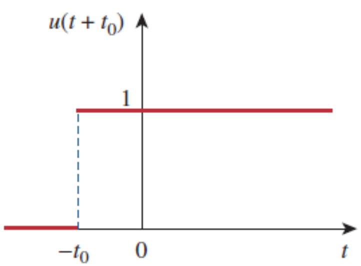
- The unit step function ($u(t)$) is 0 for negative values of t and 1 for positive values of t.



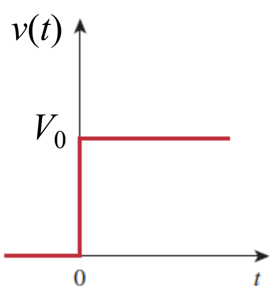
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$



$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

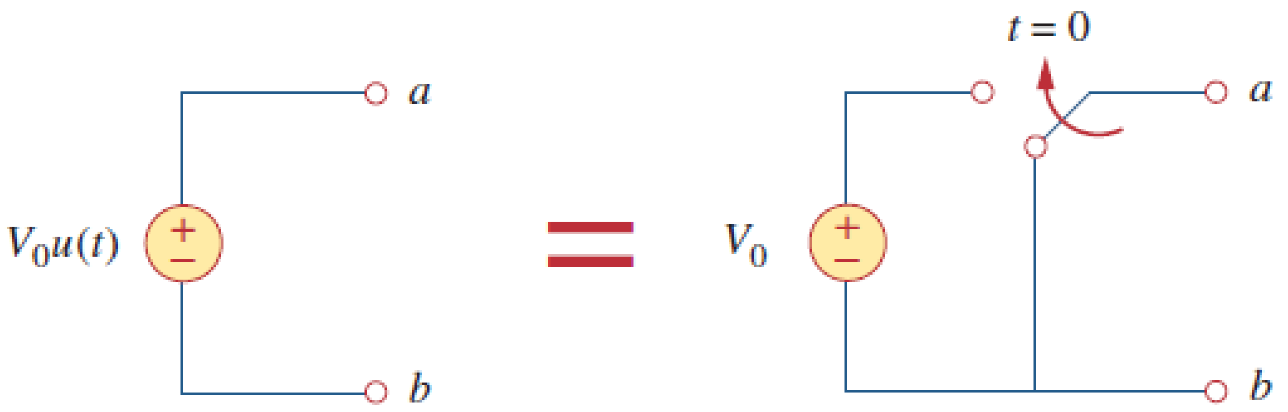


$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

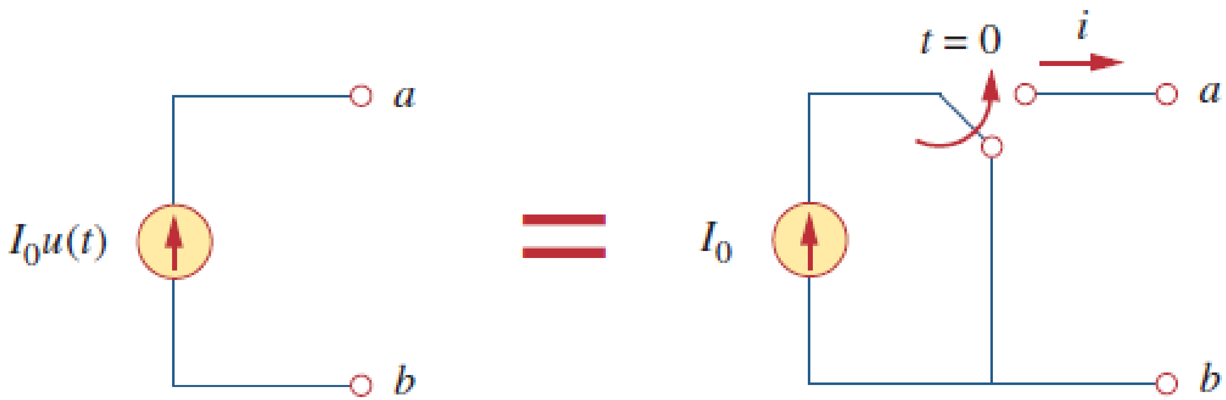
$$v(t) = V_0 u(t - t_0)$$

Unit Step Function

- Voltage source of $V_0u(t)$ and its equivalent circuit.

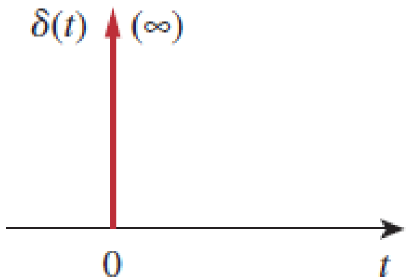


- Current source of $I_0u(t)$ and its equivalent circuit.



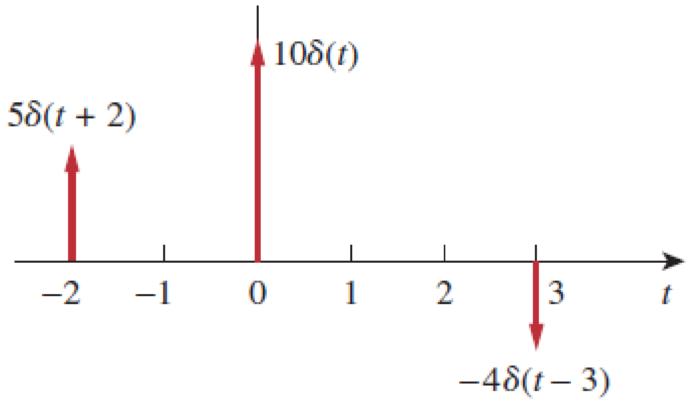
Unit Impulse Function

- The derivative of the unit step function $u(t)$ is the unit impulse function ($\delta(t)$)



$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0, & t < 0 \\ \text{Undefined}, & t = 0 \\ 0, & t > 0 \end{cases}$$

$$\int_{0^-}^{0^+} \delta(t)\, dt = 1$$



$x(t) = 5\delta(t+2) + 10\delta(t) - 4\delta(t-3)$

Integration of Unit Functions

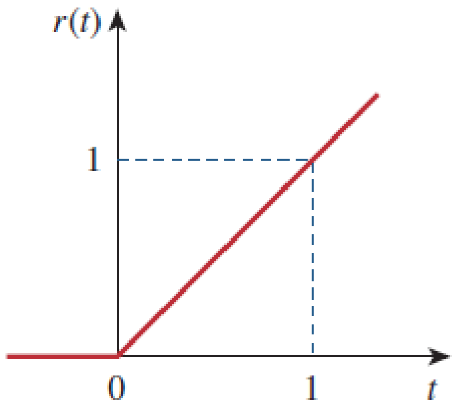
- To illustrate how the impulse function affects other functions, let us evaluate the integral

$$\int_a^b f(t)\delta(t - t_0)\,dt$$

$$\int_a^b f(t)\delta(t - t_0)\,dt = \int_a^b f(t_0)\delta(t - t_0)\,dt$$
$$= f(t_0)\int_a^b \delta(t - t_0)\,dt = f(t_0)$$

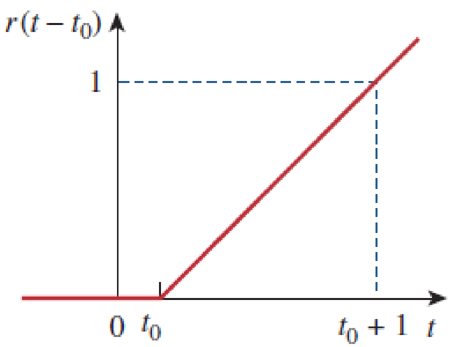
- This is a highly useful property of the impulse function known as the sampling or shifting property.

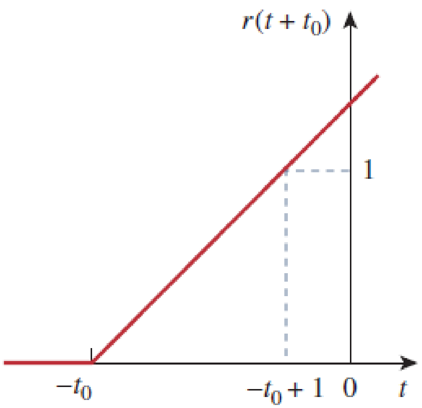
Unit Ramp Function



$$r(t) = \int_{-\infty}^{t} u(\lambda)d\lambda = tu(t)$$

$$\boxed{r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}}$$





$$r(t - t_0) = \begin{cases} 0, & t \leq t_0 \\ t - t_0, & t \geq t_0 \end{cases}$$

$$r(t + t_0) = \begin{cases} 0, & t \leq -t_0 \\ t + t_0, & t \geq -t_0 \end{cases}$$

Relationships of singularity functions

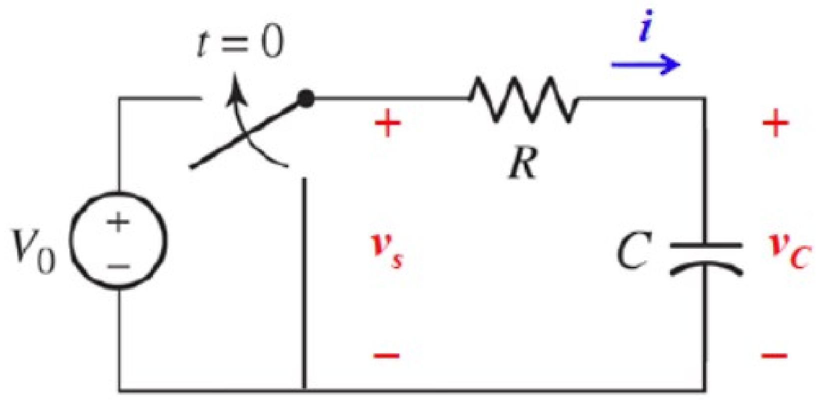
- The three singularity functions (impulse, step, and ramp) are related by differentiation as

$$\delta(t) = \frac{du(t)}{dt}, \qquad u(t) = \frac{dr(t)}{dt}$$

- or by integration as

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\,d\lambda, \qquad r(t) = \int_{-\infty}^{t} u(\lambda)\,d\lambda$$

Driven RC Circuit



Initially, (a) the source is turned off, and
(b) no energy is stored.

$$v_s = 0 \quad , \quad v_C\left(0^-\right) = 0$$

just before $t = 0$

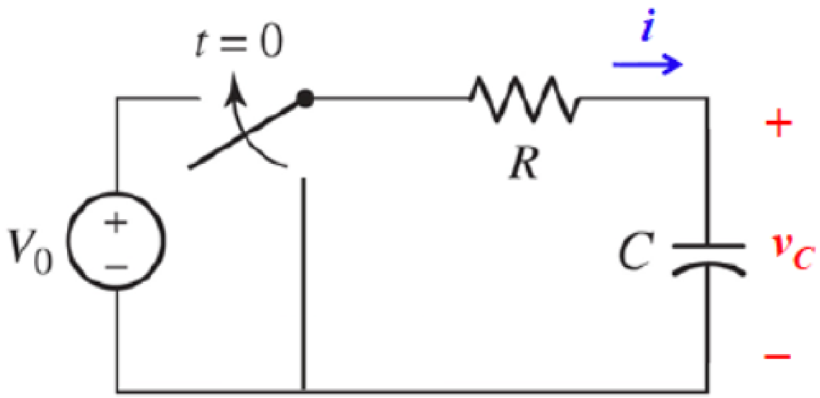
After the switch changes, $v_s = V_0 \quad , \quad v_C\left(0^+\right) = 0$

just after $t = 0$

The capacitor has not yet had time to build up voltage: $v_C(t) = \dfrac{1}{C}\int_0^t i_C(\tau) \cdot d\tau$

A long time after the switch has been closed, $v_C(\infty) = V_0$ (capacitor acts as an *open circuit*)

Driven RC Circuit



$$v_C(0) = 0$$

$$v_C(\infty) = V_0$$

$$i_C = C\frac{dv_C}{dt}$$

Writing KVL after the switch changes,

$$-V_0 + iR + v_C = 0 \quad \Rightarrow \quad -V_0 + C\frac{dv_C}{dt}\cdot R + v_C = 0 \quad \Rightarrow \quad RC\frac{dv_C}{dt} + v_C = V_0$$

$$\frac{d}{dt}v_C(t) + \frac{1}{RC}\cdot v_C(t) = \frac{V_0}{RC}$$

→ homogeneous, first-order differential equation

Solution of a First Order System

- **General form of differential equation:**

$$a_1 \frac{dy(t)}{dt} + a_0 y(t) = u(t)$$

- **Initial condition:**

$$y(t = t_0) = y_0$$

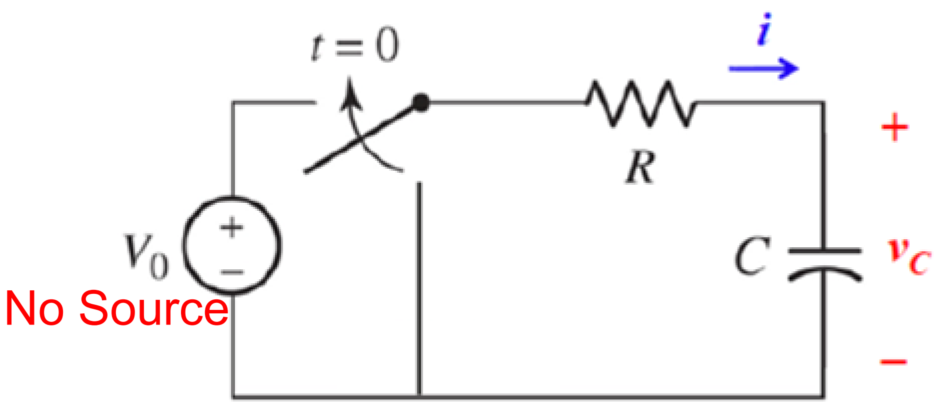
- **Solution is of the form:**

$$y(t) = y_h(t) + y_p(t)$$

- **$y_h(t)$ is *homogeneous* solution**
  - **Due to the system's response to initial conditions**
- **$y_p(t)$ is the *particular* solution**
  - **Due to the particular forcing function, *u(t)*, applied to the system**

Homogeneous Solution

- The <u>homogeneous solution</u> is the system's response to its *initial conditions* <u>only</u>
  - System response if no input is applied $\Rightarrow$ *u(t) = 0*
  - Also called the *unforced response*, *natural response*, or *zero input response*
  - All physical systems dissipate energy $\Rightarrow$ $y_h(t) \to 0$ as $t \to \infty$



$$\frac{dV_C(t)}{dt} + \frac{1}{RC}V_C(t) = 0$$

$$V_{CH}(t) = Ae^{st}$$

$$Ase^{st} + \frac{1}{RC}Ae^{st} = 0$$

$$Ae^{st}\left(s + \frac{1}{RC}\right) = 0$$

$$s = -\frac{1}{RC}$$

$$V_{CH}(t) = Ae^{-\frac{1}{RC}t}$$

Particular Solution

- **The particular solution is the system's response to the *input* only**
  - **The form of the particular solution is dictated by the form of the forcing function applied to the system**
  - **Also called the *forced response* or *zero state response***
  - **Since $y_h(t) \to 0$ as $t \to \infty$, *and* $y(t) = y_p(t) + y_h(t)$:**
    - **$y(t) \to y_p(t)$ as $t \to \infty$**

We know that $V_C(\infty) = V_0$ **so;** $V_{CP}(t) = V_0$

Then the total solution will be;

$$V_C(t) = V_{CH}(t) + V_{CP}(t)$$

$$V_C(t) = Ae^{-\frac{1}{RC}t} + V_0$$

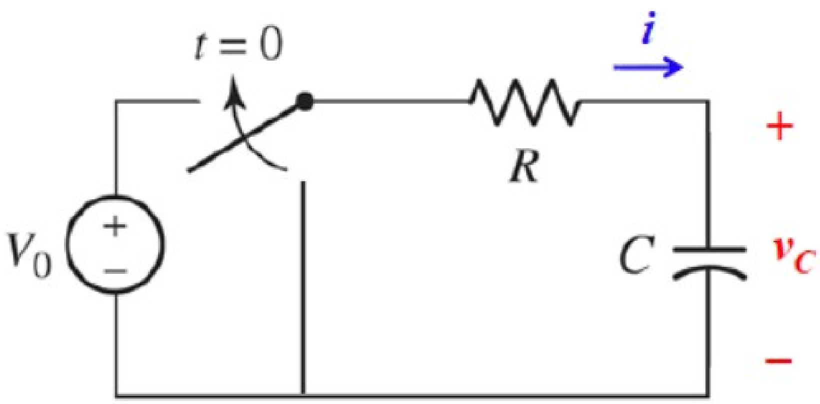
The initial condition is zero;

$$t \to 0$$

$$V_C(0) = 0 = Ae^0 + V_0$$

$$A = -V_0 \to V_C(t) = V_0 - V_0 e^{-\frac{1}{RC}t} \qquad for\ t > 0$$

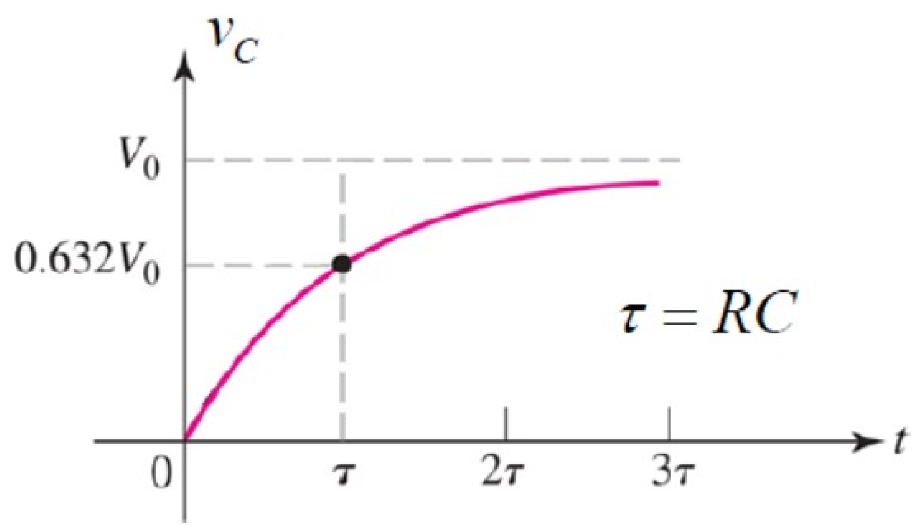
Total Solution of Driven RC circuit



$$\frac{d}{dt}v_C(t)+\frac{1}{RC}\cdot v_C(t)=\frac{V_0}{RC}$$

$$v_C(0)=0 \qquad v_C(\infty)=V_0$$

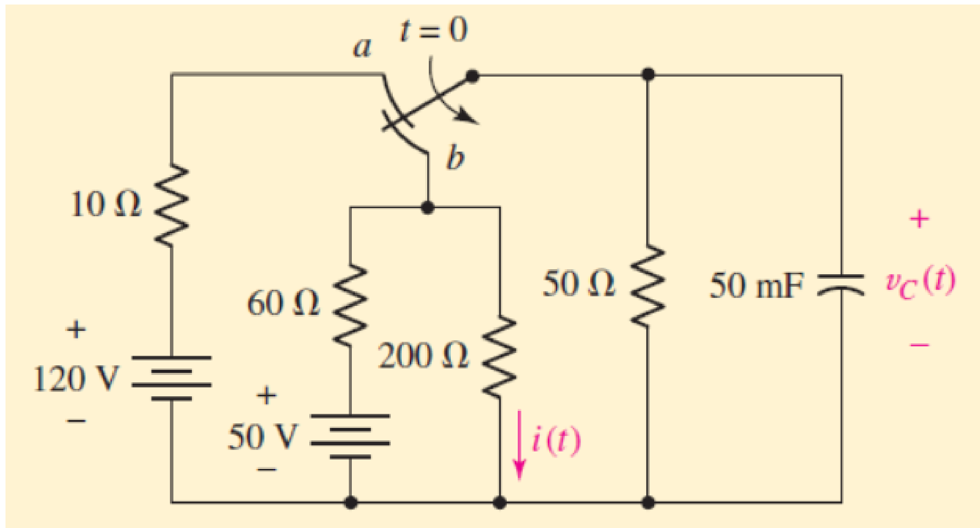
$$\boxed{v_C(t)=V_0\left(1-e^{-t/RC}\right)}$$

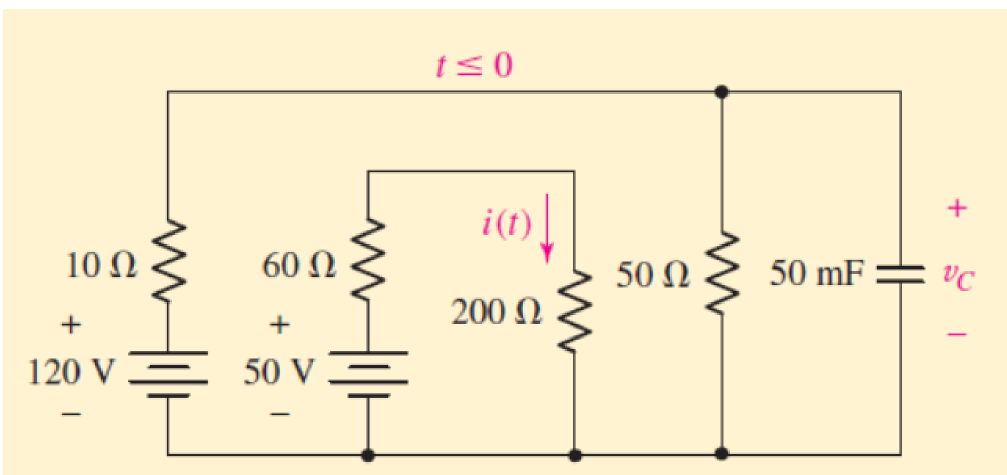


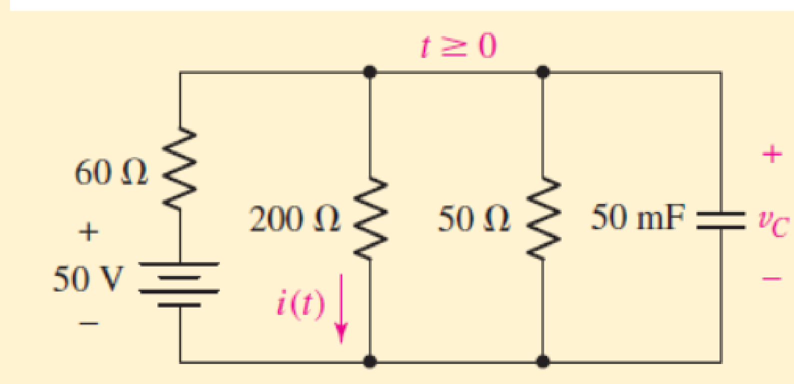
τ = **time constant** of the circuit,
a measure of how much time
is required to charge/discharge

Example 5

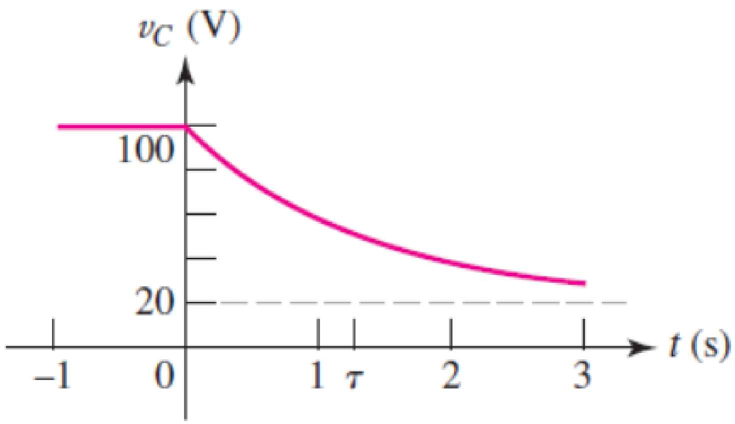
Find the capacitor voltage $v_C(t)$ and the current $i(t)$ in the 200 Ω resistor

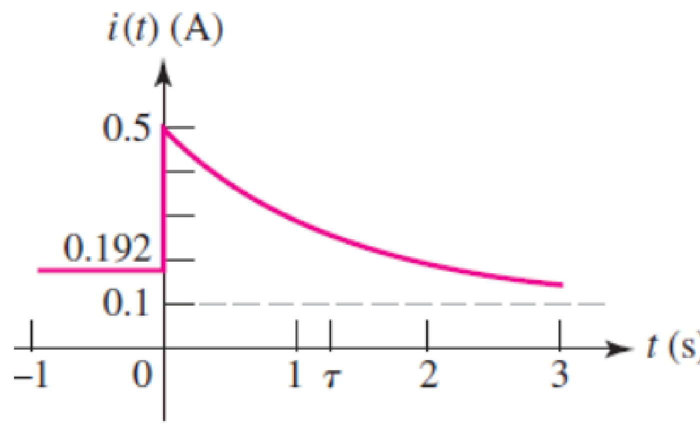






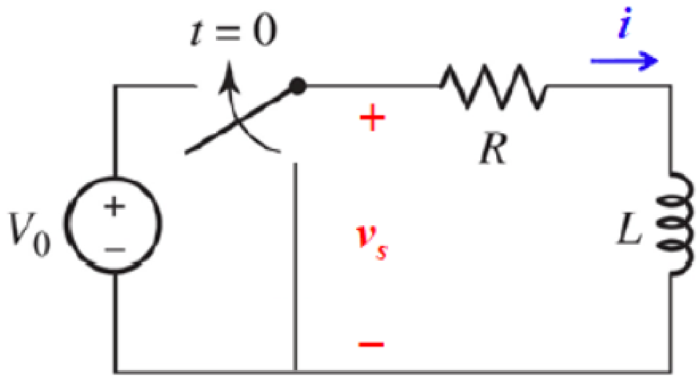
…Example 5





four numbers are needed to write the functional form of the response for this single energy-storage-element circuit, or to prepare the sketch: the constant value prior to switching (0.1923 ampere), the instantaneous value just after switching (0.5 ampere), the constant forced response (0.1 ampere), and the time constant (1.2 s). The appropriate negative exponential function is then easily written or drawn.

Driven RL Circuit



Initially, (a) the source is turned off, and
(b) no energy is stored, so
(c) no current flows.

$$v_s = 0 \quad , \quad i\left(0^-\right) = 0$$

just before $t = 0$

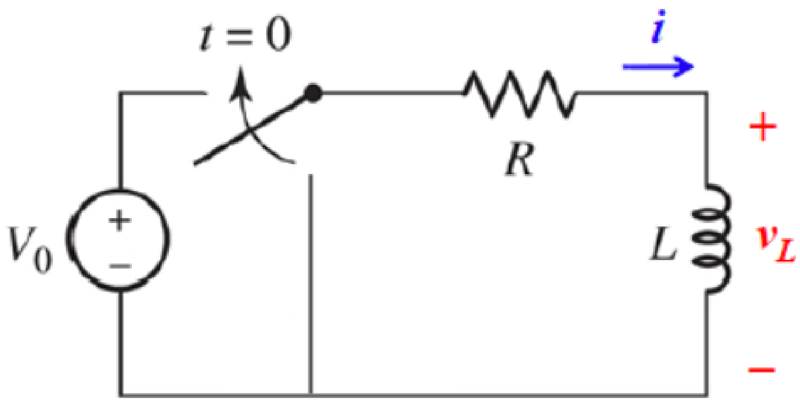
After the switch changes, $v_s = V_0 \quad , \quad i\left(0^+\right) = 0$

just after $t = 0$

The inductor has not yet had time to build up current: $i_L\left(t\right) = \dfrac{1}{L}\int_0^t v_L\left(\tau\right)\cdot d\tau$

A long time after the switch has been closed, $i\left(\infty\right) = V_0 / R$ (inductor acts as a *short circuit*)

Driven RL Circuit



$$i(0)=0$$

$$i(\infty)=V_0/R$$

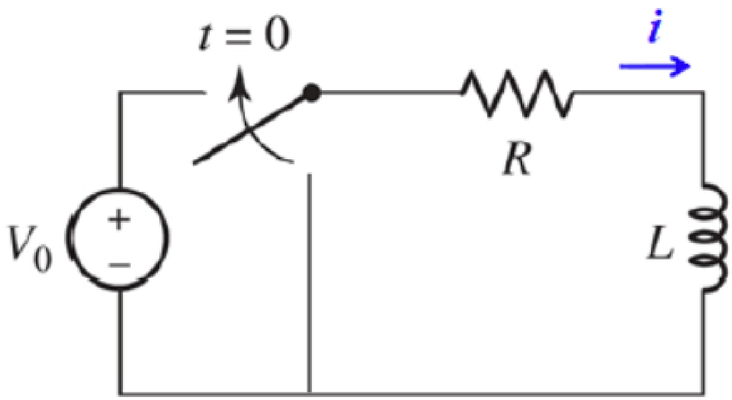
$$v_L=L\frac{di}{dt}$$

Writing KVL after the switch changes,

$$-V_0+iR+v_L=0 \quad \Rightarrow \quad v_L+R\cdot i=V_0 \quad \Rightarrow \quad L\cdot\frac{d}{dt}i(t)+R\cdot i(t)=V_0$$

$$\frac{d}{dt}i(t)+\frac{R}{L}\cdot i(t)=\frac{V_0}{L}$$ → homogeneous, first-order differential equation

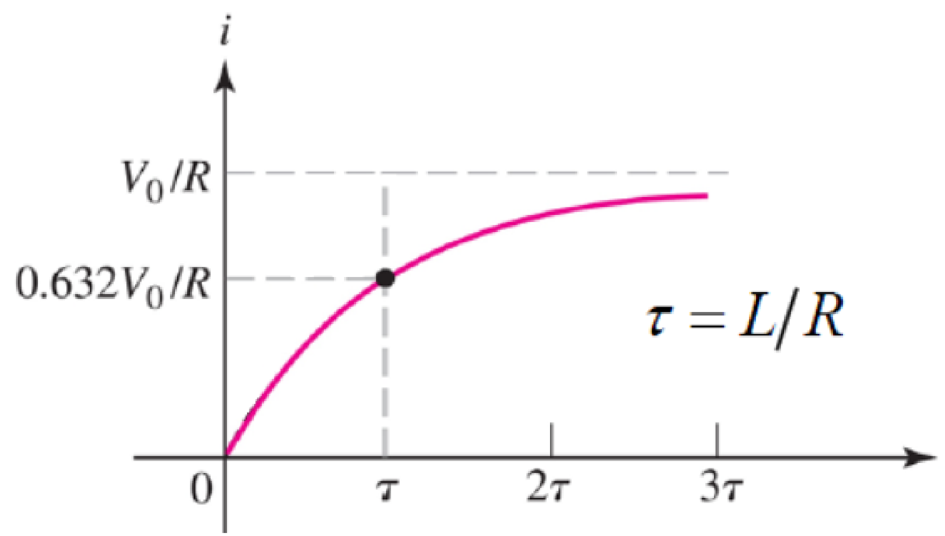
Driven RL Circuit



$$\frac{d}{dt} i(t) + \frac{R}{L} \cdot i(t) = \frac{V_0}{L}$$

$$i(0) = 0 \qquad i(\infty) = V_0 / R$$

$$\boxed{i(t) = \frac{V_0}{R}\left(1 - e^{-t \cdot R/L}\right)}$$

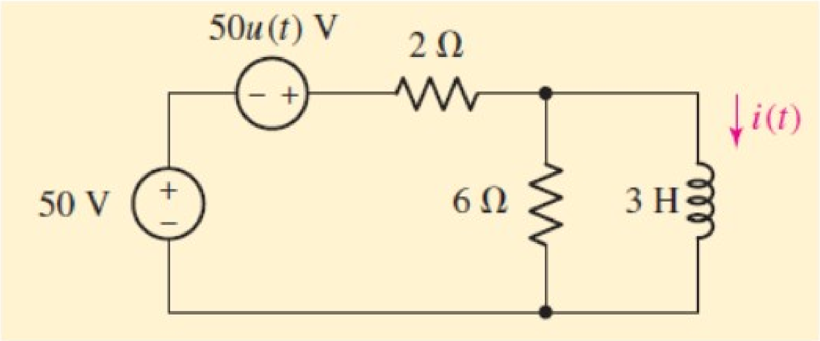


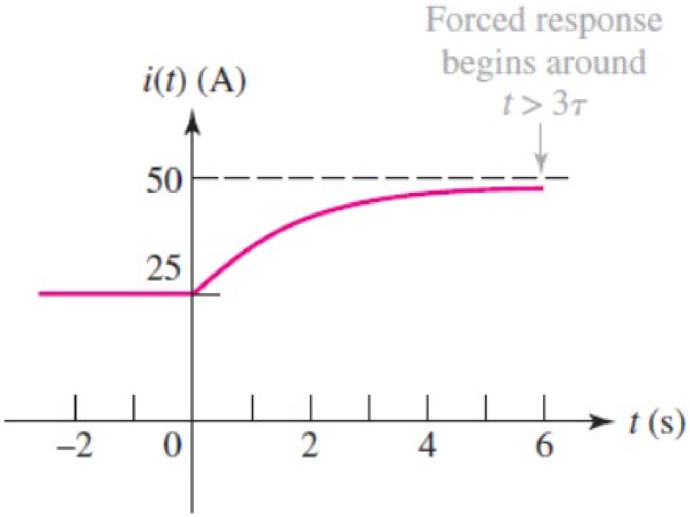
$$\tau = L/R$$

τ = **time constant** of the circuit,
a measure of how much time
is required to charge/discharge

Example 6

Determine *i*(t) for all values of time in the circuit

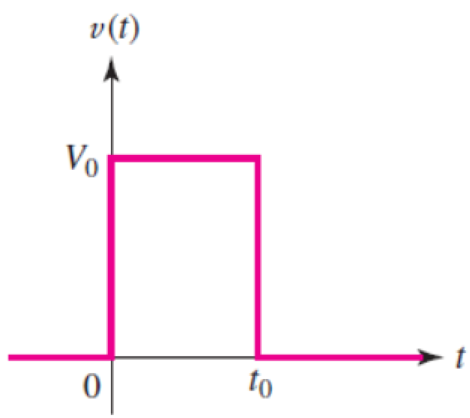




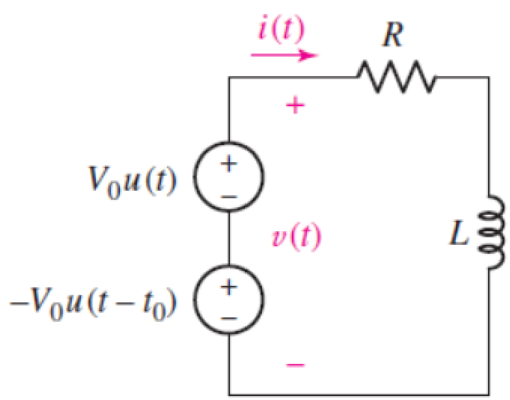
The response *i(t) of the circuit*

Example 7

Find the current response in a simple series *RL* circuit when the forcing function is a rectangular voltage pulse of amplitude V_0 and duration t_0.

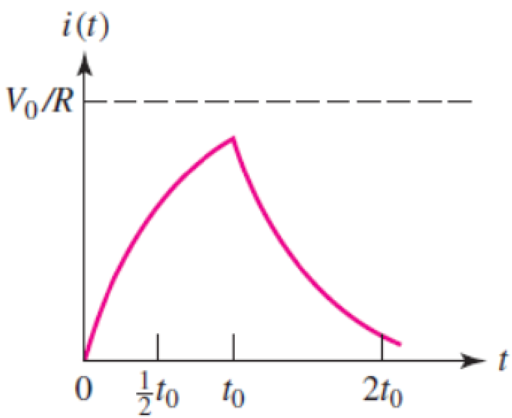


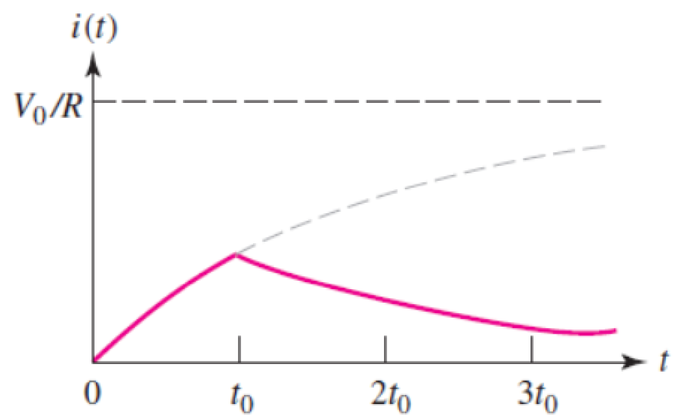
A rectangular voltage pulse which is to be used as the forcing function in a simple series *RL* circuit.



The series *RL circuit, showing the* representation of the forcing function by the series combination of two independent voltage-step sources.

…Example 7





the time constant is only one half as large as the length of the applied pulse; the rising portion of the exponential has therefore almost reached V_0/R before the decaying exponential begins. ($\tau = t_o/2$)

the time constant is twice t_0 and the response never has a chance to reach the larger amplitudes. ($\tau = 2t_o$)

General Equations

- When a voltage or current source changes its magnitude at t = 0s in a simple RC or RL circuit.
  - Equations for a simple RC circuit

$$V_C(t) = V_C(\infty) + \left[V_C(0) - V_C(\infty)\right]e^{-t/\tau}$$

$$I_C(t) = \frac{C}{\tau}\left[V_C(\infty) - V_C(0)\right]e^{-t/\tau}$$

$$\tau = RC$$

  - Equations for a simple RL circuit

$$I_L(t) = I_L(\infty) + \left[I_L(0) - I_L(\infty)\right]e^{-t/\tau}$$

$$V_L(t) = \frac{L}{\tau}\left[I_L(\infty) - I_L(0)\right]e^{-t/\tau}$$

$$\tau = L / R$$

General Solution Procedure

(1) Identify the circuit as a transient *RL* or *RC* circuit. (e.g. *not* an *RLC* circuit)

(2) Assume that the circuit response (v or i) will follow the general solution:

$$x(t) = x(\infty) + [x(0) - x(\infty)]e^{-\frac{t}{\tau}}$$

(3) If necessary, determine equivalent inductance (L_{eq}) or capacitance (C_{eq}) by appropriate series/parallel combinations.

(4) Solve for the initial inductor current $i(0)$ or capacitor voltage $v(0)$.

(5) Solve for the final inductor current $i(\infty)$ or capacitor voltage $v(\infty)$.

(6) Determine the Thevenin equivalent resistance seen by the energy-storage element (inductor or capacitor), R_{th} . Compute $\tau = L/R_{th}$ or $\tau = R_{th}C$.

(7) Substitute results from steps 4, 5, 6 into the general solution (step 2).

(8) To solve for v/i that is <u>not</u> for the inductor/capacitor, use circuit analysis (KVL, KCL, etc.) to solve for that particular v/i .