BME2312 - Analog Electronics

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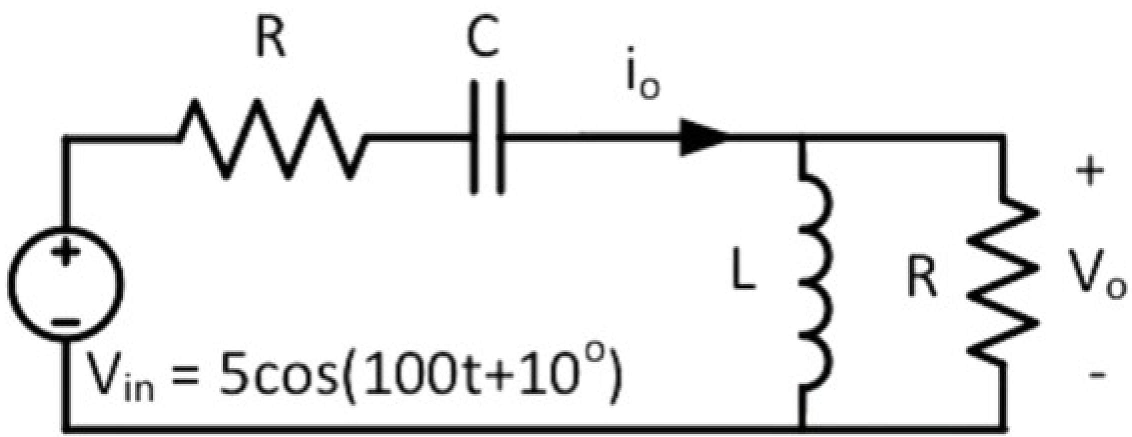
LECTURE 2

Transfer Function
Frequency Response
Bode Plots

Transfer Function

Objective:

- Introduce transfer functions to find an input-to-output relationship for a circuit that holds for all input frequencies



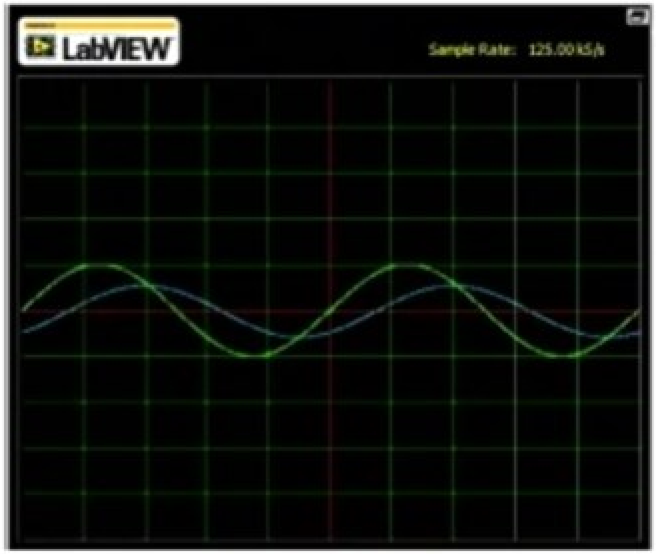
Transfer Function

Builds Upon:

- Phasors
- Impedance Method

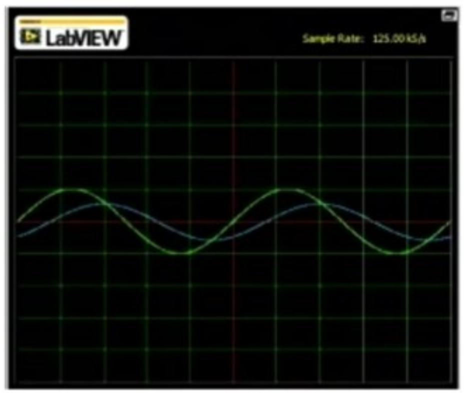
$$Z_R = R \qquad Z_c = \frac{1}{j\omega C} \qquad Z_L = j\omega L$$

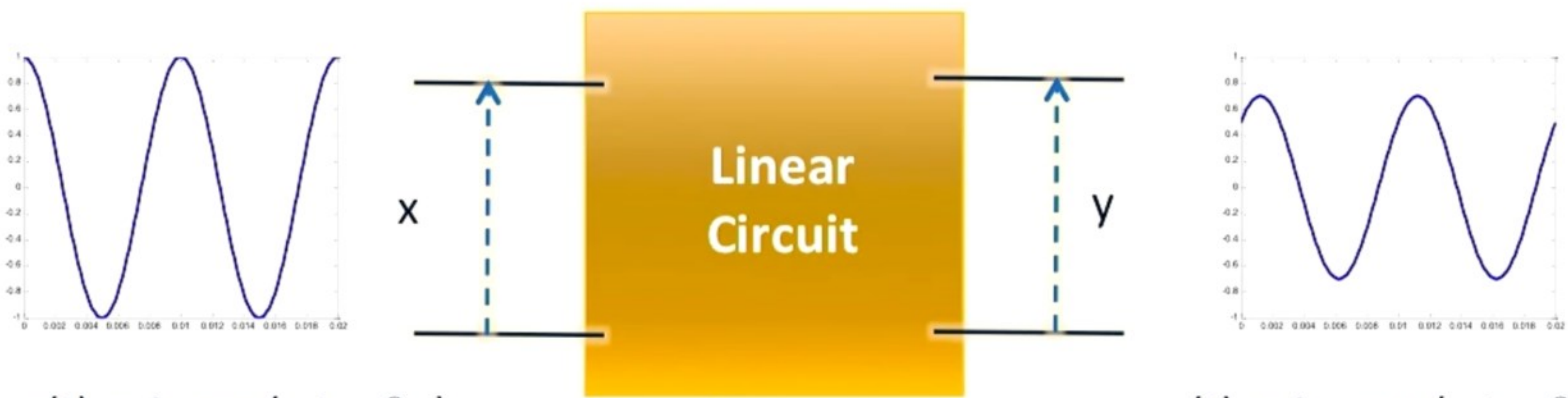
Input-Output relation of Circuits



Transfer Function

Behavior of Sinusoids in Linear Circuits



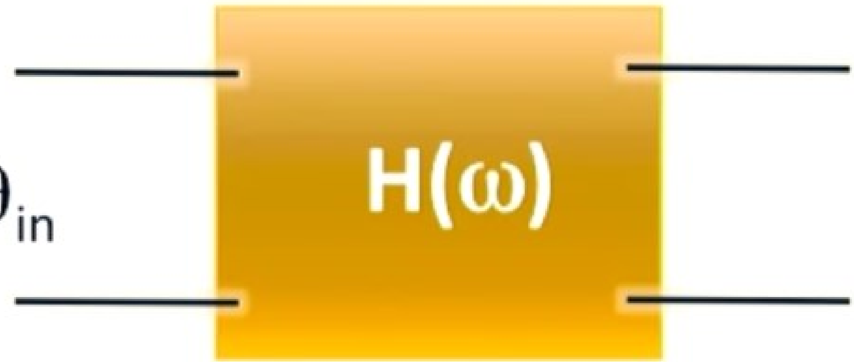


$x(t) = A_{in}\cos(\omega t + \theta_{in})$ $y(t) = A_{out}\cos(\omega t + \theta_{out})$

Transfer Function

Transfer function $H(\omega)$: Ratio of output phasor to input phasor

$\mathbf{X} = A_{in}\angle\theta_{in}$

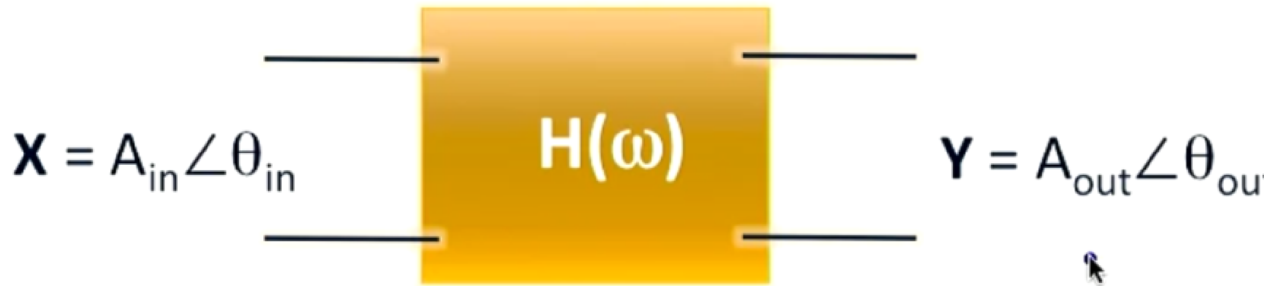


$\mathbf{Y} = A_{out}\angle\theta_{out}$

$$H(\omega) = \frac{\mathbf{Y}}{\mathbf{X}} = \frac{A_{out}\angle\theta_{out}}{A_{in}\angle\theta_{in}}$$

Transfer Function

Transfer Function

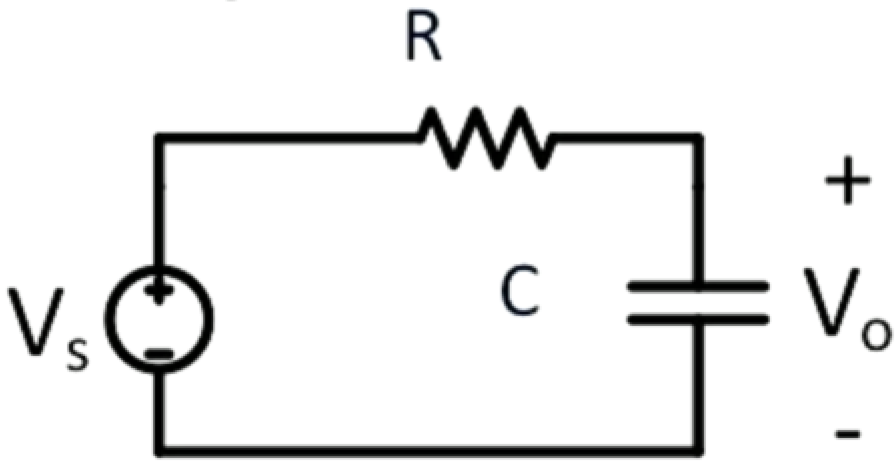


$$H(\omega) = \frac{A_{out}\angle\theta_{out}}{A_{in}\angle\theta_{in}} = \frac{A_{out}}{A_{in}}\angle\theta_{out} - \theta_{in}$$

$$\boxed{\left|H(\omega)\right| = \frac{A_{out}}{A_{in}} \qquad \angle H(\omega) = \theta_{out} - \theta_{in}}$$

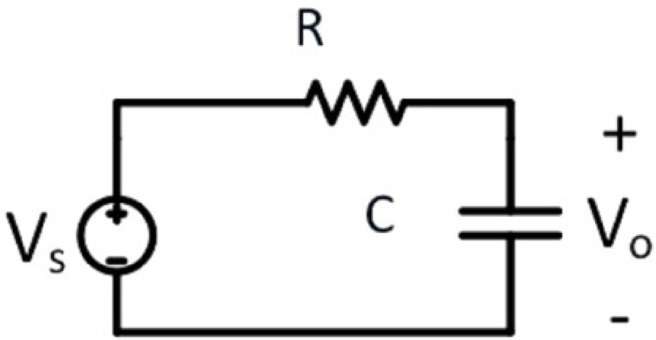
$$\boxed{A_{out} = \left|H(\omega)\right| A_{in} \qquad \theta_{out} = \angle H(\omega) + \theta_{in}}$$

Example 1

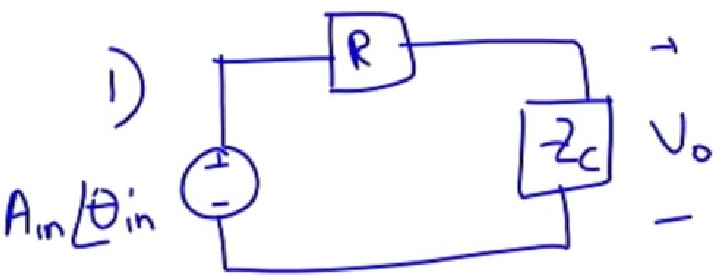


- Find the transfer function.

Solution 1



1)



$$Z_c = \frac{-j}{\omega C}$$

2) $$V_o = \frac{Z_c}{R+Z_c} A_{in}\angle\theta_{in}$$

$$A_{out}\angle\theta_{out} = \frac{1/j\omega C}{R + 1/j\omega C} A_{in}\angle\theta_{in}$$

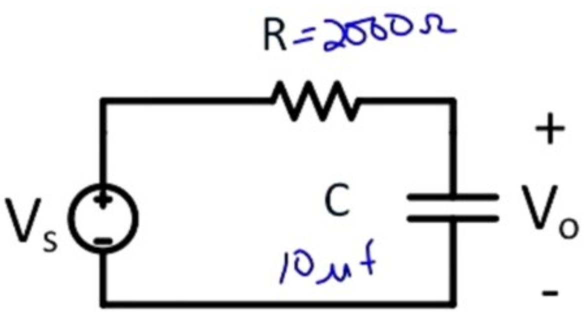
$$H(\omega) = \frac{1/j\omega C}{R + 1/j\omega C} = \boxed{\frac{1}{RCj\omega + 1}}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

$$\angle H(\omega) = -\tan^{-1}(RC\omega)$$

Solution 1

- If we put circuit element values



RC = .02

Input Signal magnitude and phase

$A_i = 2$ $\theta_i = 0$

$$H(\omega) = \frac{1}{1 + RC\omega j}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (RC\omega)^2}}$$

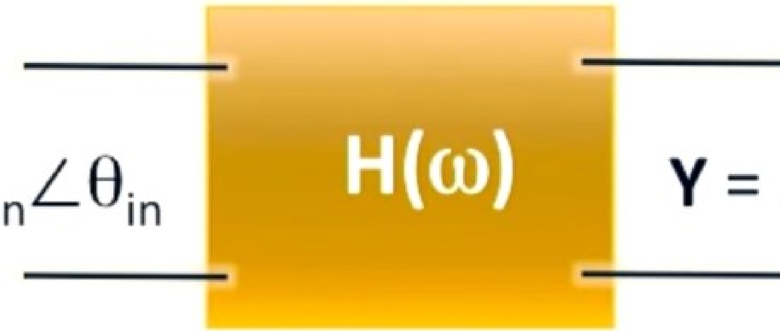
$$\angle H(\omega) = -\tan^{-1}(RC\omega)$$

ω	$\lvert H(\omega) \rvert$	$\angle H(\omega)$	$A_o = A_i \lvert H(\omega) \rvert$	$\theta_o = \theta_i + \angle H(\omega)$
10	0.83	-11.3°	2(.83) = 1.66	-11.3
500	0.49	-84.2°	2(.49) = .98	-84.2°

in $2\cos(10t) \xrightarrow{} $ out $1.66\cos(10t - 11.3°)$

in $2\cos(500t) \xrightarrow{} $ out $.98\cos(500t - 84.2°)$

Key Concepts about Transfer Function



$\mathbf{X} = A_{in}\angle\theta_{in}$ $\mathbf{Y} = A_{out}\angle\theta_{out}$

$x(t) = A_{in}\cos(\omega t + \theta_{in})$ $y(t) = A_{out}\cos(\omega t + \theta_{out})$

- **Transfer Function**

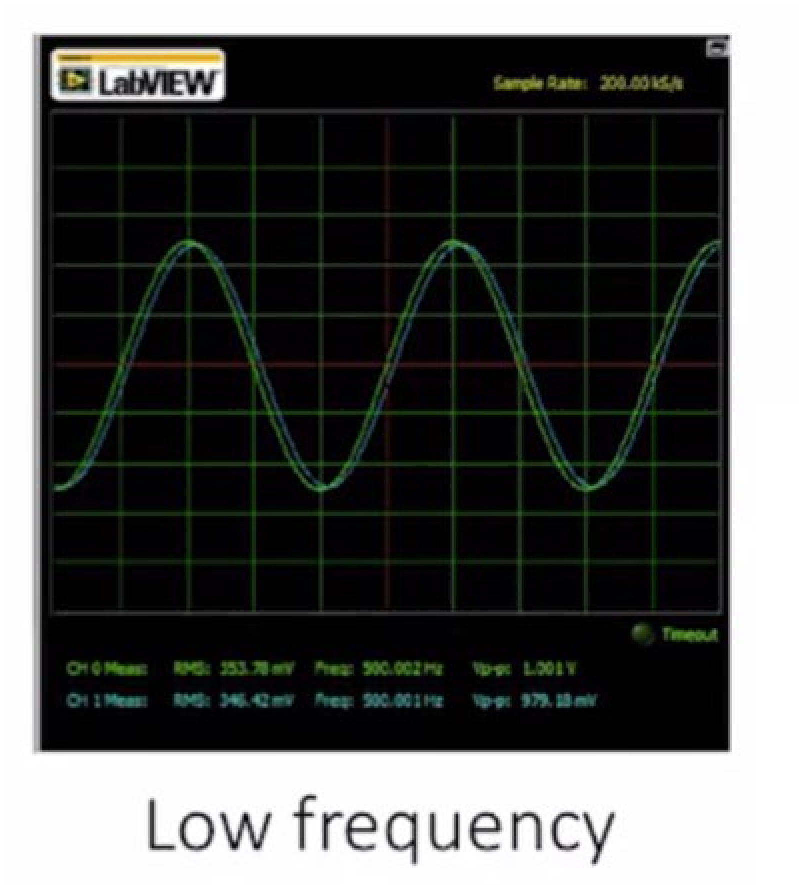
$$\mathbf{H}(\omega) = \frac{A_{out}\angle\theta_{out}}{A_{in}\angle\theta_{in}}$$

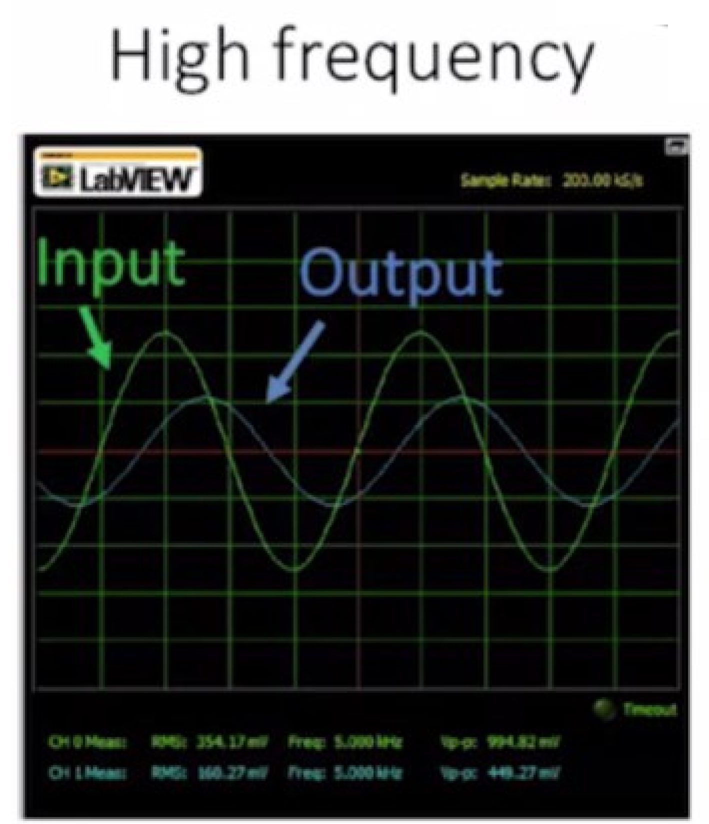
$$A_{out} = |H(\omega)|A_{in} \qquad \theta_{out} = \angle H(\omega) + \theta_{in}$$

Overview – Frequency Response

- We will focus a lot on the frequency characteristics of circuits.

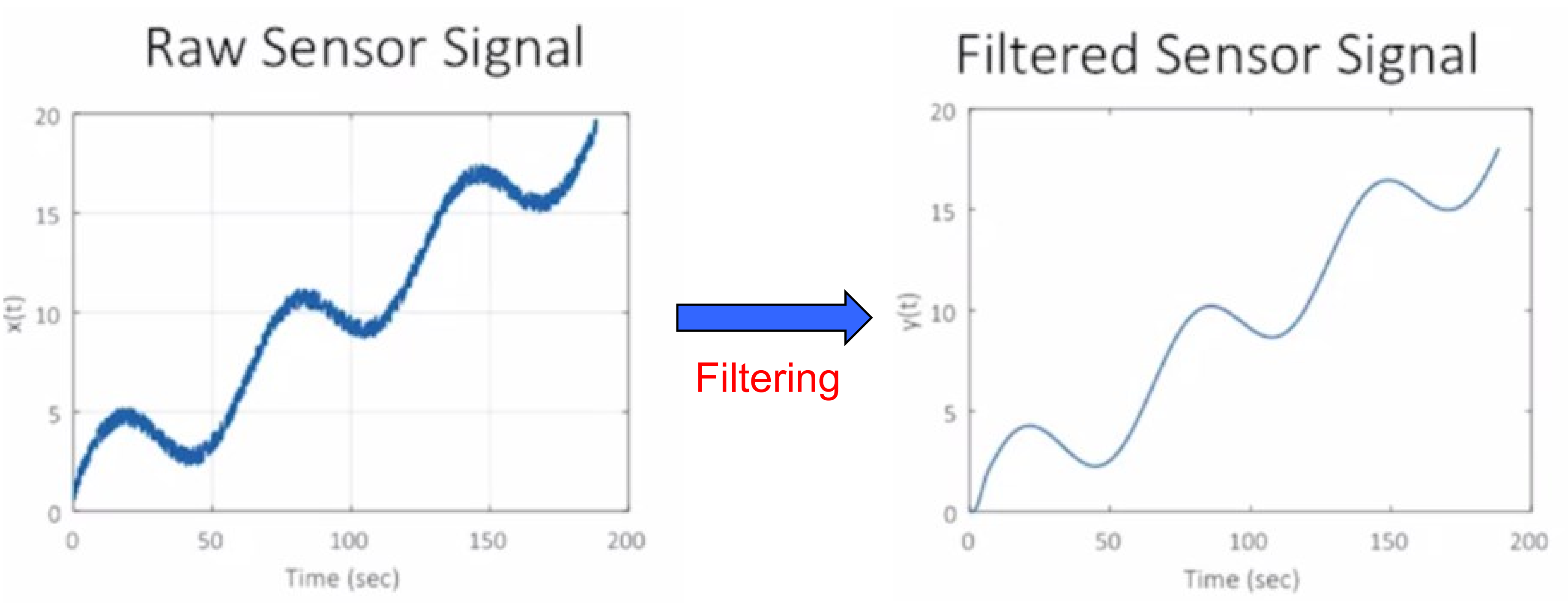
Dependence on Frequency





Overview - Frequency Response

- Then, we will see the filtering idea.

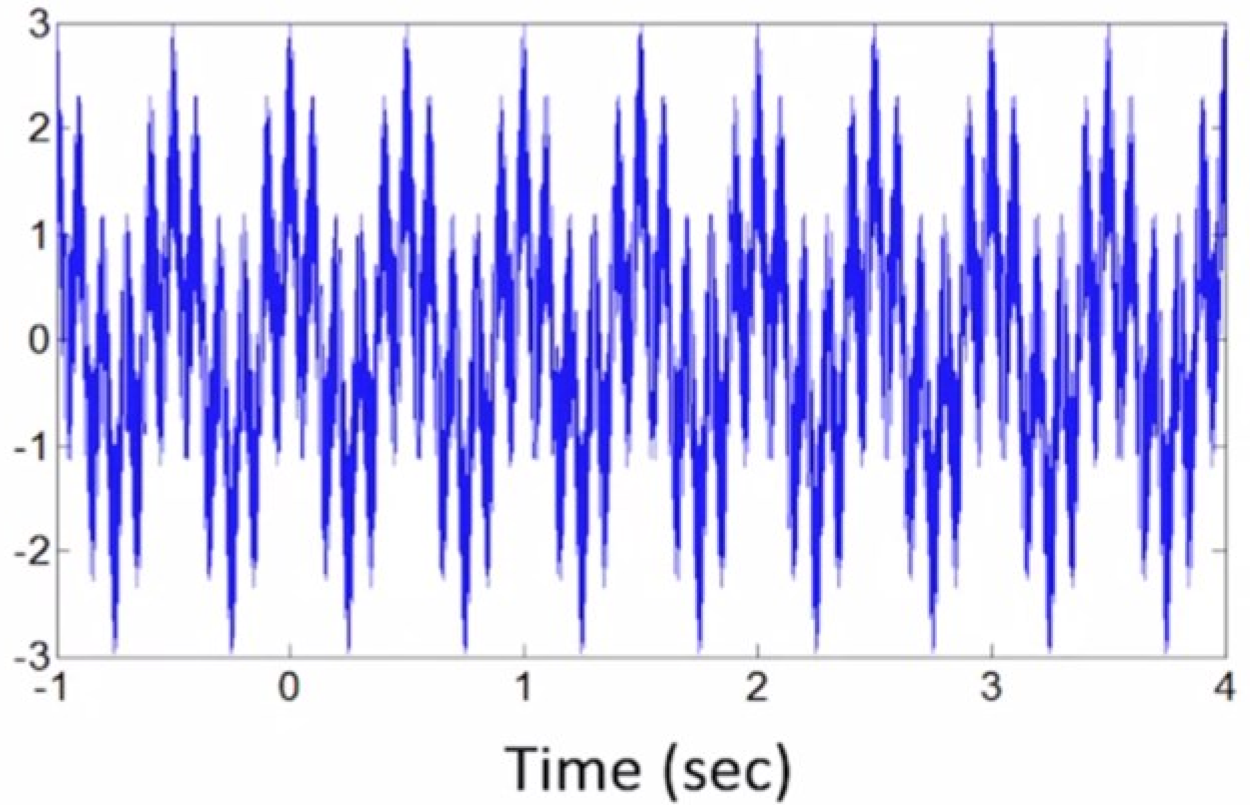


Frequency Content of Signals

Frequency Spectrum

Objective:

- Examine the frequency content of signals



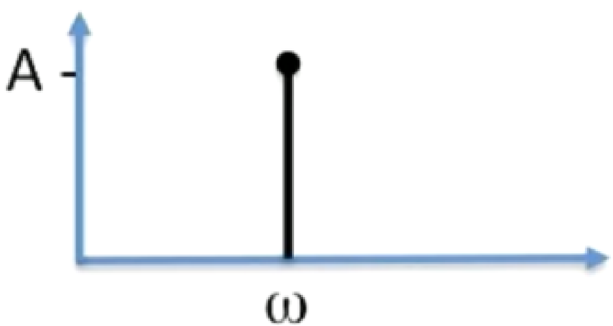
Frequency Content of Signals

Builds Upon:

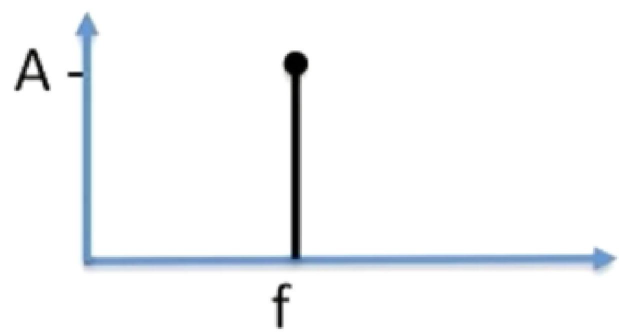
- Sinusoids:

  $v(t) = A\cos(\omega t+\theta)$ or $v(t) = A\cos(2\pi ft+\theta)$

Frequency Spectrum: plot of Amplitude versus frequency



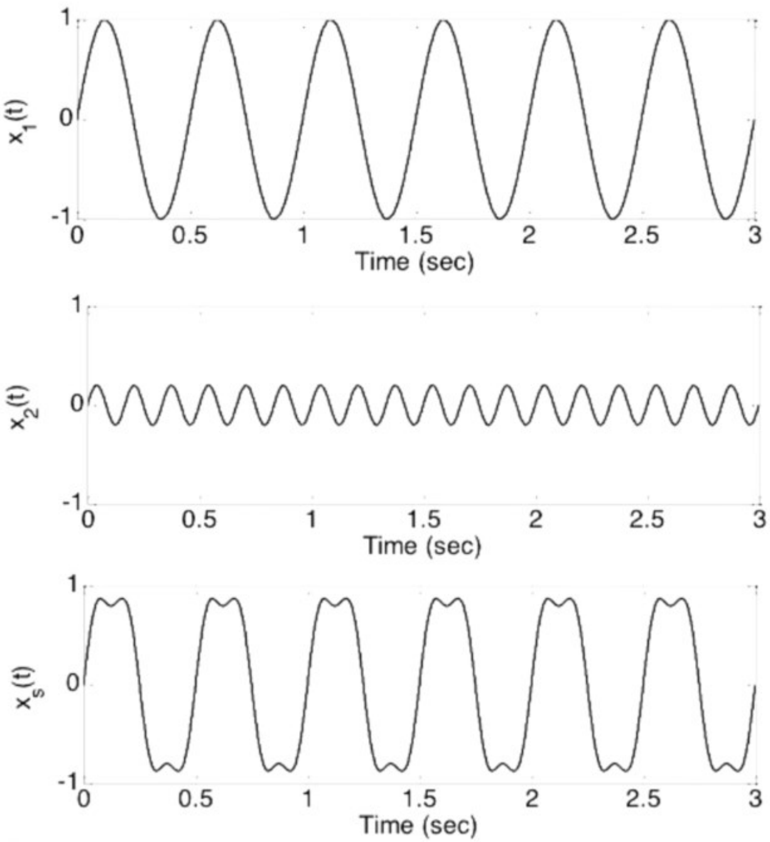
Frequency (rad/sec)



Frequency (Hz)

Frequency Content of Signals

Sum of Sinusoids

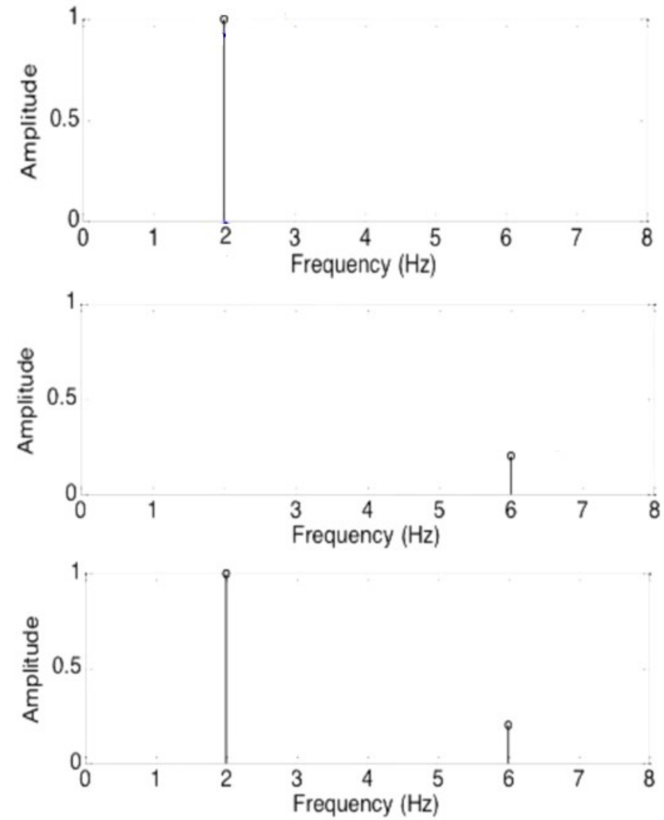


$x_1 = \sin(2\pi 2t)$

$x_2 = 0.2\sin(2\pi 6t)$

$x_s = x_1 + x_2$

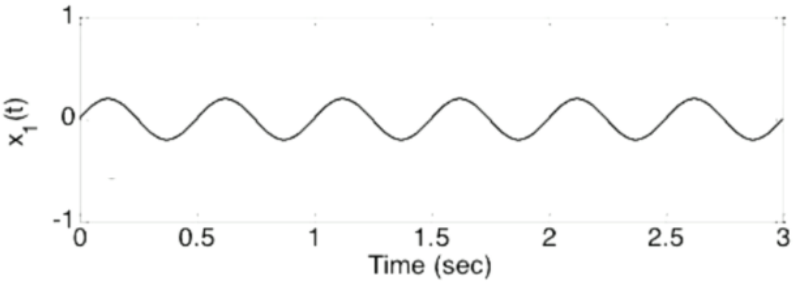
Frequency Spectrum



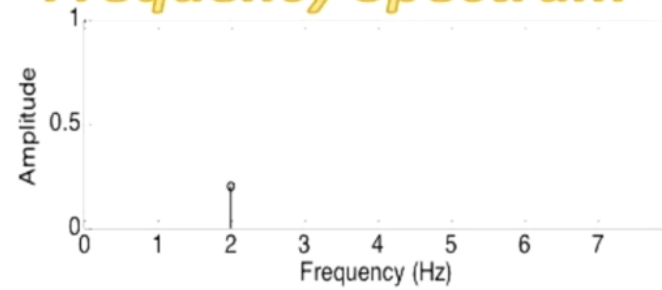
Frequency Content of Signals

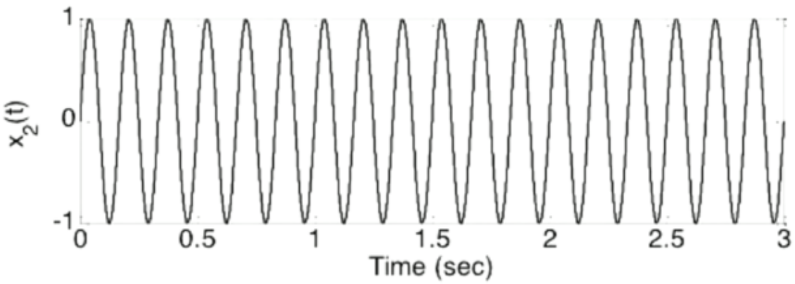
Sum of Sinusoids

Frequency Spectrum

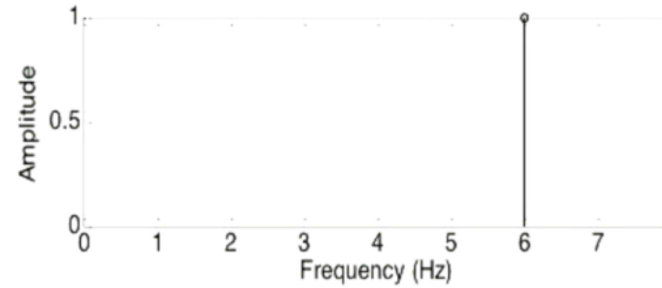


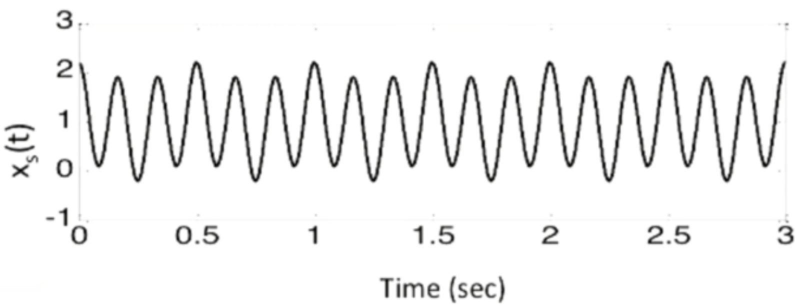
$x_1 = 0.2\sin(2\pi 2t)$



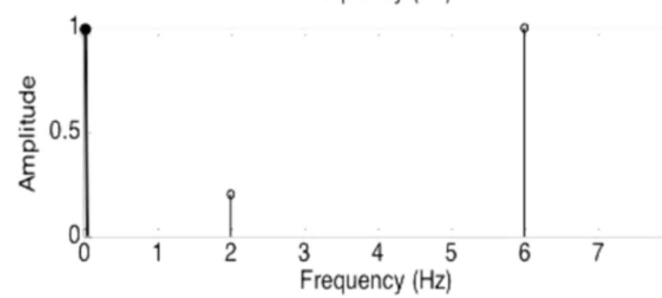


$x_2 = \sin(2\pi 6t)$



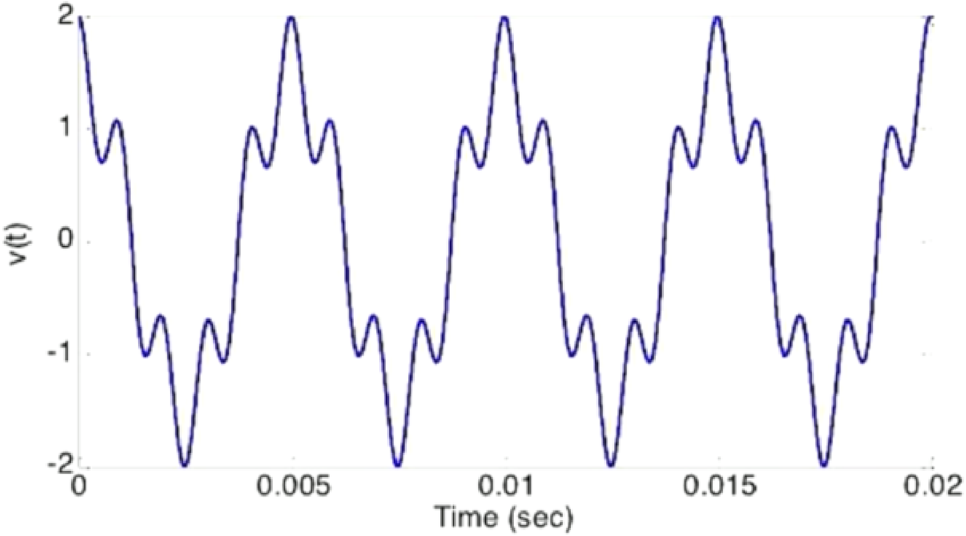


$x_s = x_1 + x_2 + 1$



Example 2

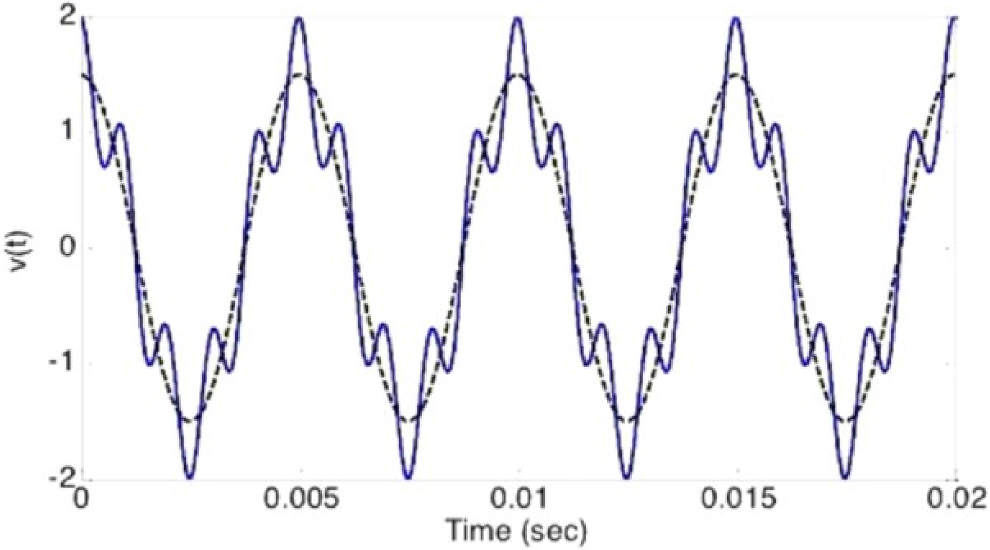
Examples

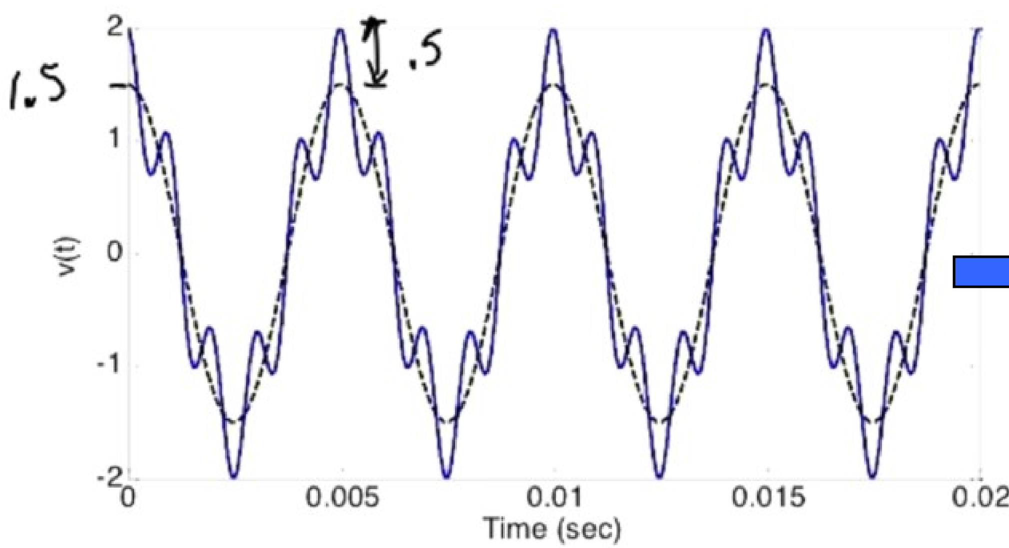


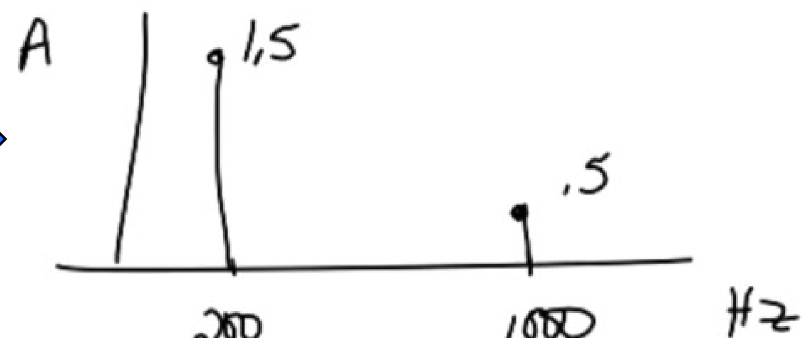
f_1 = 200Hz, f_2= 1000Hz

What are the magnitudes for two signals?

Solution 2

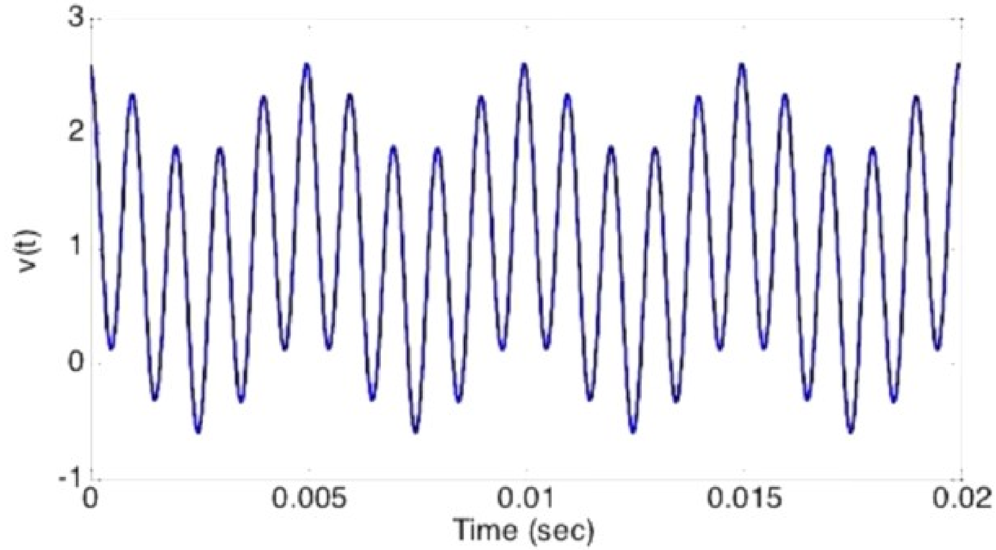

v(t)
Time (sec)
2
1
0
-1
-2
0
0.005
0.01
0.015
0.02


1.5
.5
v(t)
Time (sec)
2
1
0
-1
-2
0
0.005
0.01
0.015
0.02


A
1,5
.5
200
1000
Hz

Example 3

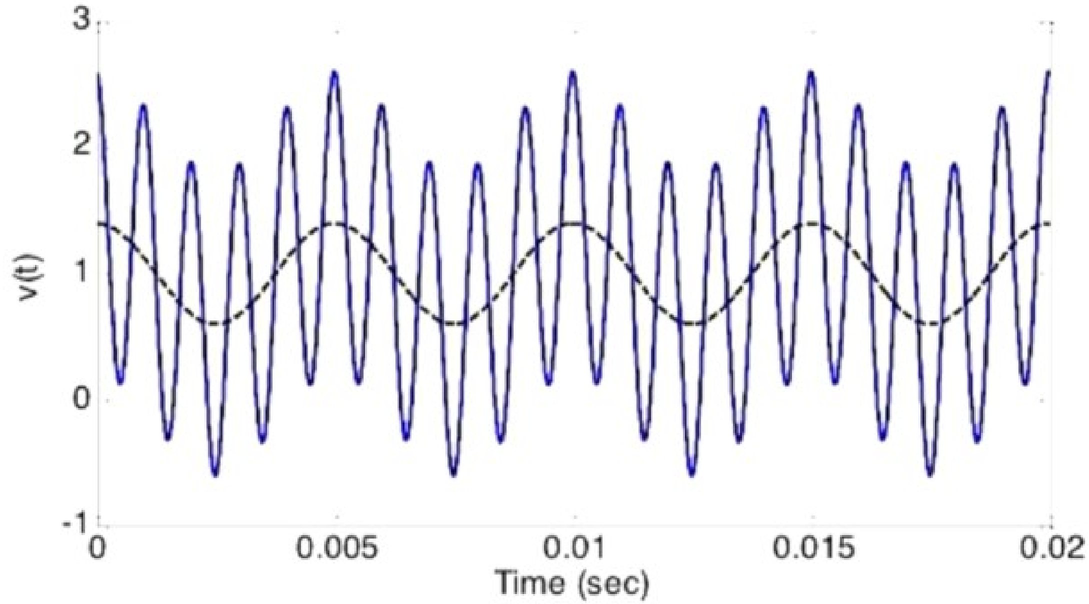
Examples



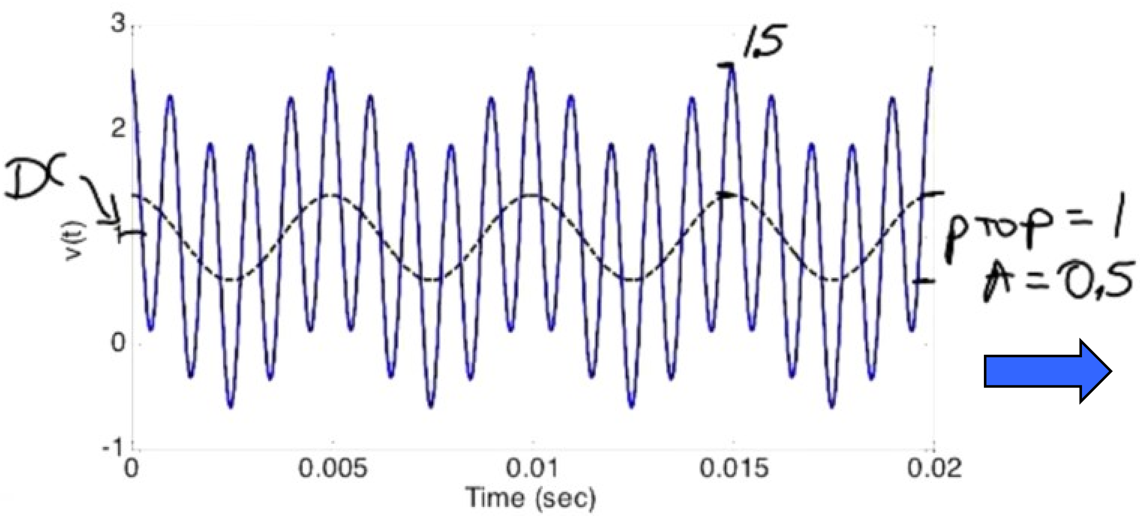
f_1 = 200Hz, f_2= 1000Hz

What are the magnitudes for two signals?

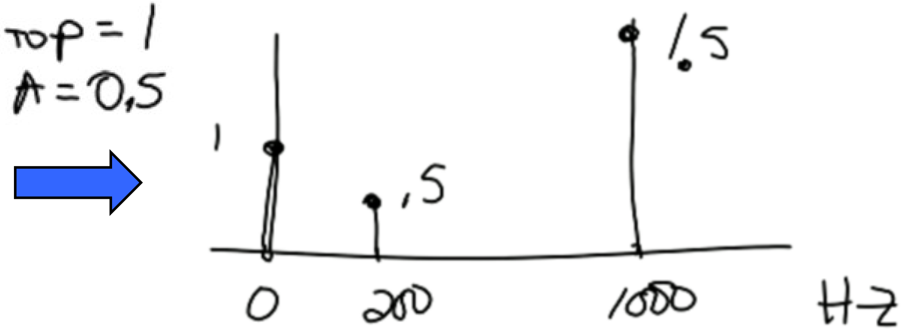
Solution 3



v(t)
Time (sec)
3
2
1
0
-1
0
0.005
0.01
0.015
0.02



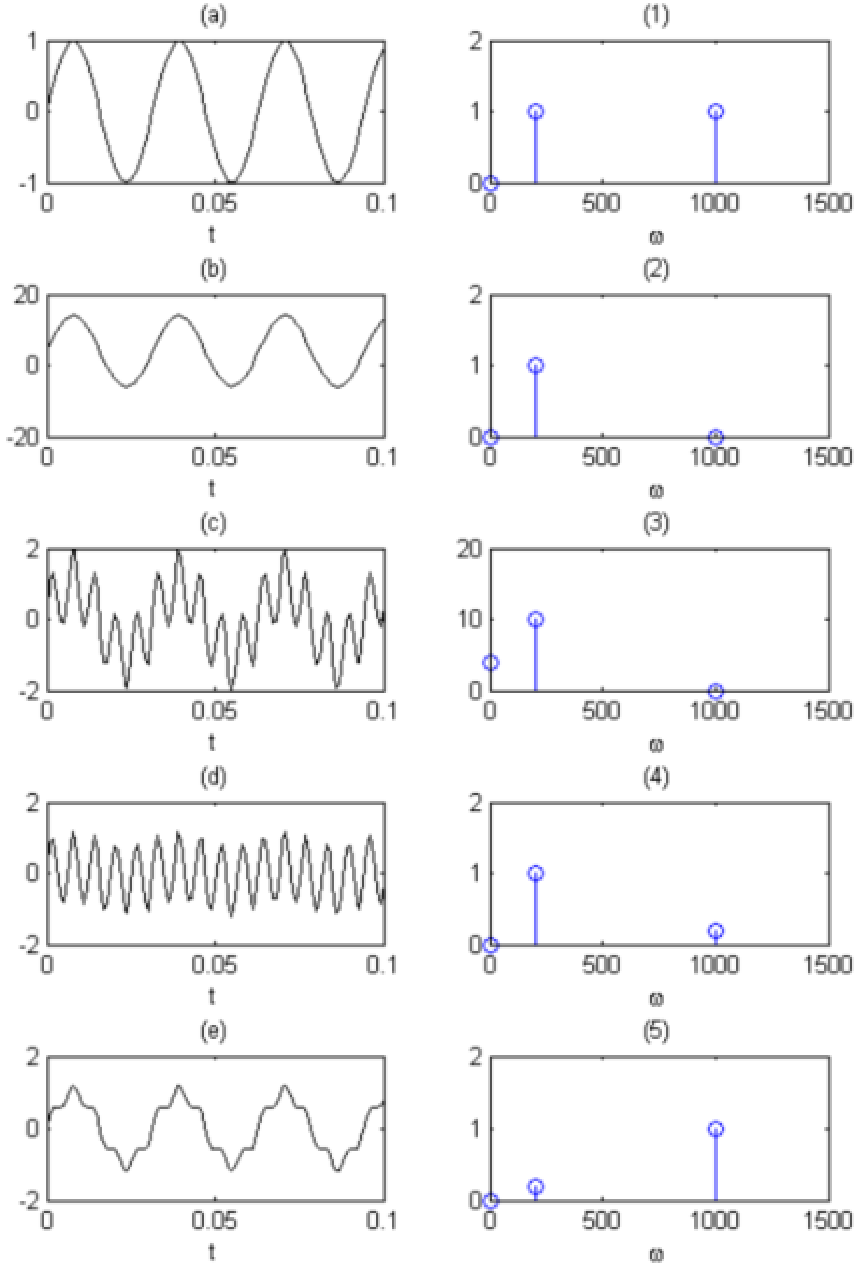
DC
1.5
ptop = 1
A = 0.5
v(t)
Time (sec)
3
2
0
-1
0
0.005
0.01
0.015
0.02



1
.5
1.5
0
200
1000
Hz

Consider the time domain waveforms below on the left. Match waveforms (a) - (e) to their respective frequency spectrum representation on the right.

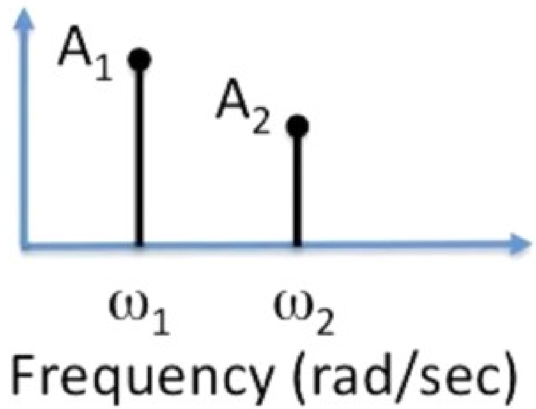
Example 4



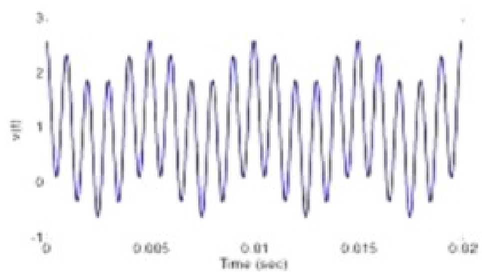
Key Concepts

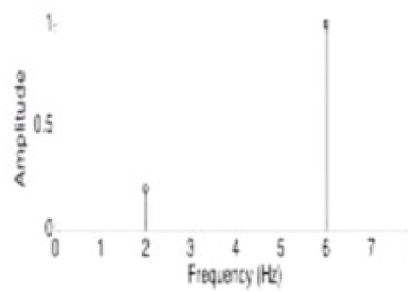
- Frequency spectrum

$$v(t) = A_1\cos(\omega_1 t+\theta)+A_2\cos(\omega_2 t+\theta)$$



- Time domain and frequency domain

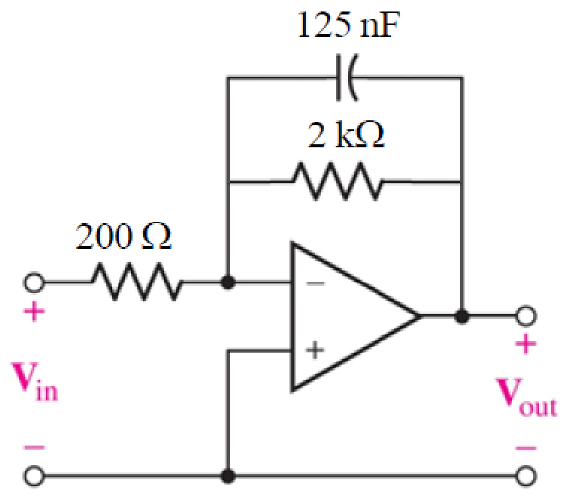




Frequency Response of a Circuit - An example

Sketch $|\mathbf{V}_{out}/\mathbf{V}_{in}|$ vs. ω for $1 \text{ rad/s} < \omega < 100 \text{ krad/s}$.

Determine $v_{out}(t)$ for $v_{in}(t) = \cos(10^4 t)$ V .



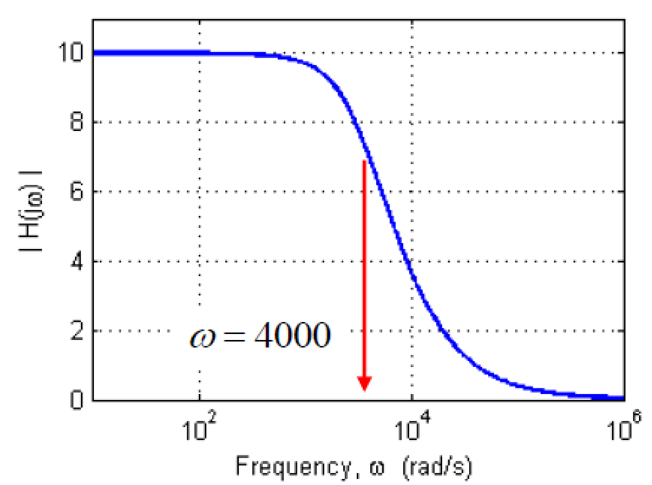
$$\frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_i} = \frac{R_f \,\|\, 1/j\omega C_f}{R_i} = -\frac{R_f}{R_i} \cdot \frac{1/j\omega C_f}{R_f + 1/j\omega C_f}$$

$$= -\frac{R_f}{R_i} \cdot \frac{1/R_f C_f}{j\omega + 1/R_f C_f} = -10 \cdot \frac{4000}{j\omega + 4000}$$

$$\left|\frac{\mathbf{V}_{out}}{\mathbf{V}_{in}}\right| = 10 \cdot \frac{4000}{\sqrt{\omega^2 + 4000^2}}$$

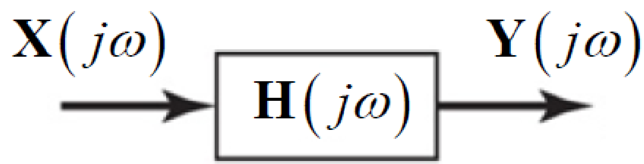
$$\mathbf{V}_{out}\left(\omega = 10^4\right) = \frac{4 \cdot 10^4}{j10^4 + 4000}\mathbf{V}_{in} = \frac{4}{j + 0.4} = 3.7\angle -68° \text{ V}$$

$$v_{out}\left(t\right) = 3.7\cos\left(10^4 t - 68°\right) \text{ V}$$



Frequency Response

Let $\mathbf{X}(j\omega)$ be the phasor form of the input to a circuit, and let $\mathbf{Y}(j\omega)$ be the phasor form of the output from a circuit, expressed as a ratio:



$$\mathbf{H}(j\omega) = \frac{\mathbf{Y}(j\omega)}{\mathbf{X}(j\omega)}$$

→ $\mathbf{H}(j\omega)$ is the circuit's **transfer function** a.k.a. its **frequency response**

The **gain** of a circuit is the ratio of the output amplitude to input amplitude:

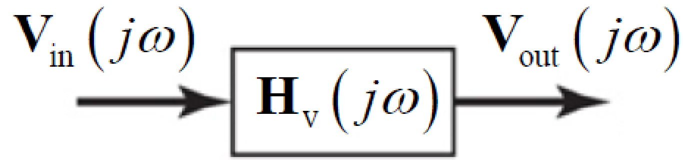
$$\text{gain} = A = \frac{|\mathbf{Y}(j\omega)|}{|\mathbf{X}(j\omega)|}$$

The **phase shift** of a circuit is the difference in the phase of the output with respect to the input:

$$\text{phase} = \measuredangle\mathbf{Y}(j\omega) - \measuredangle\mathbf{X}(j\omega)$$

Voltage Transfer Function

Let $\mathbf{V}_{in}(j\omega)$ be the phasor form of the <u>voltage input</u> to a circuit, and let $\mathbf{V}_{out}(j\omega)$ be the phasor form of the <u>voltage output</u> from a circuit, expressed as a ratio:



$$\mathbf{H}_{v}(j\omega) = \frac{\mathbf{V}_{out}(j\omega)}{\mathbf{V}_{in}(j\omega)}$$

→ $\mathbf{H}_{v}(j\omega)$ is the circuit's **voltage transfer function**

The **voltage gain** of a circuit is the ratio of the output amplitude to input amplitude:

$$\text{gain} = A_{v} = \frac{\left|\mathbf{V}_{out}(j\omega)\right|}{\left|\mathbf{V}_{in}(j\omega)\right|}$$

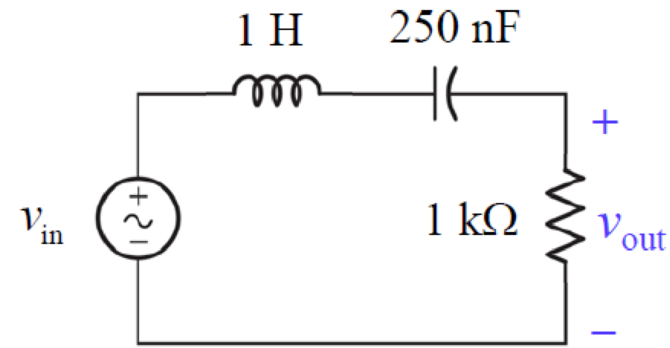
The **phase shift** of a circuit is the difference in the phase of the output with respect to the input:

$$\text{phase} = \measuredangle\mathbf{V}_{out}(j\omega) - \measuredangle\mathbf{V}_{in}(j\omega)$$

Example: Freq Response

Plot the amplitude and phase of the voltage transfer function for this circuit for

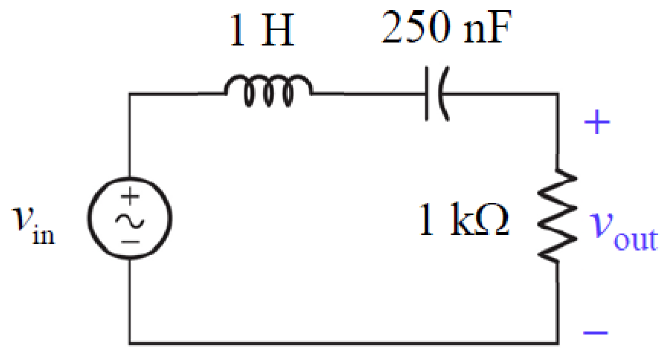
$100\text{ Hz} < f < 1\text{ kHz}$ (linear)



Example: Freq Response

Plot the amplitude and phase of the voltage transfer function for this circuit for

$$100 \text{ Hz} < f < 1 \text{ kHz} \quad \text{(linear)}$$




```
L = 1;
C = 250e-9;
R = 1000;

f = linspace(100,1000,5e2);
omega = 2*pi*f;

Z_L = j*omega*L;
Z_C = -j./(omega*C);

H = R ./ (R + Z_C + Z_L);
```

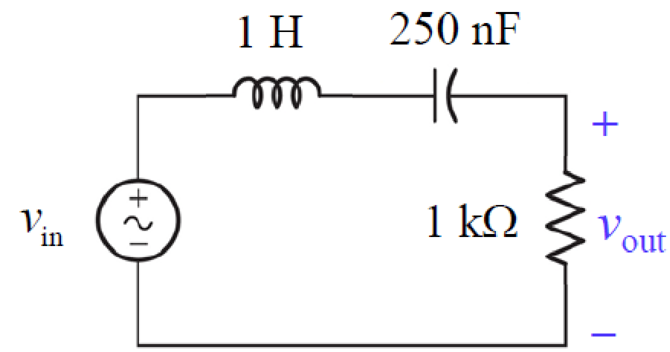
```
>> abs(H(1))

   ans =

                                      0.1717

>> angle(H(1))*180/pi

   ans =

                                     80.1138


>> abs(H(length(H)))

   ans =

                                      0.1744

>> angle(H(length(H))) * 180/pi

   ans =

                                    -79.9571
```

$$\mathbf{H}_v(100) \;\; = 0.17\angle 80.1^\circ$$
$$\mathbf{H}_v(500) \;\; = 0.47\angle -62.4^\circ$$
$$\mathbf{H}_v(1000) = 0.17\angle -80.0^\circ$$

Example: Freq Response

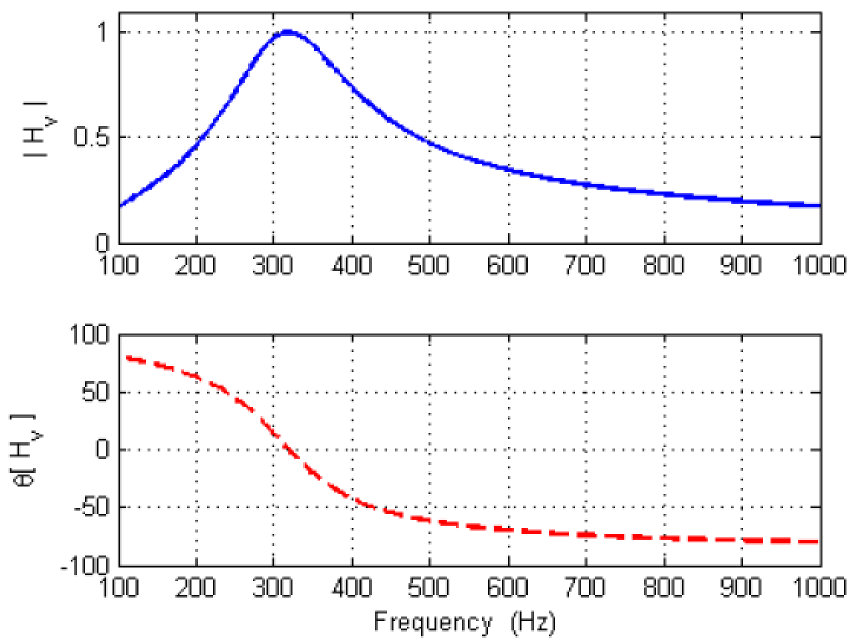
Plot the amplitude and phase of the voltage transfer function for this circuit for

$100\text{ Hz} < f < 1\text{ kHz}$ (linear)




```
L = 1;
C = 250e-9;
R = 1000;

f = linspace(100,1000,5e2);
omega = 2*pi*f;

Z_L = j*omega*L;
Z_C = -j./(omega*C);

H = R ./ (R + Z_C + Z_L);
```

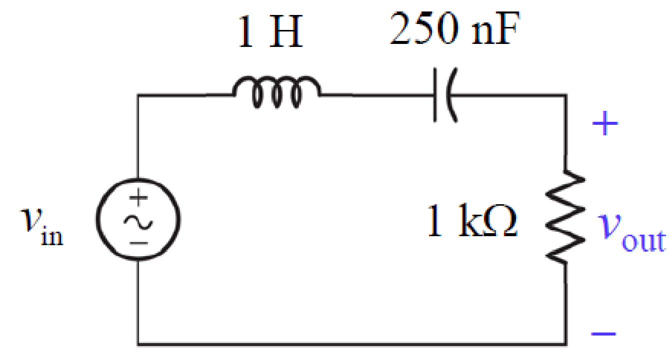
```
figure(1)
subplot(2,1,1)
plot(f,abs(H),'b-','LineWidth',2)
ylabel('| H_v |')
axis([-Inf Inf 0 1.1])
grid
subplot(2,1,2)
plot(f,angle(H)*180/pi,'r--','LineWidth',2)
ylabel('\theta[ H_v ]')
axis([-Inf Inf -100 100])
xlabel('Frequency  (Hz)')
grid
```



Example: Freq Response

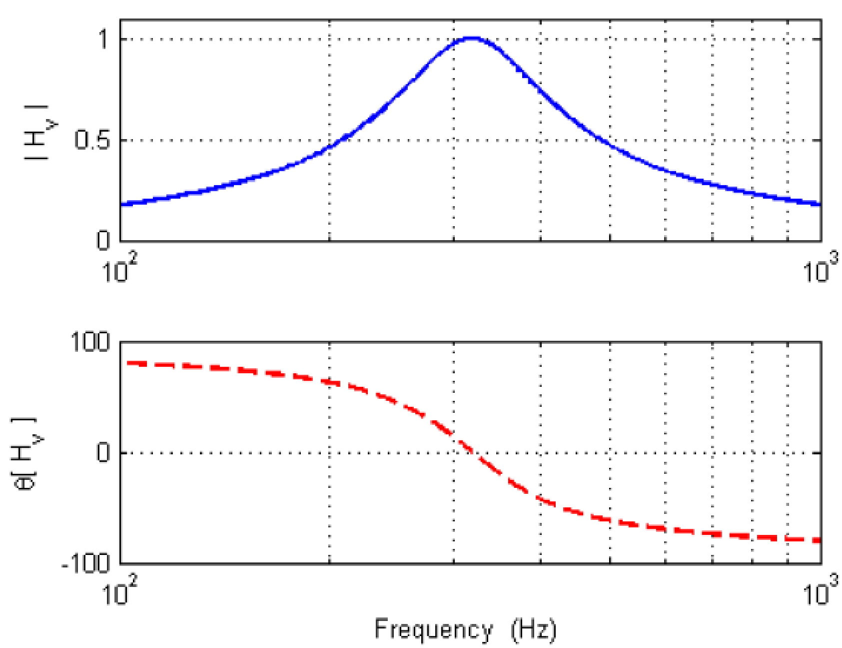
Plot the amplitude and phase of the voltage transfer function for this circuit for

$$100 \text{ Hz} < f < 1 \text{ kHz} \quad \text{(linear)}$$




```
L = 1;
C = 250e-9;
R = 1000;

f = logspace(2,3,5e2);
omega = 2*pi*f;

Z_L = j*omega*L;
Z_C = -j./(omega*C);

H = R ./ (R + Z_C + Z_L);
```

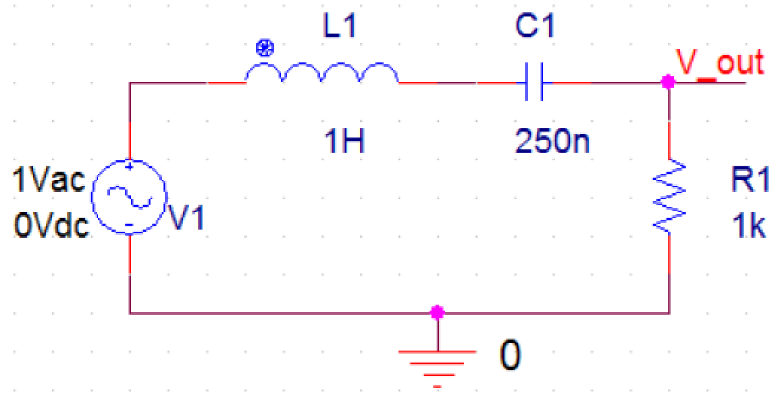
```
figure(1)
subplot(2,1,1)
semilogx(f,abs(H),'b-','LineWidth',2)
ylabel('| H_v |')
axis([1e2 1e3 0 1.1])
grid
subplot(2,1,2)
semilogx(f,angle(H)*180/pi,'r--','LineWidth',2)
ylabel('\theta[ H_v ]')
axis([1e2 1e3 -100 100])
xlabel('Frequency  (Hz)')
grid
```

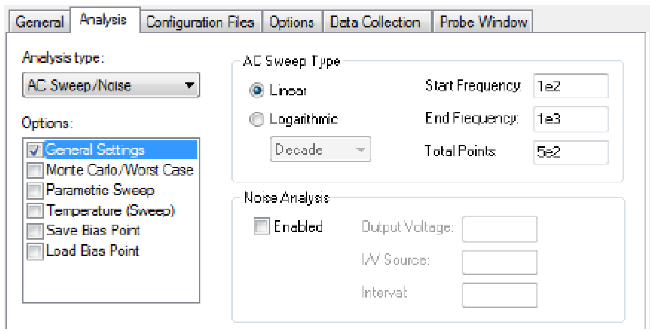


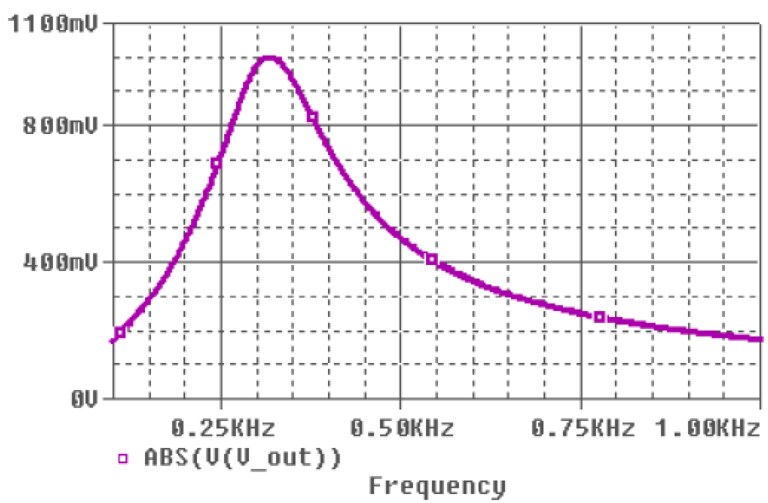
Example: Freq Response, PSpice

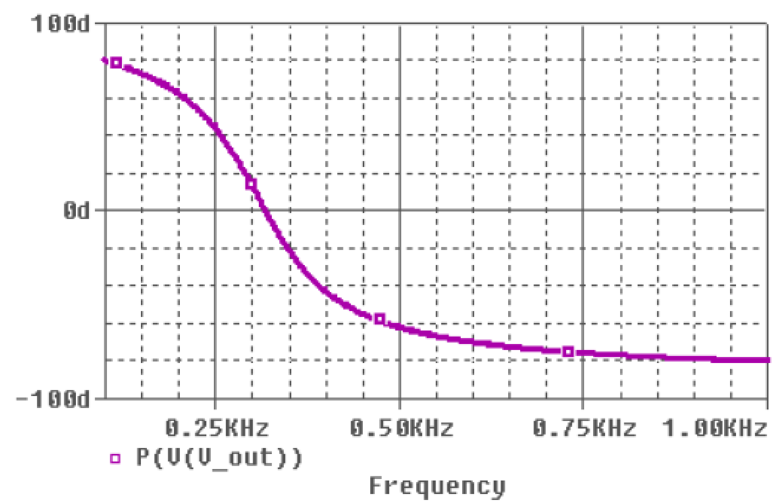
Plot the amplitude and phase of the voltage transfer function for this circuit for

$$100 \text{ Hz} < f < 1 \text{ kHz} \quad \text{(linear)}$$





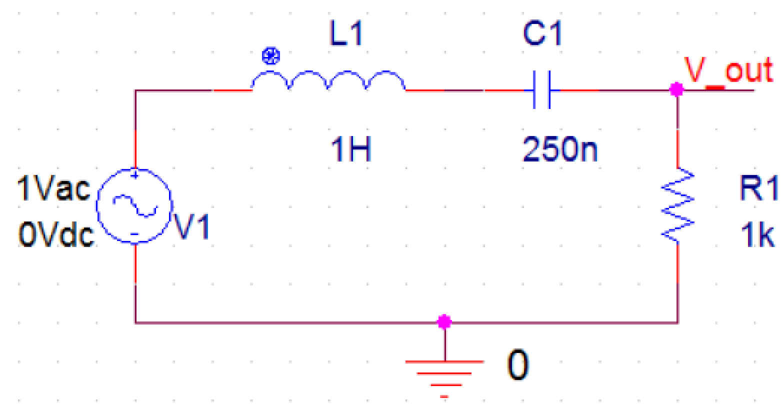


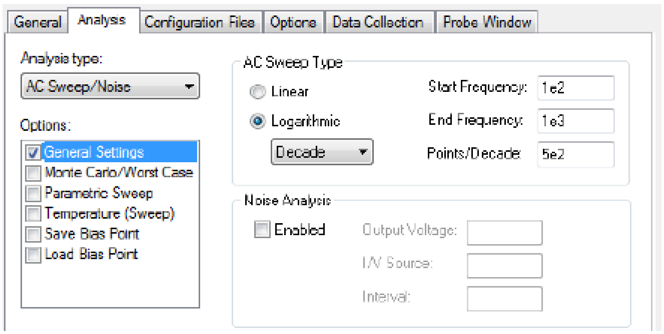


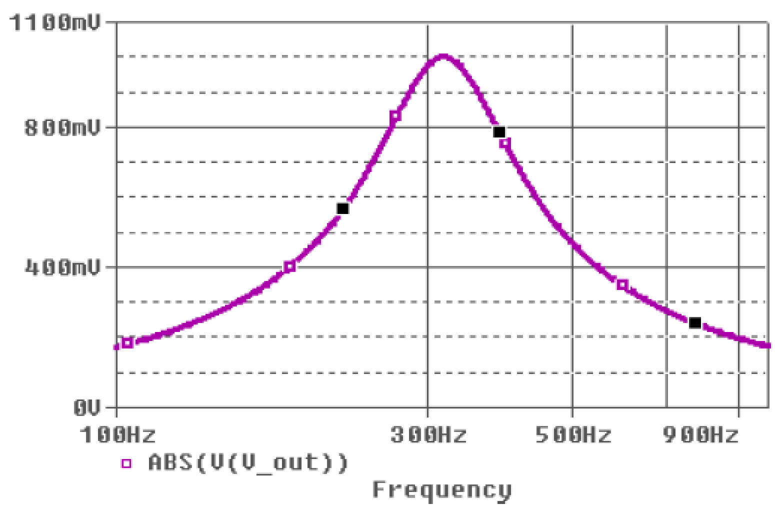
Example: Freq Response, PSpice

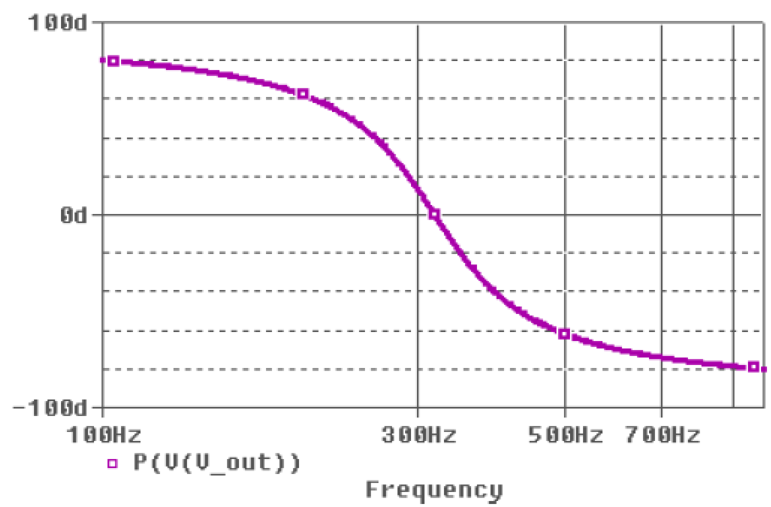
Plot the amplitude and phase of the voltage transfer function for this circuit for

$$100 \text{ Hz} < f < 1 \text{ kHz} \quad \text{(logarithmic)}$$









Frequency Response - Sample 5

Consider the circuit shown in Figure 13.2-4a. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage $v_o(t)$ across the series connection of the capacitor and the 16-kΩ resistor. The network function that represents this circuit has the form

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}} \tag{13.2-4}$$

The network function depends on two parameters, z and p. The parameter z is called the zero of the circuit and the parameter p is called the pole of the circuit. Determine the values of z and of p for the circuit in Figure 13.2-4a.

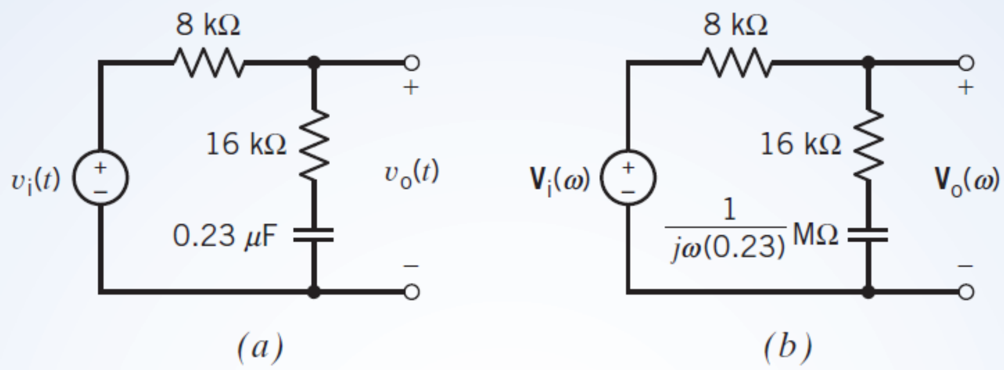


FIGURE 13.2-4 The circuit considered in Example 13.2-1 represented (a) in the time domain and (b) in the frequency domain.

Frequency Response - Solution 5

We will analyze the circuit to determine its network function and then put the network function into the form given in Eq. 13.2-4. A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 13.2-4*b* shows the frequency-domain representation of the circuit from Figure 13.2-4*a*.

The impedances of the capacitor and the 16-kΩ resistor are connected in series in Figure 13.2-4*b*. The equivalent impedance is

$$\boldsymbol{Z}_{\mathrm{e}}(\omega) = 16,000 + \frac{10^6}{j(0.23)\omega}$$

The equivalent impedance is connected in series with the 8-kΩ resistor. $\mathbf{V}_{\mathrm{i}}(\omega)$ is the voltage across the series impedances, and $\mathbf{V}_{\mathrm{o}}(\omega)$ is the voltage across the equivalent impedance $\mathbf{Z}_{\mathrm{e}}(\omega)$. Apply the voltage division principle to get

$$\mathbf{V}_{\mathrm{o}}(\omega) = \frac{16,000 + \dfrac{10^6}{j(0.23)\omega}}{8000 + 16,000 + \dfrac{10^6}{j(0.23)\omega}}\mathbf{V}_{\mathrm{i}}(\omega) = \frac{10^6 + j(0.23)\omega(16,000)}{10^6 + j(0.23)\omega(24,000)}\mathbf{V}_{\mathrm{i}}(\omega)$$

$$= \frac{\dfrac{10^6 + j(3680)\omega}{10^6}}{\dfrac{10^6 + j(5520)\omega}{10^6}}\mathbf{V}_{\mathrm{i}}(\omega) = \frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega}\mathbf{V}_{\mathrm{i}}(\omega)$$

Frequency Response - Solution 5

Divide both sides of this equation by $\mathbf{V}_\mathrm{i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_\mathrm{o}(\omega)}{\mathbf{V}_\mathrm{i}(\omega)} = \frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega}$$

Equating the network functions given by Eq. 13.2-4 and 13.2-5 gives

$$\frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$

Comparing these network functions gives

$$z = \frac{1}{0.00368} = 271.74 \text{ rad/s} \quad \text{and} \quad p = \frac{1}{0.00552} = 181.16 \text{ rad/s}$$

Frequency Response - Sample 6

Consider the circuit shown in Figure 13.2-5a. The input to the circuit is the voltage of the voltage source $v_i(t)$. The output is the voltage $v_o(t)$ across the series connection of the inductor and the 2-Ω resistor. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_o(\omega)}{\mathbf{V}_i(\omega)} = 0.2\frac{1 + j\frac{\omega}{5}}{1 + j\frac{\omega}{25}} \tag{13.2-6}$$

Determine the value of the inductance L.

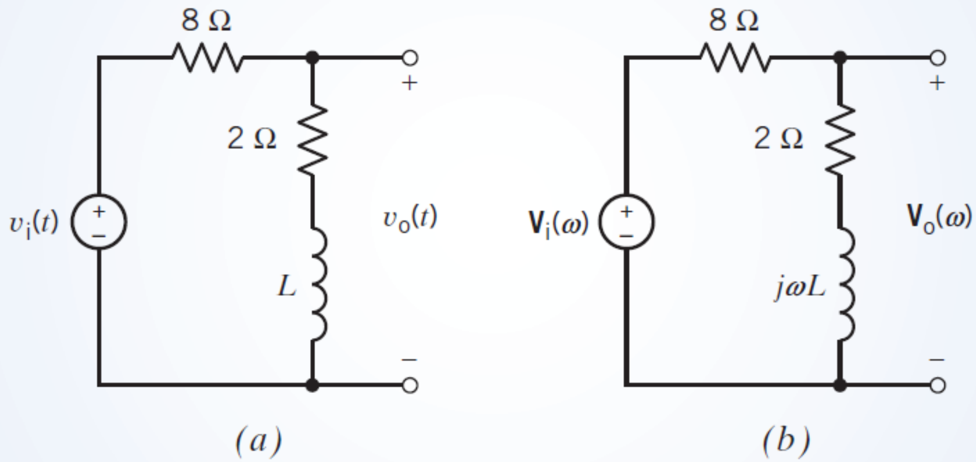


FIGURE 13.2-5 The circuit considered in Example 13.2-2 represented (a) in the time domain and (b) in the frequency domain.

Frequency Response - Solution 6

The circuit has been represented twice, by a circuit diagram and by a network function. The unknown inductance L appears in the circuit diagram but not in the given network function. We can analyze the circuit to determine its network function. This second network function will depend on the unknown inductance. We will determine the value of the inductance by equating the two network functions.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 13.2-5*b* shows the frequency-domain representation of the circuit from Figure 13.2-5*a*.

The impedances of the inductor and the 2-Ω resistor are connected in series in Figure 13.2-5*b*. The equivalent impedance is

$$\mathbf{Z}_{\mathrm{e}}(\omega) = 2 + j\omega L$$

The equivalent impedance is connected in series with the 8-Ω resistor. $\mathbf{V}_{\mathrm{i}}(\omega)$ is the voltage across the series impedances, and $\mathbf{V}_{\mathrm{o}}(\omega)$ is the voltage across the equivalent impedance $\mathbf{Z}_{\mathrm{e}}(\omega)$. Apply the voltage division principle to get

$$\mathbf{V}_{\mathrm{o}}(\omega) = \frac{2 + j\omega L}{8 + 2 + j\omega L}\mathbf{V}_{\mathrm{i}}(\omega) = \frac{2 + j\omega L}{10 + j\omega L}\mathbf{V}_{\mathrm{i}}(\omega)$$

Frequency Response - Solution 6

Next, we put the network function into the form specified by Eq. 13.2-6. Factoring 2 out of both terms in the numerator and factoring 10 out of both terms in the denominator, we get

$$\mathbf{H}(\omega) = \frac{2\left(1 + j\omega\frac{L}{2}\right)}{10\left(1 + j\omega\frac{L}{10}\right)} = 0.2\frac{1 + j\omega\frac{L}{2}}{1 + j\omega\frac{L}{10}} \tag{13.2-7}$$

Equating the network functions given by Eqs. 13.2-6 and 13.2-7 gives

$$0.2\frac{1 + j\omega\frac{L}{2}}{1 + j\omega\frac{L}{10}} = 0.2\frac{1 + j\frac{\omega}{5}}{1 + j\frac{\omega}{25}}$$

Comparing these network functions gives

$$\frac{L}{2} = \frac{1}{5} \quad \text{and} \quad \frac{L}{10} = \frac{1}{25}$$

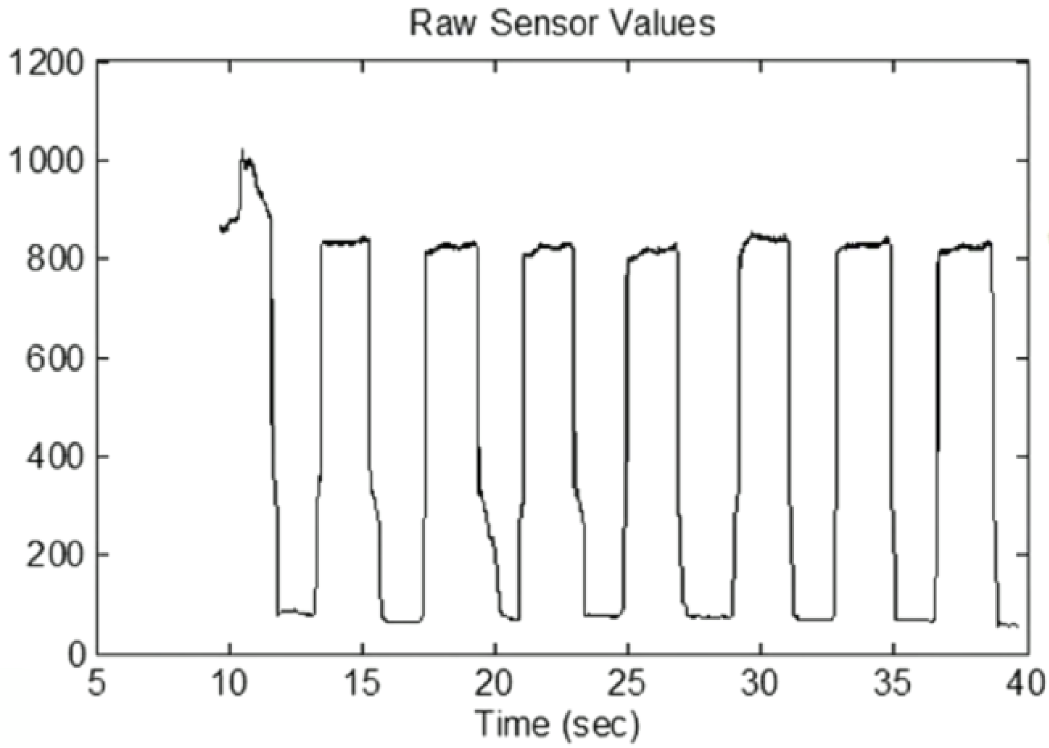
The values of L obtained from these equations must agree, and they do. (If they do not, we've made an error.) Solving each of these equations gives $L = 0.4\,\mathrm{H}$.

Spectrum Of Real-World Signals

- a signal that comes form IR sensor.

Objective:

- Examine log scales and harmonics in real signals

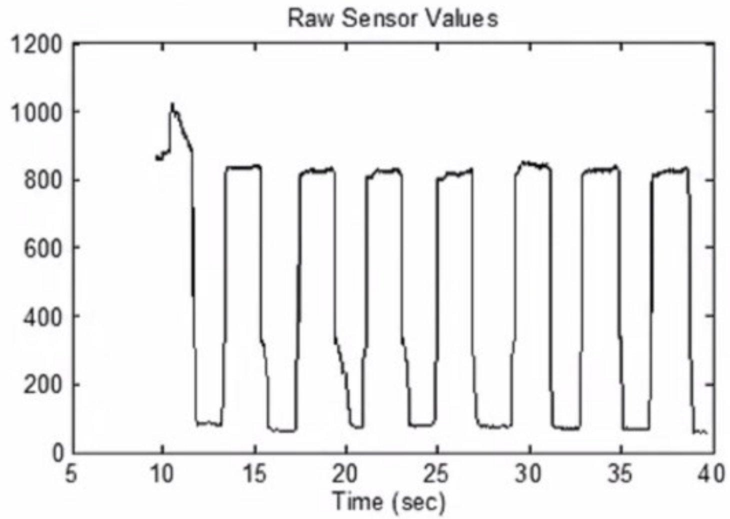


- measures the intensity of light.
- high levels – light on
- low levels – light off
- There is noise on the signal (measurement noise)

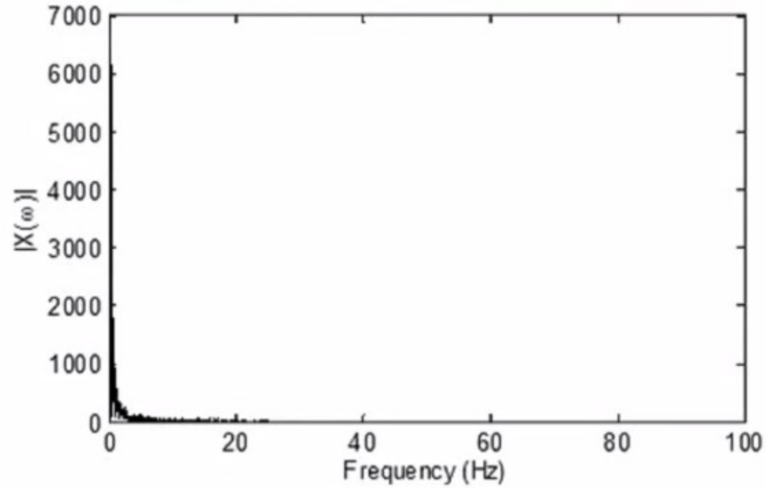
- Noise comes from a certain freq.
- Let's find the frequency and remove it (filtering)

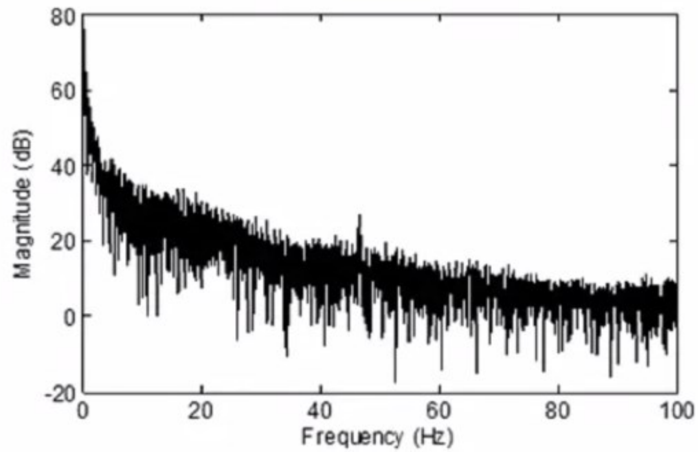
Spectrum Of Real-World Signals

Example of log scale



Spectrum Analyzer Instrument: Linear Scale





Spectrum Analyzer Instrument: Log Scale

Logarithms

$$N = b^x \quad \Leftrightarrow \quad \boxed{\log_b N = x}$$

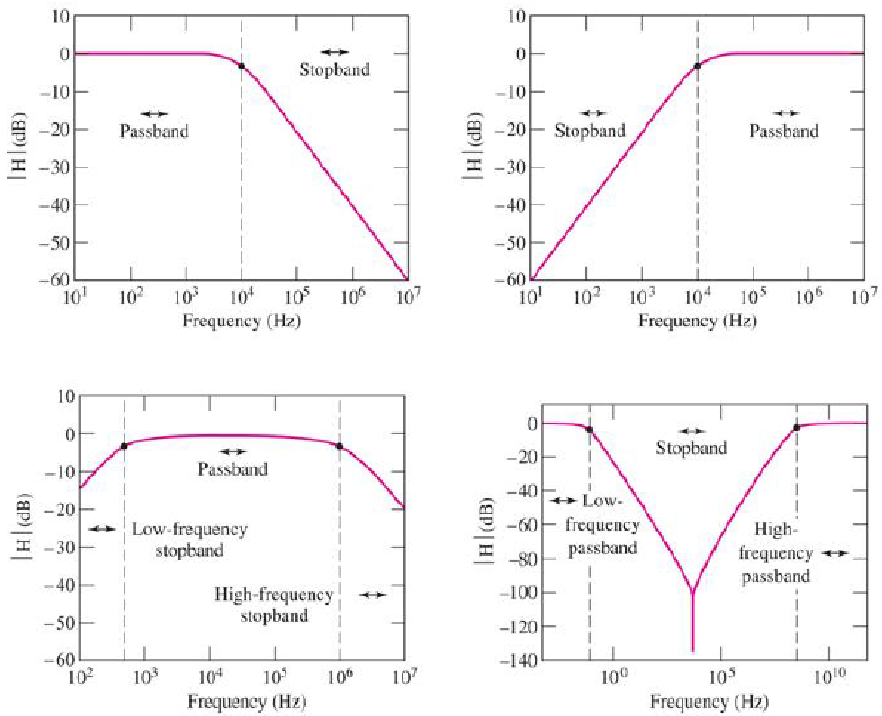
where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

-- a way to easily write/compare numbers that are very large and/or very small, simultaneously

-- an alternative to scientific notation

using $b = 10$ ("base-10")…

N	$\Leftrightarrow$	x
0.000001	$\Leftrightarrow$	-6
0.001	$\Leftrightarrow$	-3
1	$\Leftrightarrow$	0
1,000	$\Leftrightarrow$	3
1,000,000	$\Leftrightarrow$	6



Logarithms

$$N = b^x \quad \Leftrightarrow \quad \boxed{\log_b N = x}$$

where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

$b = 2$ ("base-2")			$b = e \approx 2.718$ ("base-e")			$b = 10$ ("base-10")			$b = 16$ ("base-16")		
N	$\Leftrightarrow$	x	N	$\Leftrightarrow$	x	N	$\Leftrightarrow$	x	N	$\Leftrightarrow$	x
2	$\Leftrightarrow$	1	$e^{-5} \approx 1\%$	$\Leftrightarrow$	-5	$10^{-7.5}$	$\Leftrightarrow$	-7.5	16	$\Leftrightarrow$	1
4	$\Leftrightarrow$	2	$e^{-3} \approx 5\%$	$\Leftrightarrow$	-3	$10^{-5.0}$	$\Leftrightarrow$	-5.0	256	$\Leftrightarrow$	2
8	$\Leftrightarrow$	3	$e^{-1} \approx 37\%$	$\Leftrightarrow$	-1	0.5	$\Leftrightarrow$	-0.3	4096	$\Leftrightarrow$	3
16	$\Leftrightarrow$	4	1	$\Leftrightarrow$	0	1	$\Leftrightarrow$	0	64K	$\Leftrightarrow$	4
32	$\Leftrightarrow$	5	e	$\Leftrightarrow$	1	2	$\Leftrightarrow$	0.3	1M	$\Leftrightarrow$	5
64	$\Leftrightarrow$	6	$e^3 \approx 20$	$\Leftrightarrow$	3	$10^{5.0}$	$\Leftrightarrow$	5.0	16M	$\Leftrightarrow$	6
128	$\Leftrightarrow$	7	$e^5 \approx 150$	$\Leftrightarrow$	5	$10^{7.5}$	$\Leftrightarrow$	7.5	256M	$\Leftrightarrow$	7

Logarithms

$$N = b^x \quad \Leftrightarrow \quad \boxed{\log_b N = x}$$

where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

```
>> log(.05)

ans =   -2.9957

>> log(.37)

ans =   -0.9943

>> log(exp(1))

ans =         1

>> log(20)

ans =    2.9957
```

$b = e \approx 2.718$

("base-e")

N	$\Leftrightarrow$	x
$e^{-5} \approx 1\%$	$\Leftrightarrow$	-5
$e^{-3} \approx 5\%$	$\Leftrightarrow$	-3
$e^{-1} \approx 37\%$	$\Leftrightarrow$	-1
1	$\Leftrightarrow$	0
e	$\Leftrightarrow$	1
$e^{3} \approx 20$	$\Leftrightarrow$	3
$e^{5} \approx 150$	$\Leftrightarrow$	5

$b = 10$

("base-10")

N	$\Leftrightarrow$	x
$10^{-7.5}$	$\Leftrightarrow$	-7.5
$10^{-5.0}$	$\Leftrightarrow$	-5.0
0.5	$\Leftrightarrow$	-0.3
1	$\Leftrightarrow$	0
2	$\Leftrightarrow$	0.3
$10^{5.0}$	$\Leftrightarrow$	5.0
$10^{7.5}$	$\Leftrightarrow$	7.5

```
>> log10(.00001)

ans =          -5

>> log10(.5)

ans =     -0.3010

>> log10(2)

ans =      0.3010

>> log10(1e5)

ans =           5
```

Arithmetic with Logarithms

$$N = b^x \quad \Leftrightarrow \quad \log_b N = x$$

where N = positive number ("linear value")
b = the **base** of the logarithm
x = the **exponent** of the logarithm

$$\log_b(x \cdot y) = \log_b x + \log_b y$$

$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b(x) = \frac{\log_n(x)}{\log_n(b)}$$

Examples:

$$\log_2(4 \cdot 16) =$$

$$\log_3(81/9) =$$

$$10^x = 3 \cdot 10^6$$

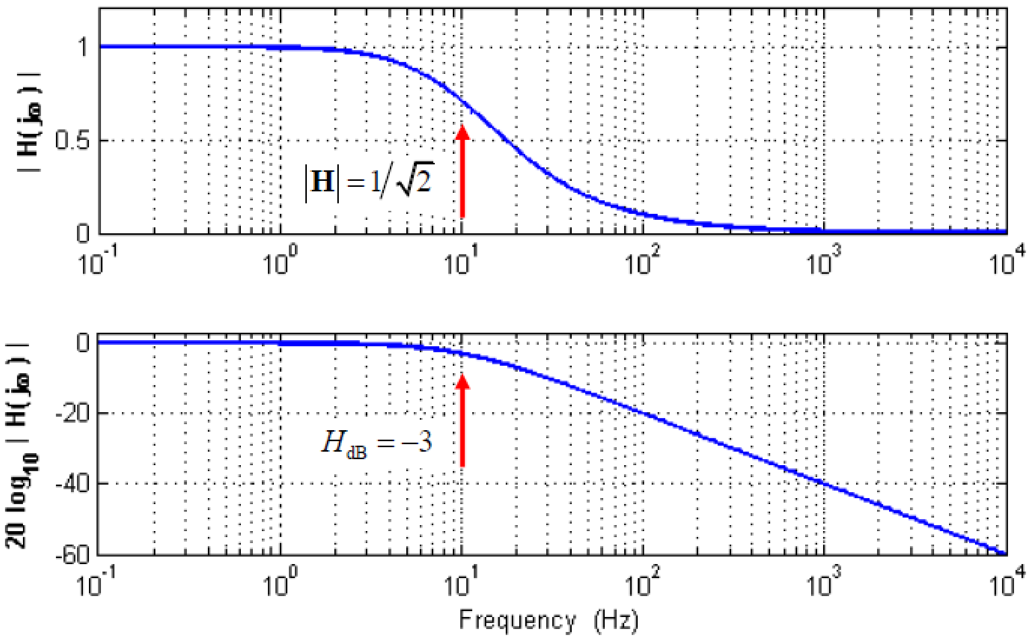
Decibels

The **decibel** scale is a logarithmic scale that uses base $b = 10$.

$$\log_{10}(N) = x$$

By convention, a transfer function in decibels is

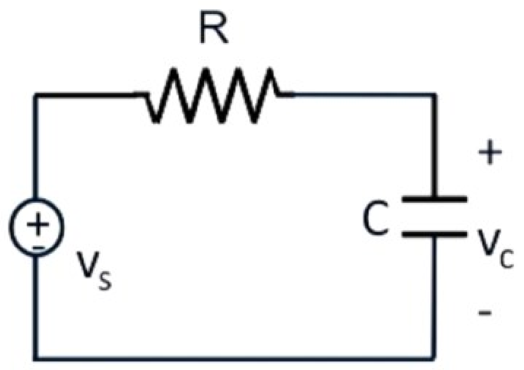
$$20 \cdot \log_{10}|\mathbf{H}(j\omega)| = H_{dB}$$



$\lvert\mathbf{H}\rvert$		H_{dB}
10^3	$\Leftrightarrow$	$+60$
10^2	$\Leftrightarrow$	$+40$
10	$\Leftrightarrow$	$+20$
2	$\Leftrightarrow$	$+6$
$\sqrt{2} \approx 1.4$	$\Leftrightarrow$	$+3$
1	$\Leftrightarrow$	0
$1/\sqrt{2} \approx 0.7$	$\Leftrightarrow$	-3
0.5	$\Leftrightarrow$	-6
0.1	$\Leftrightarrow$	-20
10^{-2}	$\Leftrightarrow$	-40
10^{-3}	$\Leftrightarrow$	-60

Frequency Plot - Linear

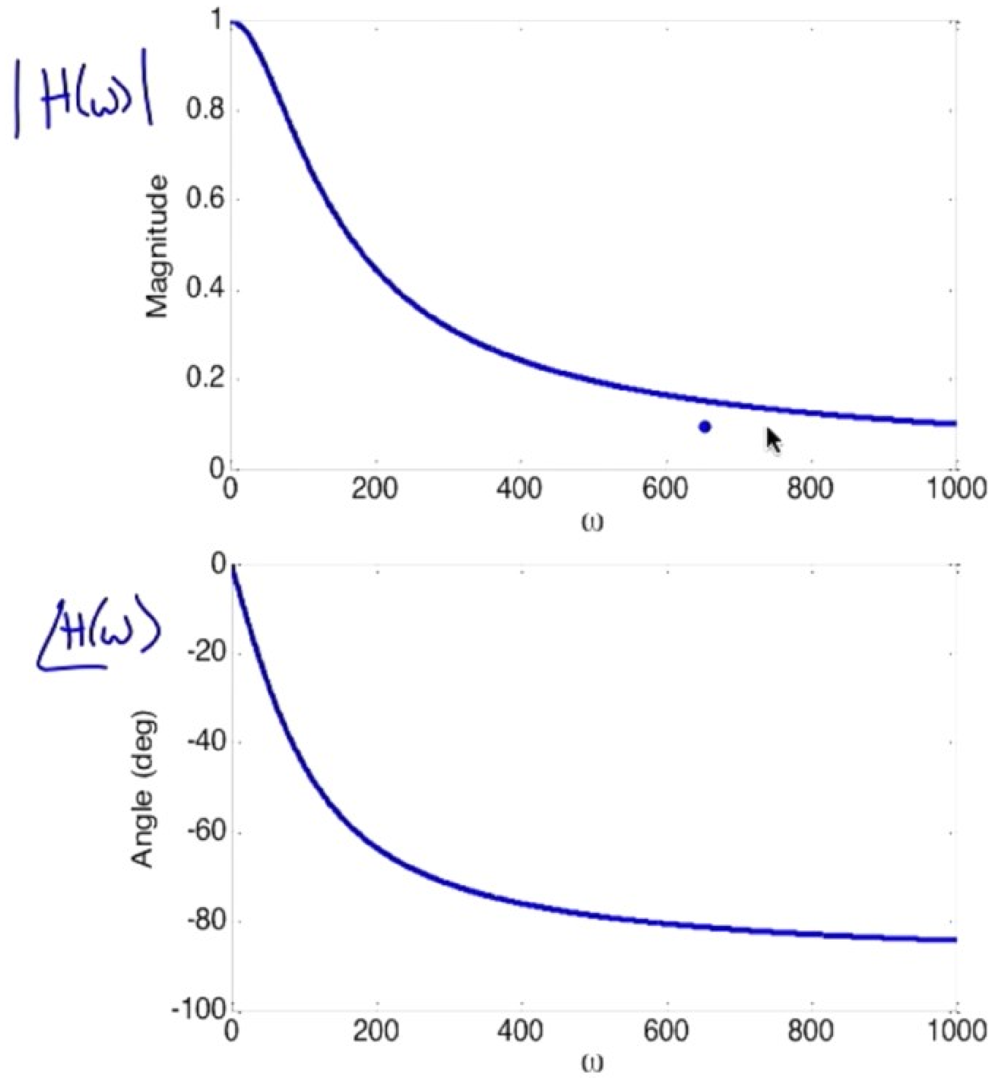
Transfer Function



$$H(\omega) = \frac{1}{1 + j\omega RC}$$

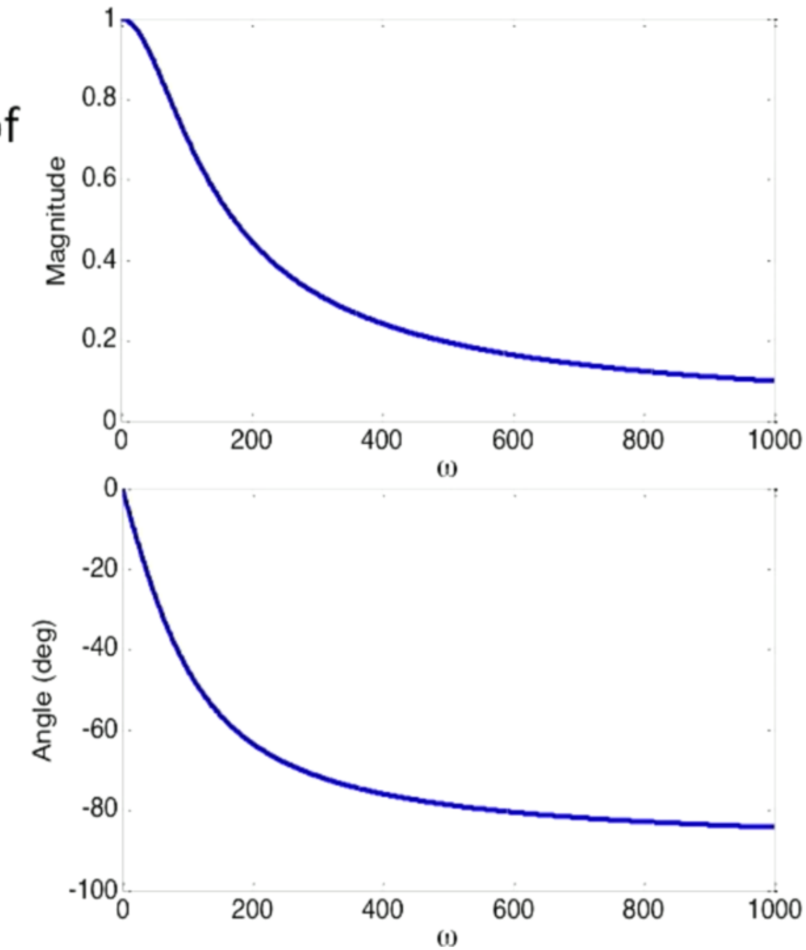
$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$



Example 7

A circuit has the frequency response plot shown. What is steady-state response, $v_o(t)$, to an input of $v_{in}(t) = 2 + \cos(200t)$?



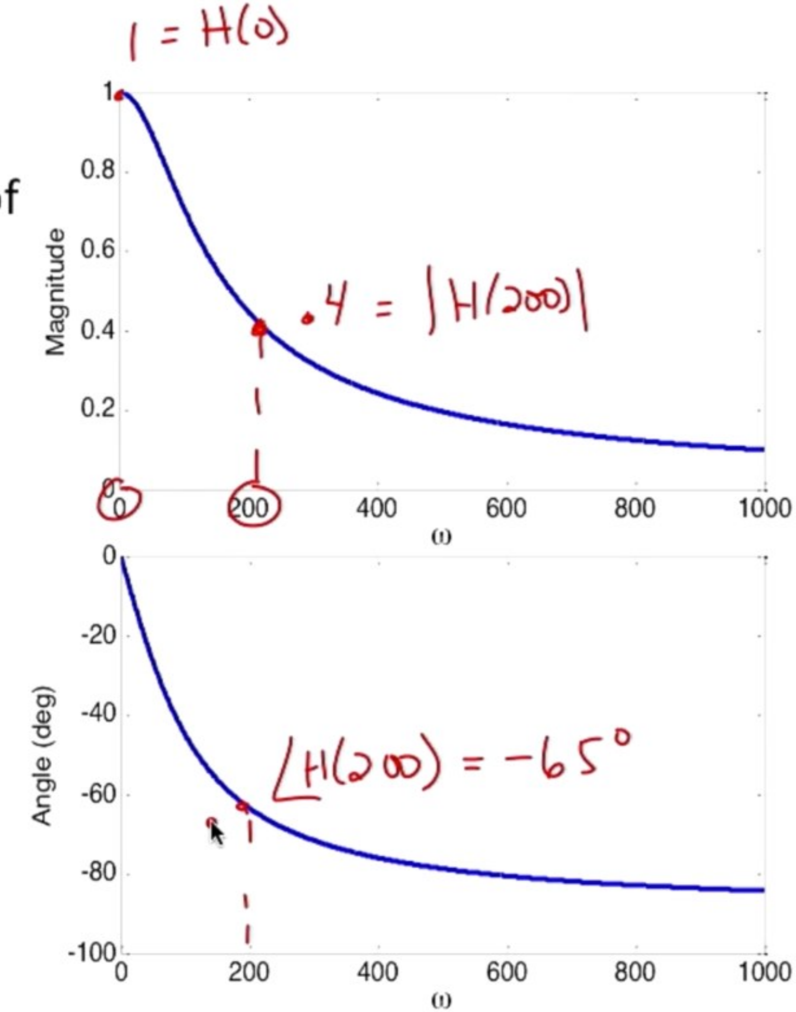
Solution 7

A circuit has the frequency response plot shown. What is steady-state response, $v_o(t)$, to an input of $v_{in}(t) = 2 + \cos(200t)$?

DC

$\omega = 200$

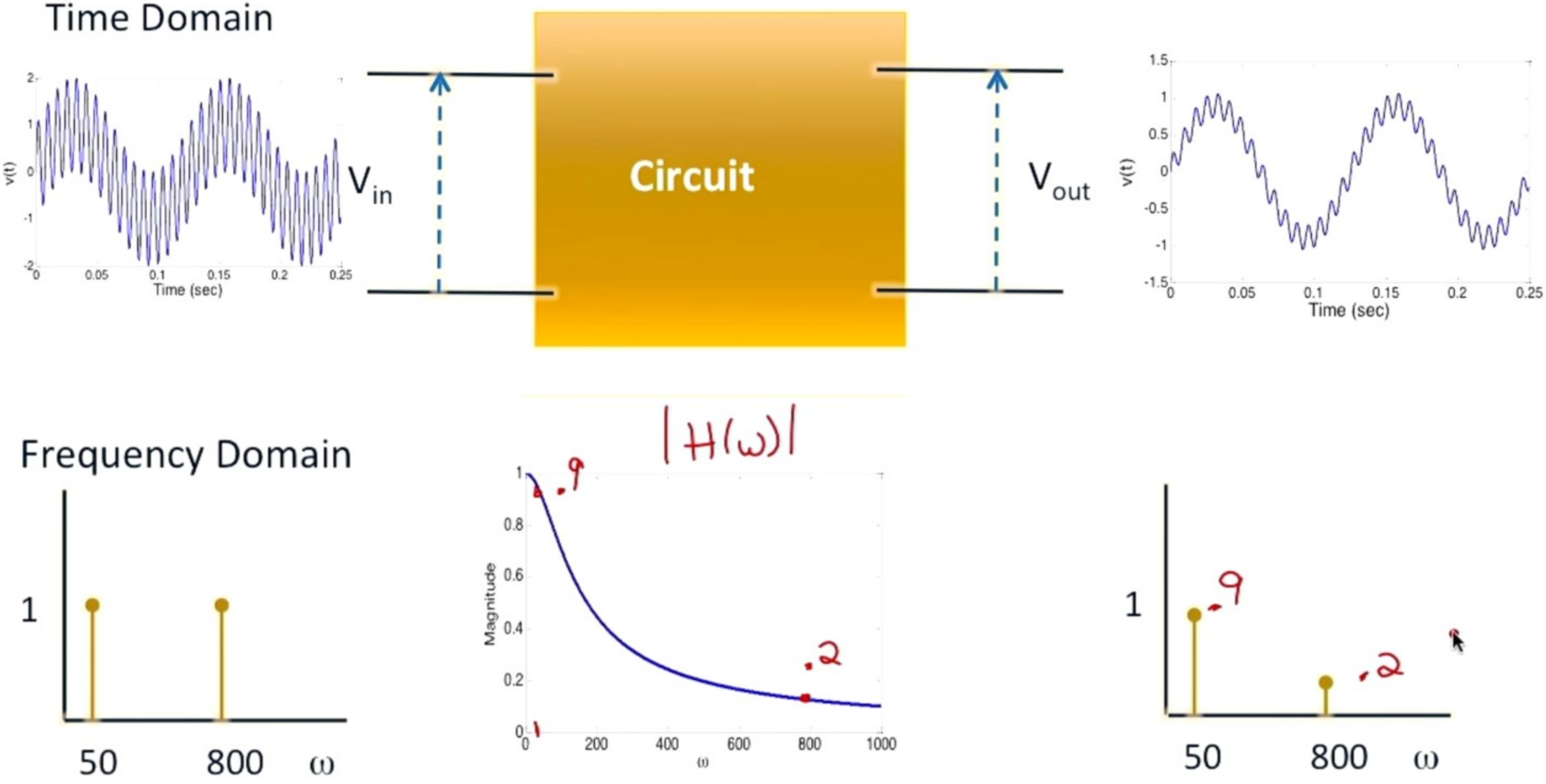
$V_{out}(t) = 2 + 0.4\cos(200t - 65°)$



Finding Output Using Linear Plot

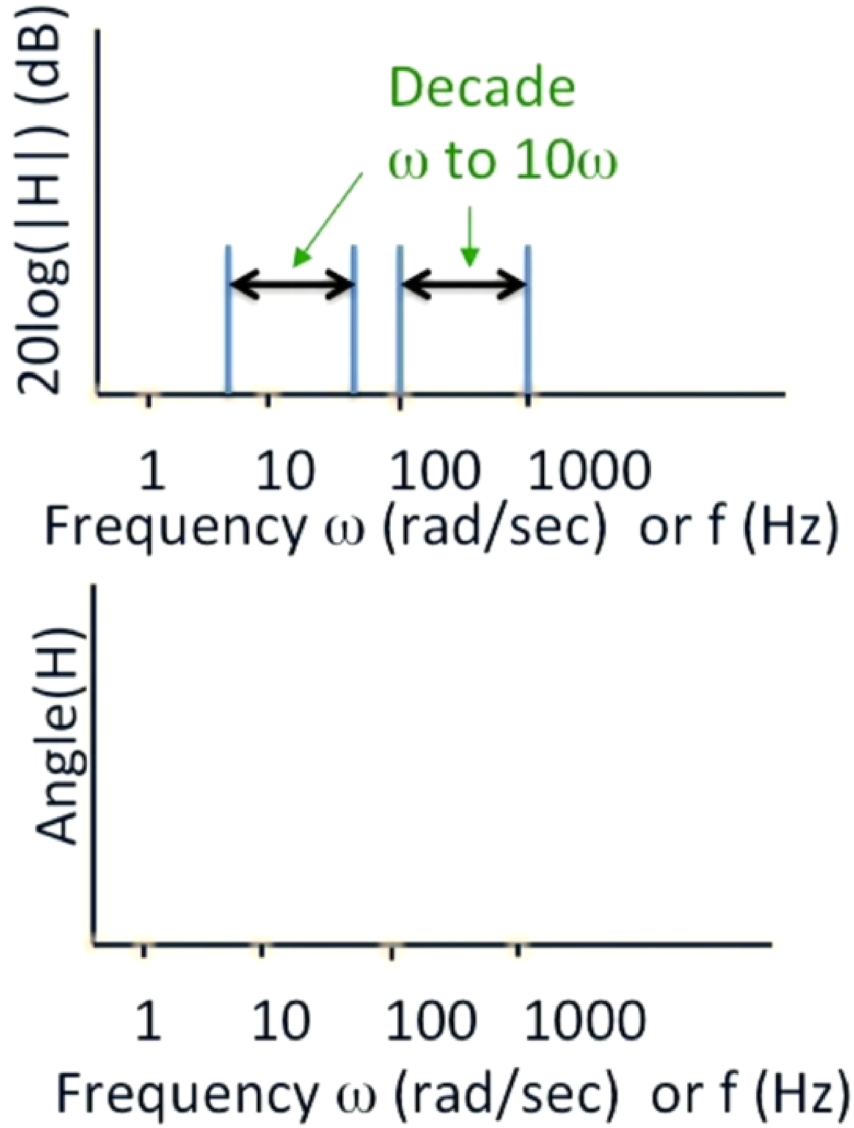
Circuit Response

Time Domain

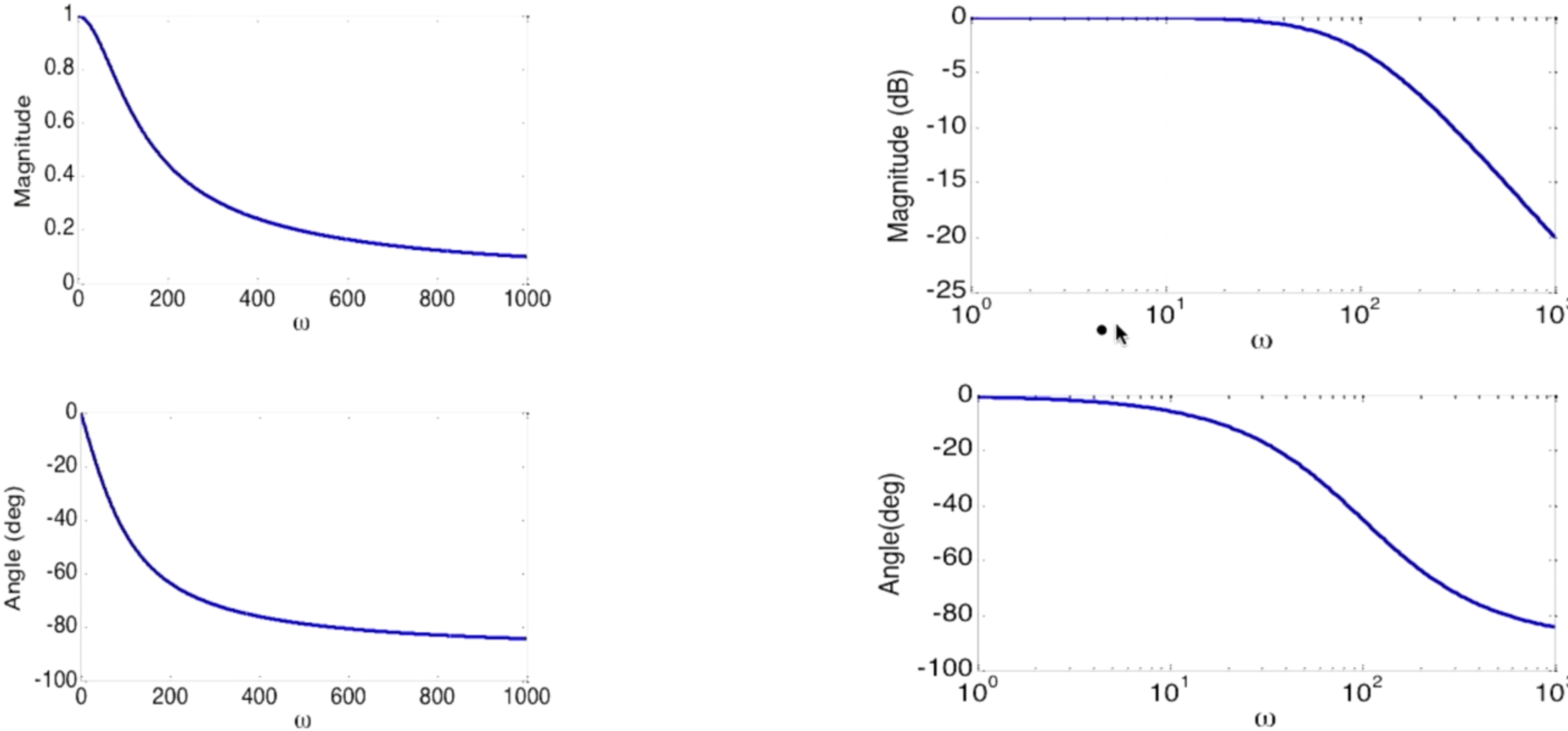


Review Finding Output Using Bode Plot

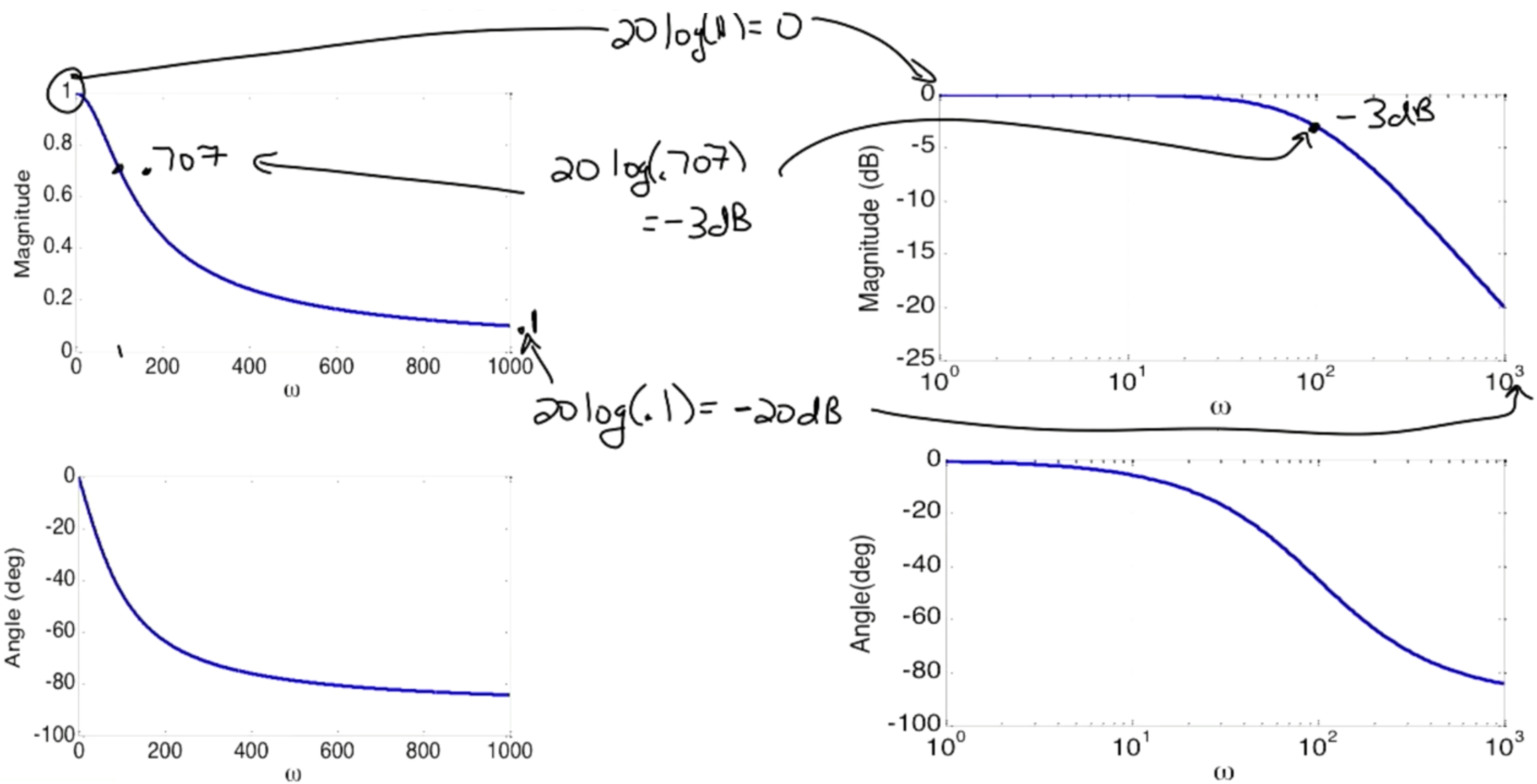
Bode Plot



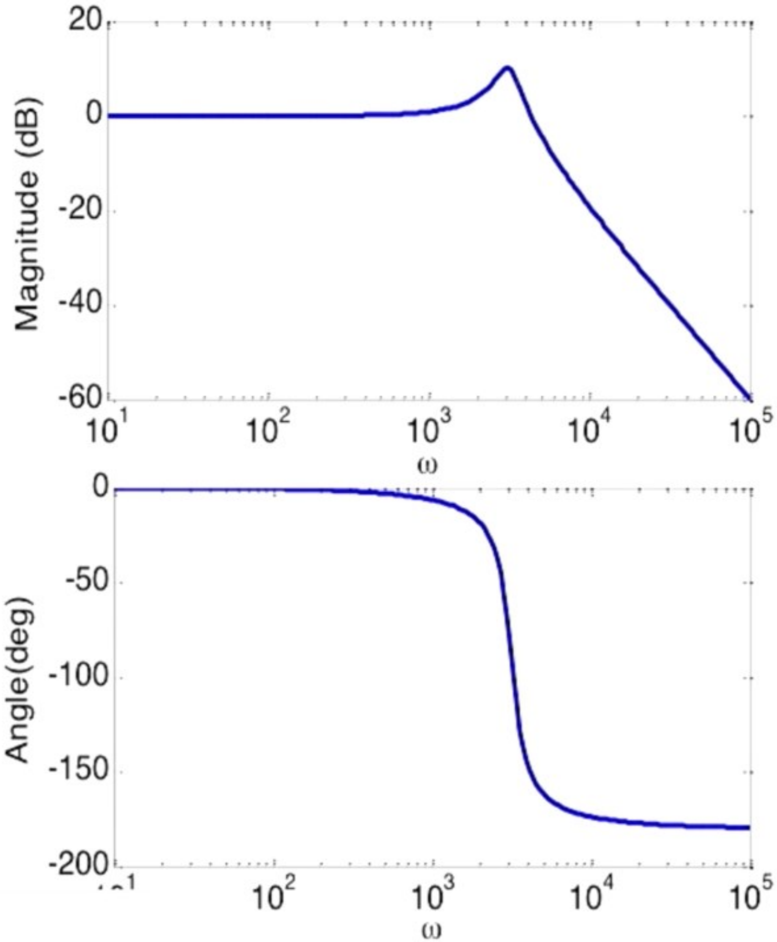
Linear Plot versus Bode Plot



Review of Circuit Analysis with AC Impedances

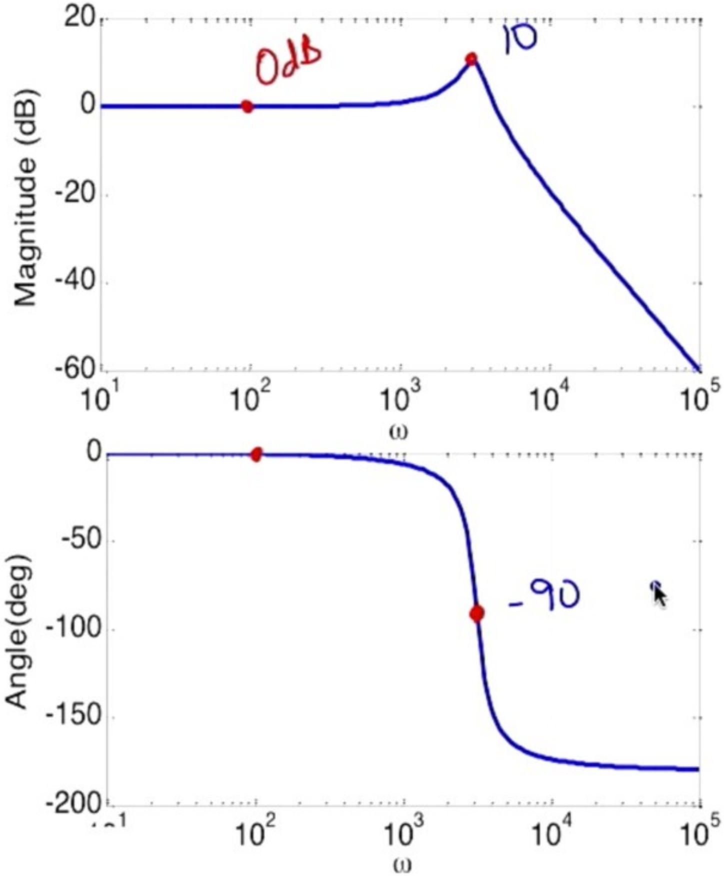


Example 8



A circuit has the Bode plot shown. What is the steady-state circuit output for an input of $v_{in}(t)=1+\cos(100t-45^{\circ})+\cos(3000t)$?

Solution 8



A circuit has the Bode plot shown. What is the steady-state circuit output for an input of $v_{in}(t)=1+\cos(100t-45^o)+\cos(3000t)$?

DC

0dB

$\omega=100$

$\omega=3000$

$$A_o = A_n\,|H(\omega)|$$

$$20\log|H(\omega)| = |H(\omega)|_{DB}$$

$$|H(\omega)| = 10^{|H(\omega)|_{DB}/20}$$

at $\omega=0$ 0dB $\rightarrow |H(0)|=1$

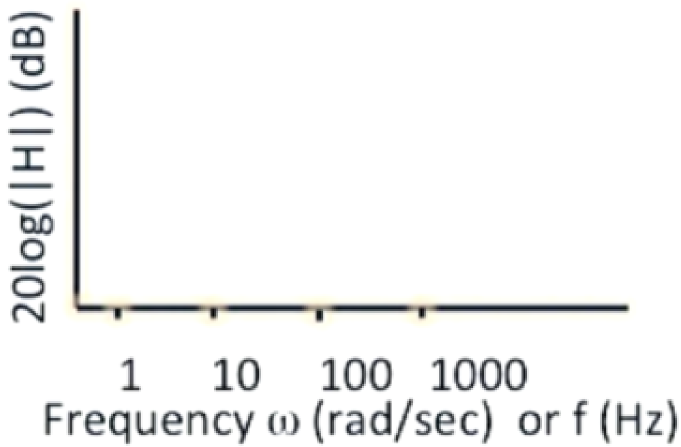
$\omega=100$ 0dB $\rightarrow |H(100)|=1$

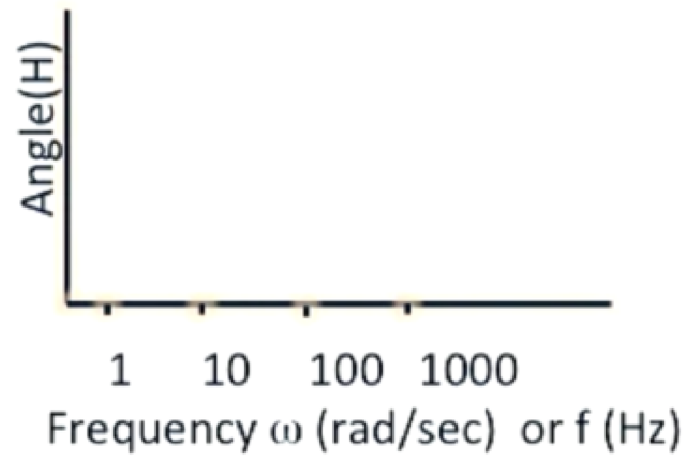
$\omega=3000$ 10dB $\rightarrow |H(3000)| = 10^{10/20} = 3.16$

$$V_o(t) = 1 + \cos(100t-45^\circ) + 3.16\cos(3000t-90^\circ)$$

Key Concepts

- Bode plot



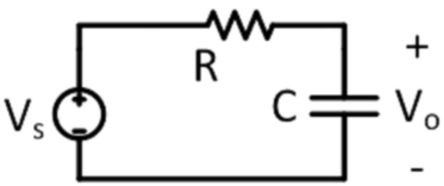


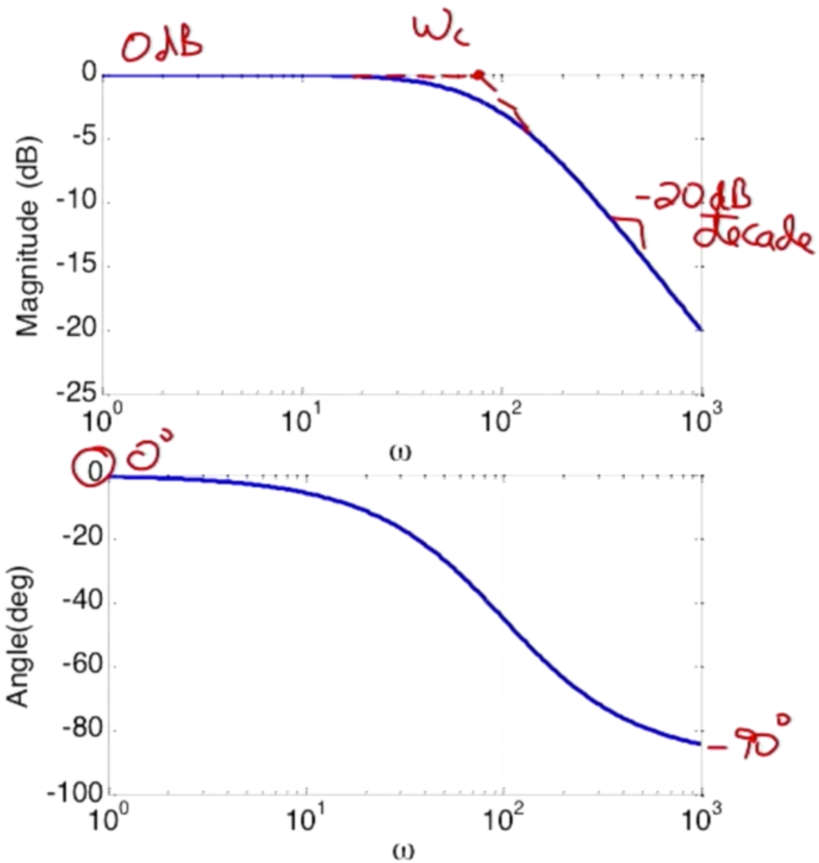
- Input – Output Behavior

$$A_{out} = |H(\omega)| A_{in} \qquad \theta_{out} = \angle H(\omega) + \theta_{in}$$

$$\text{If } |H(\omega)|_{dB} = 20\log(|H(\omega)|) \quad \text{then} \quad |H(\omega)| = 10^{|H(\omega)|_{dB}/20}$$

Bode Plot of RC Circuits





$$H(\omega) = \frac{1}{1 + j\omega RC}$$

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)$$

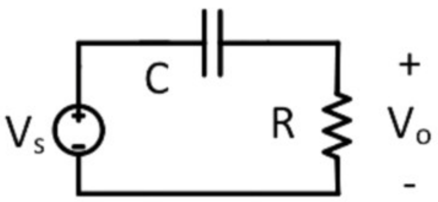
Low Frequency:
Magnitude: 0dB
Angle: 0°

High Frequency:
Magnitude: -20dB/decade
Angle: -90°

Corner Frequency:

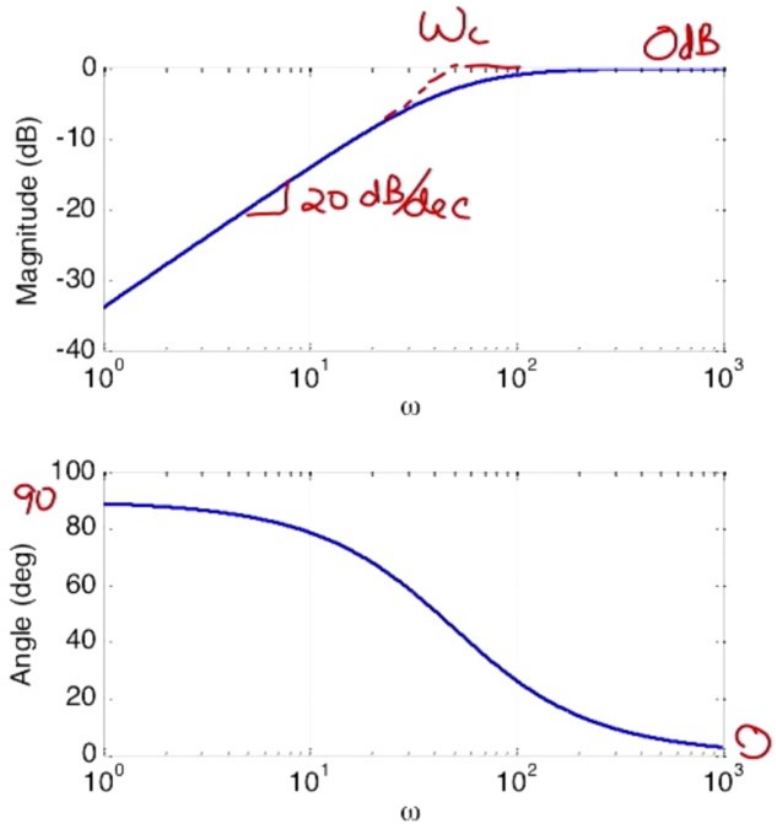
$$\omega_c = \frac{1}{RC}\ \text{rad/sec}$$

Bode Plot of RC Circuits



$$\frac{R}{R+\frac{1}{j\omega C}}$$

$$H(\omega) = \frac{j\omega RC}{1 + j\omega RC}$$



Low Frequency:
Magnitude: +20dB/decade
Angle: 90°

High Frequency:
Magnitude: 0dB
Angle: 0°

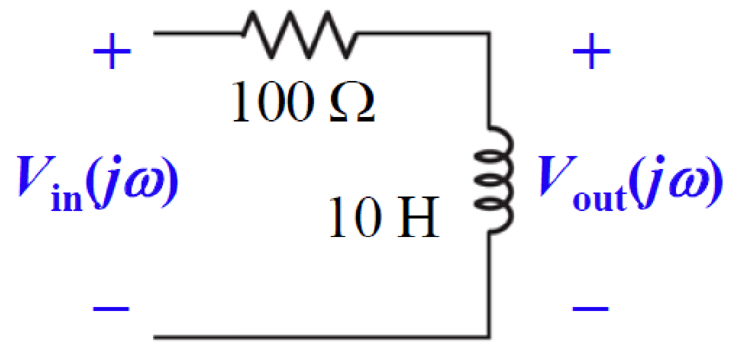
Corner Frequency:

$$\omega_c = \frac{1}{RC} \text{ rad/sec}$$

Bode Plot of RL Circuits

Determine the transfer function, $\mathbf{H}(j\omega)$ and plot $|\mathbf{H}(j\omega)|$ and $20\log_{10}|\mathbf{H}(j\omega)|$ vs. ω for

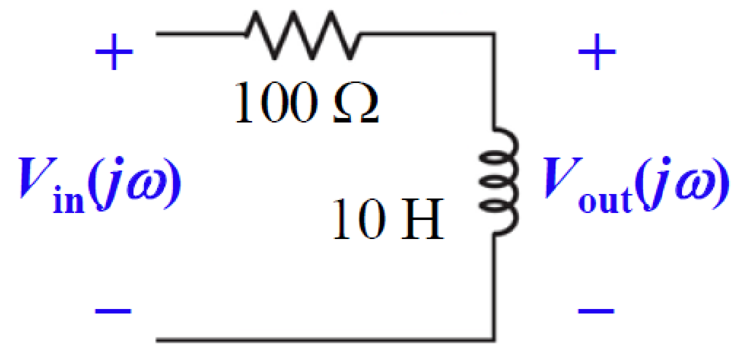
$$0.1\,\text{rad/s} < \omega < 1\,\text{krad/s}$$



Bode Plot of RL Circuits

Determine the transfer function, $\mathbf{H}(j\omega)$ and plot $|\mathbf{H}(j\omega)|$ and $20\log_{10}|\mathbf{H}(j\omega)|$ vs. ω for

$$0.1\text{ rad/s} < \omega < 1\text{ krad/s}$$




```
w = logspace(-1,3,1001);
H = j*w ./ (j*w + 10);

subplot(2,1,1)
semilogx(w,abs(H),'Linewidth',2)
axis([10^-1 10^3 0 1.1])
grid
ylabel('|H(j\omega)|');
xlabel('Frequency, \omega  (rad/s)')

subplot(2,1,2)
semilogx(w,20*log10(abs(H)),...
         'Linewidth',2)
axis([10^-1 10^3 -40 10])
grid
ylabel('20log_{10}( |H(j\omega)| )');
xlabel('Frequency, \omega  (rad/s)')
```

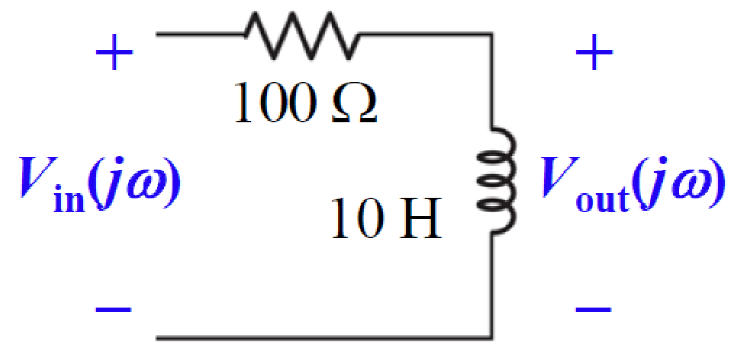
$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{j\omega}{j\omega + 10}$$

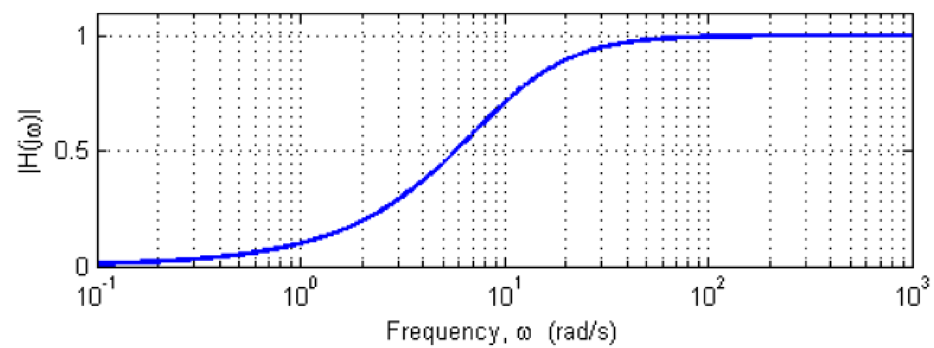
$$|\mathbf{H}(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + 10^2}}$$

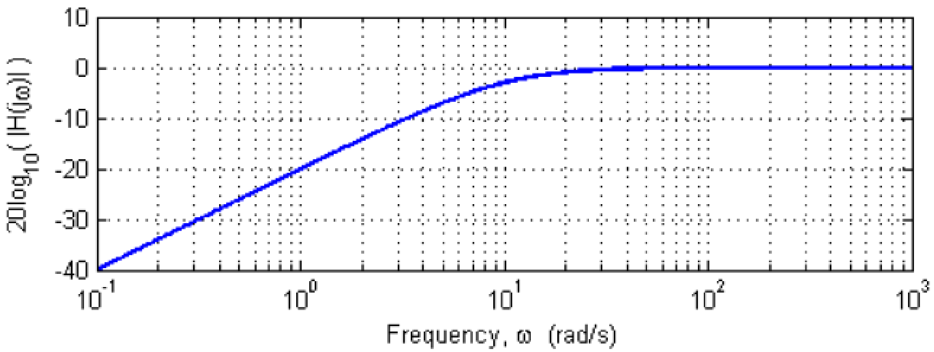
Bode Plot of RL Circuits

Determine the transfer function, $\mathbf{H}(j\omega)$ and plot $|\mathbf{H}(j\omega)|$ and $20\log_{10}|\mathbf{H}(j\omega)|$ vs. ω for

$$0.1\,\text{rad/s} < \omega < 1\,\text{krad/s}$$



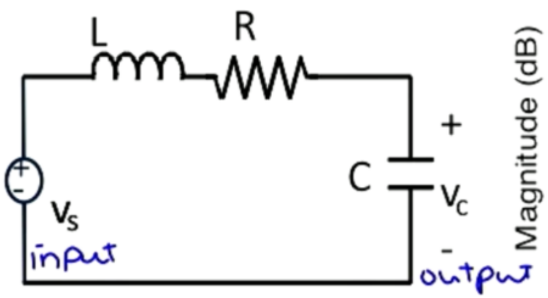




$$\mathbf{H}(j\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_{in}} = \frac{j\omega}{j\omega + 10}$$

$$|\mathbf{H}(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + 10^2}}$$

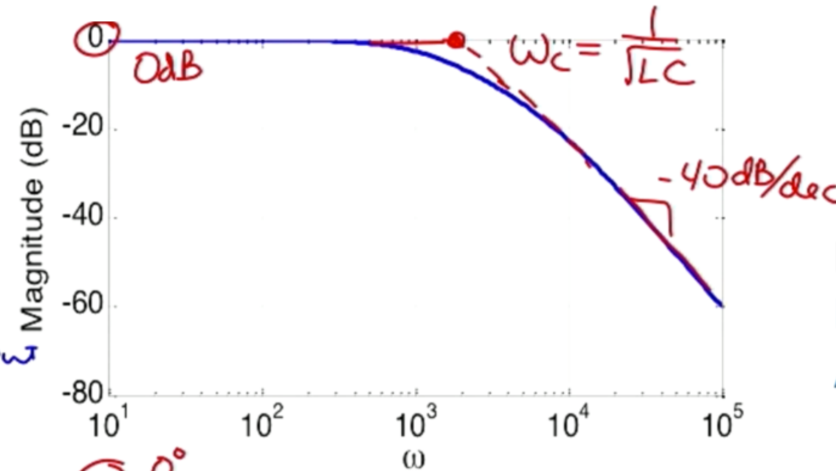
Bode Plot of RLC Circuits, Overdamped

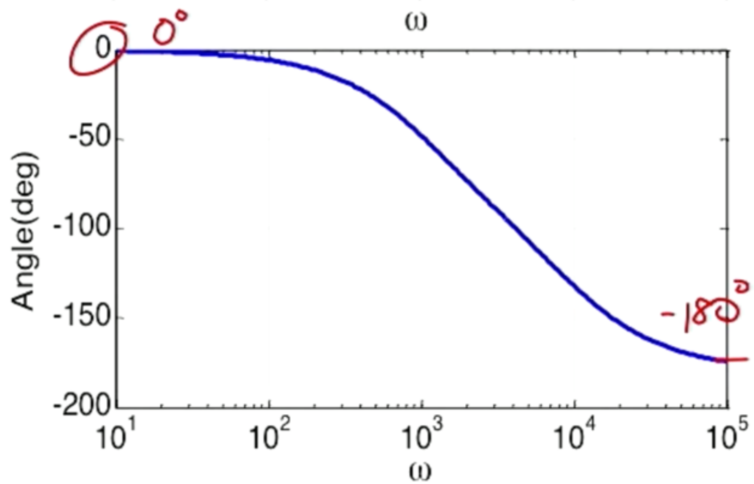


$$H(\omega) = \frac{1}{(1 - LC\omega^2) + RCj\omega}$$

$20 \log |H(\omega)|$

$\angle H(\omega)$





Low Frequency:

Magnitude: 0dB $\Rightarrow \frac{A_o}{A_i} = 1$

Angle: 0°

High Frequency:

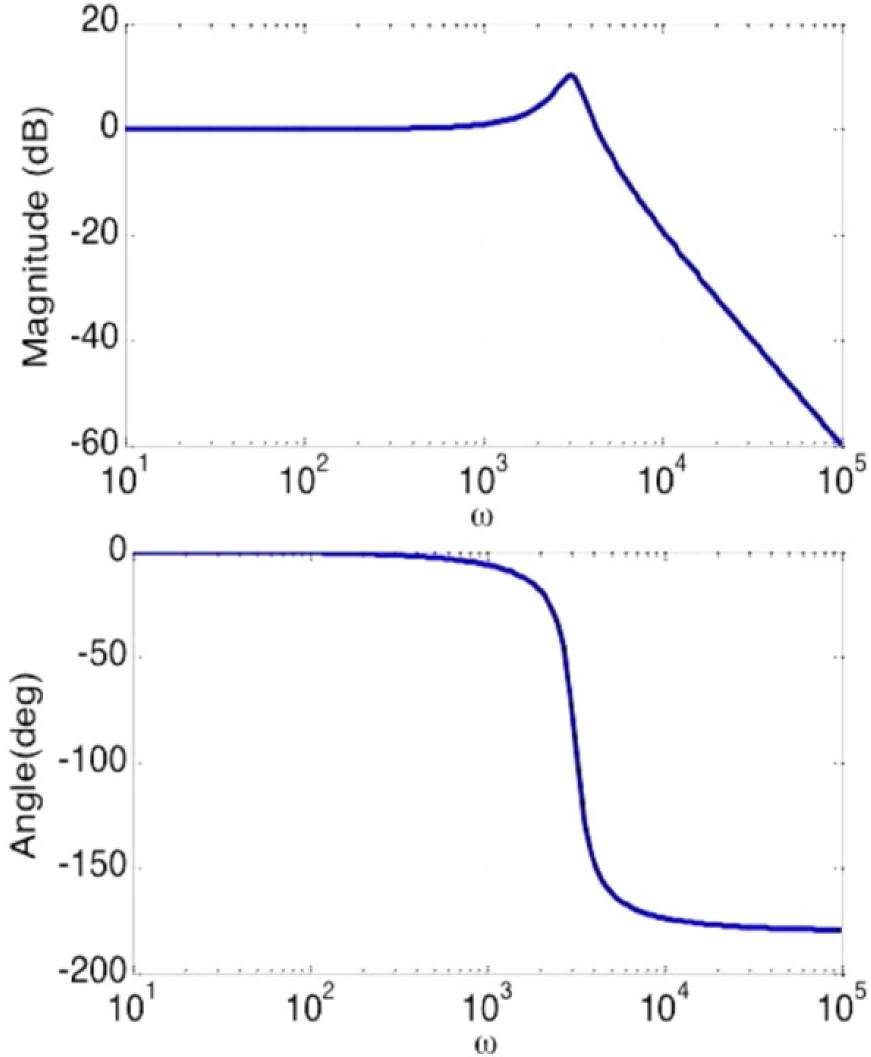
Magnitude: -40dB/decade

Angle: -180°

Corner Frequency:

$$\omega_c = \frac{1}{\sqrt{LC}}$$

Bode Plot of RLC Circuits, Underdamped



Low Frequency:

Magnitude: 0dB

Angle: 0°

High Frequency:

Magnitude: -40dB/decade

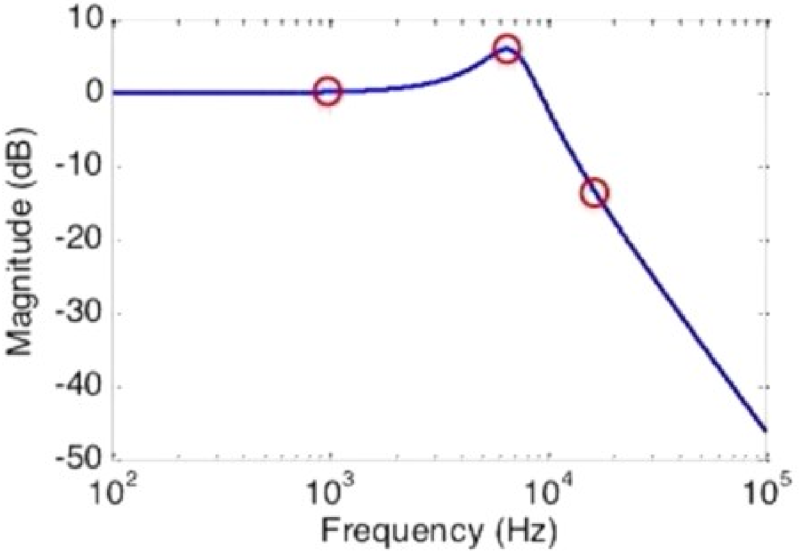
Angle: -180°

Corner Frequency:

$$\omega_c = \frac{1}{\sqrt{LC}}$$

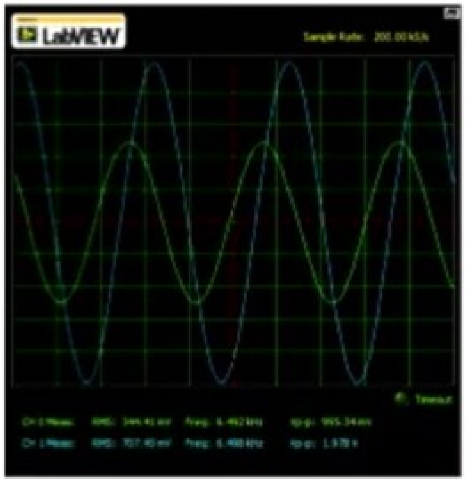
Resonance

RLC with low damping has resonant peak where $\dfrac{A_o}{A_i}$ is max.



green – input, blue - output

f=1kHz

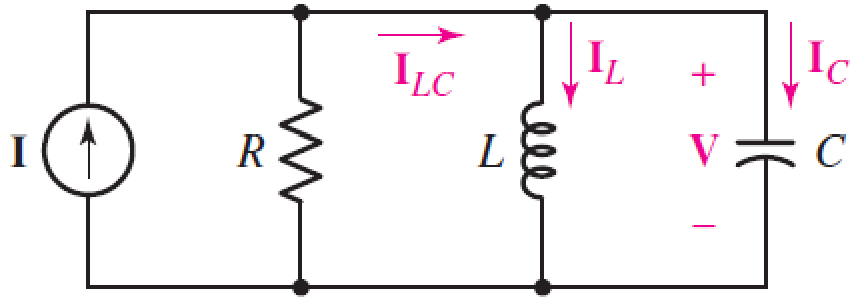


f=6.5kHz

f=15kHz

Parallel Resonant Circuit

➢ Resonance is a condition in an RLC circuit in which the capacitive and inductive reactance are equal in magnitude, thereby resulting in purely resistive impedance.



The steady-state admittance offered to the ideal current source is

$$\mathbf{Y} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

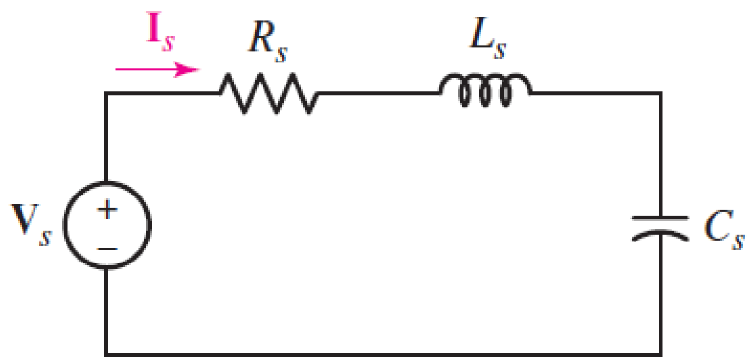
Resonance occurs when the voltage and current at the input terminals are in phase.

$$\omega C - \frac{1}{\omega L} = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{rad/s} \qquad\qquad f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \text{Hz}$$

resonant frequency ω_0

Series Resonant Circuit



Network Function

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{R + j\omega L + \dfrac{1}{j\omega C}}$$

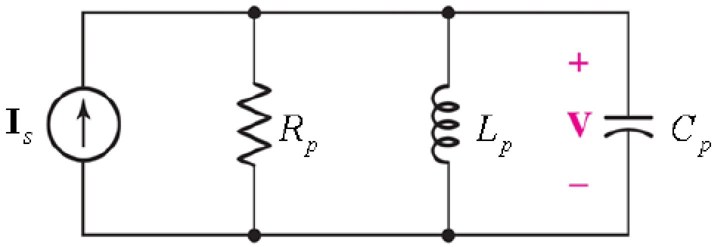
Resonance occurs when the voltage and current at the input terminals are in phase.

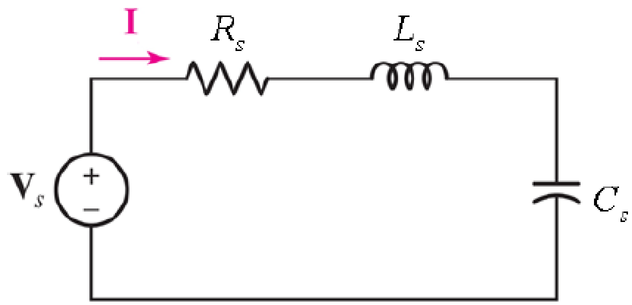
$$\mathbf{Y} = \frac{1}{R + j\left(\omega L - \dfrac{1}{\omega C}\right)}$$

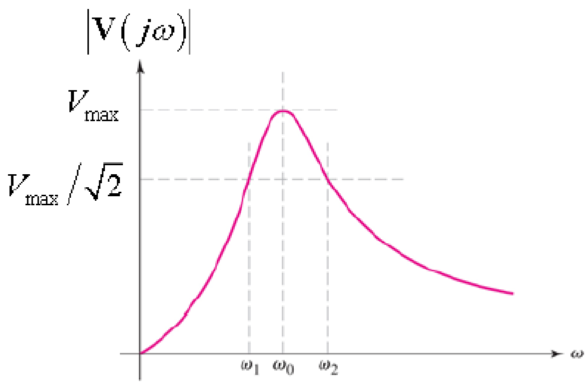
$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{rad/s} \qquad\qquad f_0 = \frac{1}{2\pi\sqrt{LC}} \quad \text{Hz}$$

resonant frequency ω_0

Quality Factor

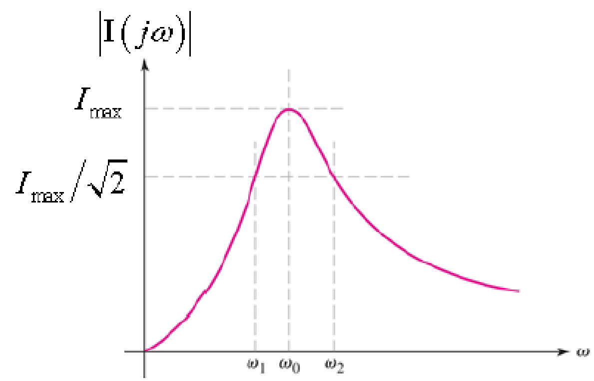






$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$Q = \frac{\omega_0}{\omega_2 - \omega_1}$$



$$Q = \omega_0 RC$$

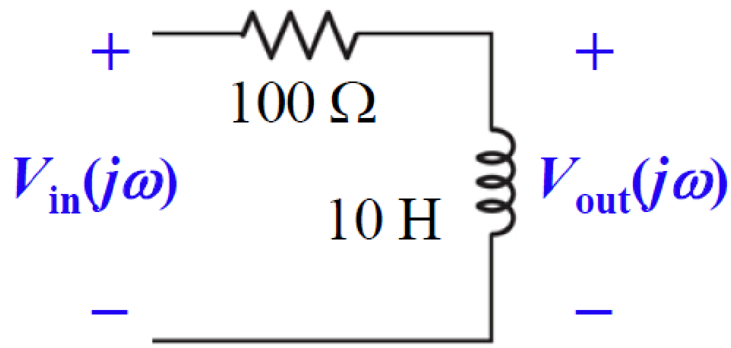
quality factor, Q ("Q factor", unitless)

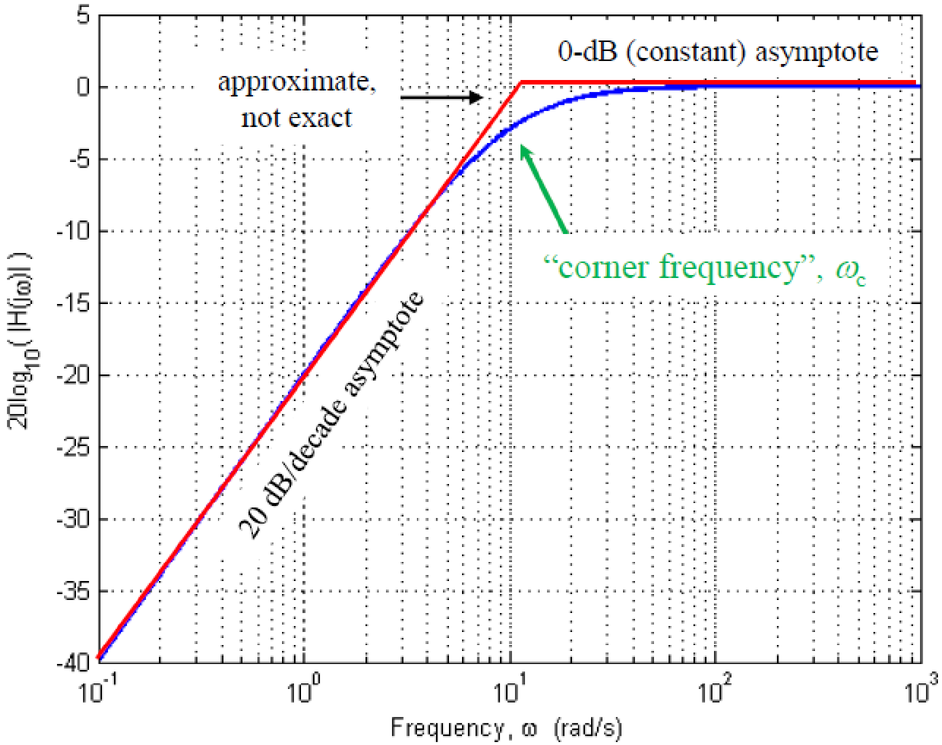
higher Q → more narrow resonance, slower transient decay

$$Q = \omega_0 \frac{L}{R}$$

How to Create Bode Amplitude Plot

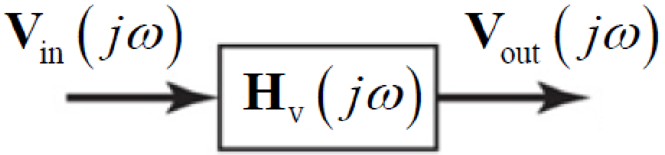
A **Bode amplitude plot** is an approximation to the logarithmic plot $20\log_{10}|\mathbf{H}(j\omega)|$ vs. ω drawn using the *asymptotes* of the exact transfer function.





(Voltage) Transfer Function

Let $\mathbf{V}_{\text{in}}(j\omega)$ be the phasor form of the voltage input to a circuit, and let $\mathbf{V}_{\text{out}}(j\omega)$ be the phasor form of the voltage output from a circuit, expressed as a ratio:



$$\mathbf{H}_{\text{v}}(j\omega) = \frac{\mathbf{V}_{\text{out}}(j\omega)}{\mathbf{V}_{\text{in}}(j\omega)}$$

$\mathbf{H}_{\text{v}}(j\omega)$ may generally be written as a function of one (factored) polynomial $\mathbf{N}(j\omega)$ divided by another (factored) polynomial $\mathbf{D}(j\omega)$.

$$\mathbf{H}_{\text{v}}(j\omega) = \frac{\mathbf{N}(j\omega)}{\mathbf{D}(j\omega)} = j\omega_0 \frac{(1+j\omega_1)(1+j\omega_2)(1+j\omega_3)\ldots(1+j\omega_N)}{(1+j\omega_\alpha)(1+j\omega_\beta)(1+j\omega_\chi)\ldots(1+j\omega_Z)}$$

zeros of $\mathbf{H}(j\omega)$ are values of $j\omega$ for which $\mathbf{N}(j\omega) = 0$ $\rightarrow$ $-j\omega_1, -j\omega_2, -j\omega_3, \ldots -j\omega_N$

poles of $\mathbf{H}(j\omega)$ are values of $j\omega$ for which $\mathbf{D}(j\omega) = 0$ $\rightarrow$ $-j\omega_\alpha, -j\omega_\beta, -j\omega_\chi, \ldots -j\omega_Z$

Standard Form

➢ A transfer function may be written in terms of factors that have real and imaginary parts.

$$H(\omega)=\frac{N(\omega)}{D(\omega)}=\frac{K(j\omega)^{\pm 1}\left(1+{j\omega}/{z_1}\right)\left[1+{j2\varsigma_1\omega}/{\omega_k}+\left({j\omega}/{\omega_k}\right)^2\right]\cdots}{\left(1+{j\omega}/{p_1}\right)\left[1+{j2\varsigma_1\omega}/{\omega_n}+\left({j\omega}/{\omega_n}\right)^2\right]\cdots}$$

➢ This representation is called the STANDARD FORM. It has several different factors. We can draw the Bode plots by plotting each of the terms of the transfer function separately and then adding them.

The different factors of the transfer function are

1.) Gain term K.

2.) A pole $(j\omega)^{-1}$ or a zero $(j\omega)$ at the origin.

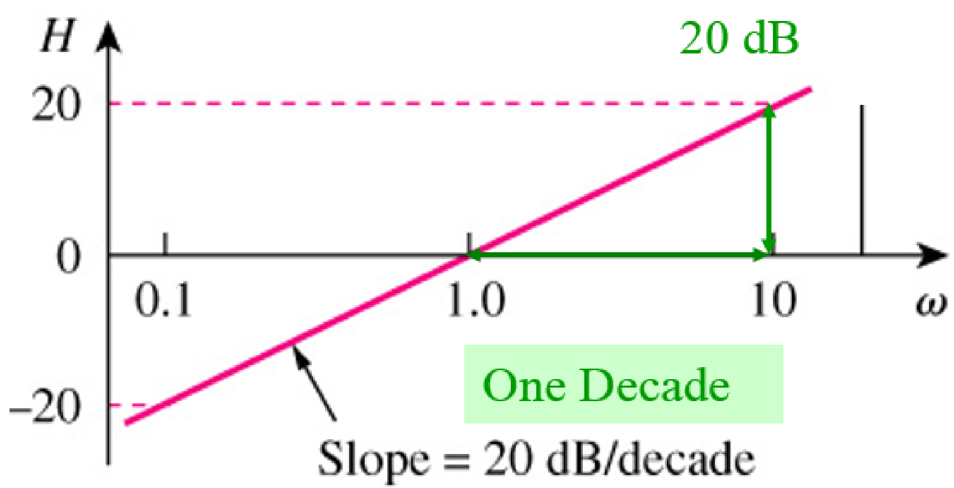
3.) A simple pole $\left(1+{j\omega}/{p_1}\right)$ or zero $\left(1+{j\omega}/{z_1}\right)$

4.) A quadratic pole $\left[1+{j2\varsigma_1\omega}/{\omega_n}+\left({j\omega}/{\omega_n}\right)^2\right]$ or zero $\left[1+{j2\varsigma_1\omega}/{\omega_k}+\left({j\omega}/{\omega_k}\right)^2\right]$

Decade

- A DECADE is an interval between 2 frequencies with a ratio of 10 (between 10 Hz and 100 Hz or between 500 Hz and 5000 Hz). 20 dB/decade means that magnitude changes 20 dB whenever the frequency changes tenfold or one decade.

- Slopes are expressed in dB/decade.



Bode Plot Procedure

- To plot the Bode plots of a given transfer function.

1.) Put the transfer function in STANDARD FORM.
2.) Write the Magnitude and phase equations from the STANDARD FORM.
3.) Plot the magnitude of each term separately.
4.) Add all magnitude terms to obtain the magnitude transfer function.
5.) Repeat 2-4 for the phase response.
6.) The total magnitude response in Decibel units is the summation and subtraction of the responses of different terms.
7.) The total phase response in degrees is the summation and subtraction of the phase responses of different terms.

STANDARD FORM

$$H(\omega)=\frac{0.4(1+j\omega/10)}{j\omega(1+j\omega/5)^2}$$

$$\begin{aligned}H_{db} &= 20\log_{10}0.4+\\ &20\log_{10}\left|1+j\omega/10\right|-\\ &20\log_{10}\left|j\omega\right|-\\ &40\log_{10}\left|1+j\omega/5\right|\end{aligned}$$

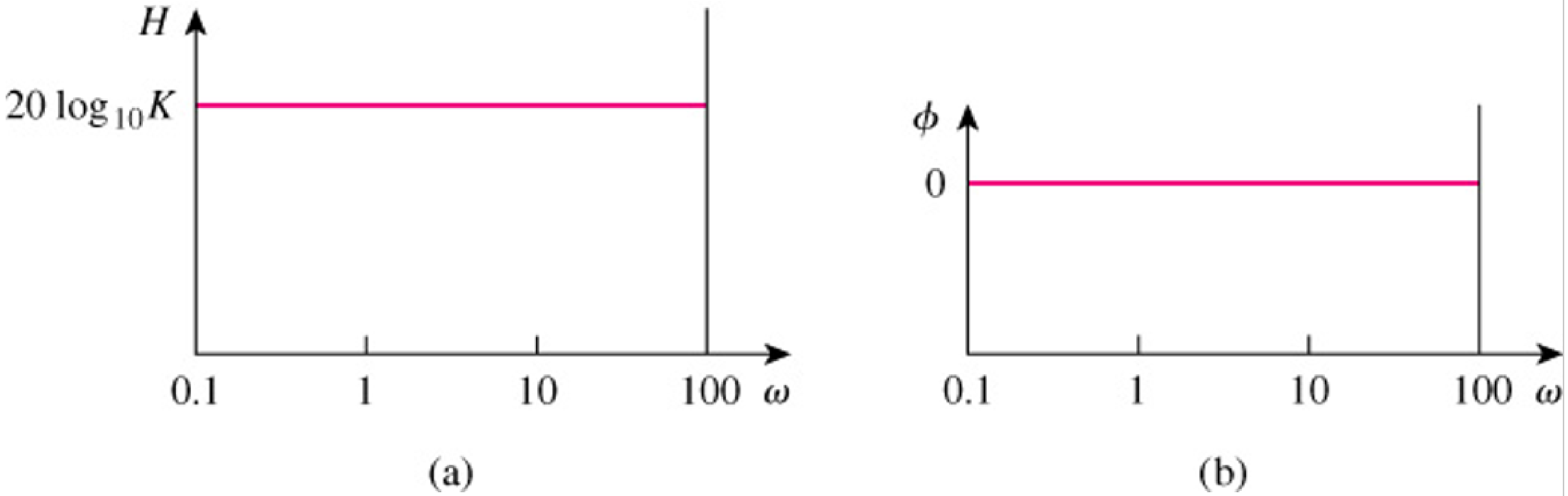
$$\begin{aligned}\phi &= 0^\circ+\tan^{-1}(\omega/10)\\ &-90^\circ-2\tan^{-1}(\omega/5)\end{aligned}$$

Term Types

- We examine how to plot different terms that may appear in a transfer function. The total response will be obtained by adding all the responses.

$$H(\omega)=\frac{N(\omega)}{D(\omega)}=\frac{K(j\omega)^{\pm1}\left(1+j\omega/z_1\right)\left[1+j2\varsigma_1\omega/\omega_k+\left(j\omega/\omega_k\right)^2\right]\cdots}{\left(1+j\omega/p_1\right)\left[1+j2\varsigma_1\omega/\omega_n+\left(j\omega/\omega_n\right)^2\right]\cdots}$$

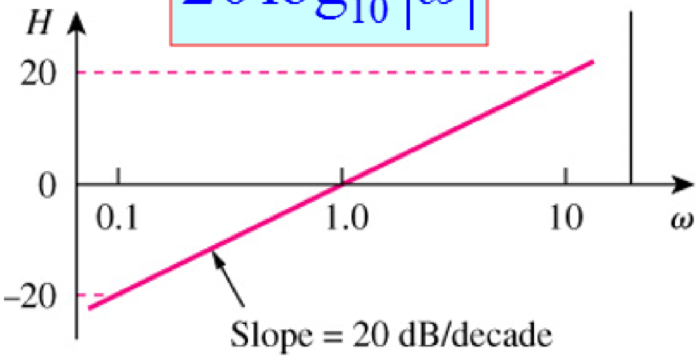
CONSTANT TERM $20\log_{10} K$

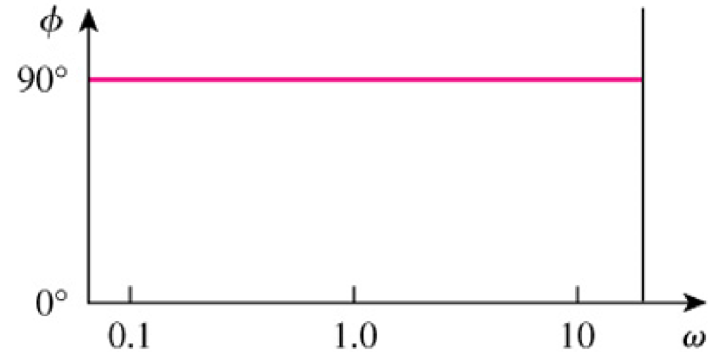


Term Types

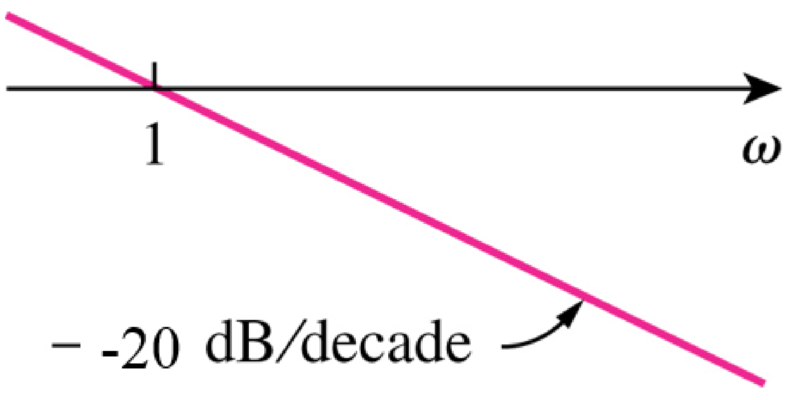
ZERO AT THE ORIGIN $(j\omega)$

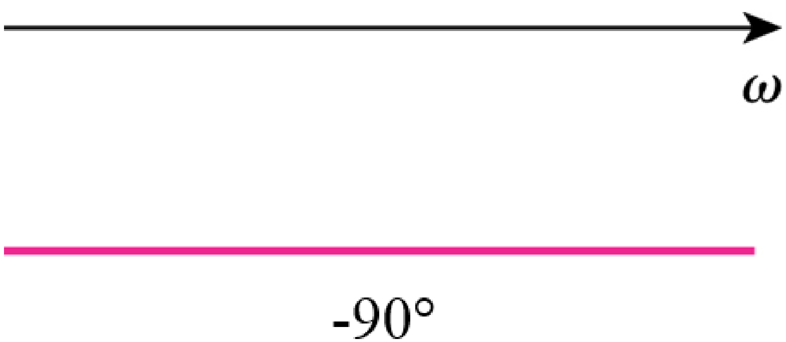
$$\boxed{20\log_{10}|\omega|}$$





POLE AT THE ORIGIN $(j\omega)^{-1}$





Term Types

Let the simplest transfer function be the **single-zero** function given by

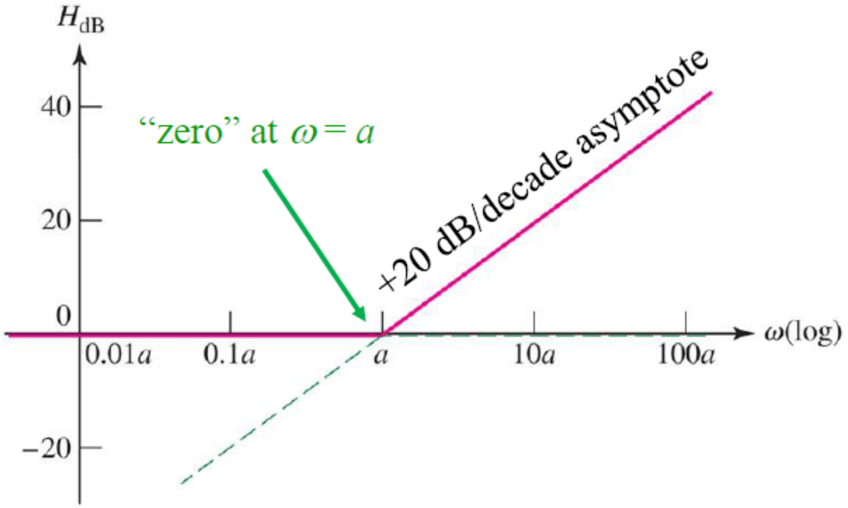
$$\mathbf{H}(j\omega) = 1 + \frac{j\omega}{a}$$

The amplitude may be written as $\quad |\mathbf{H}(j\omega)| = \sqrt{1 + \frac{\omega^2}{a^2}} \qquad H_{\text{dB}} = 20\log_{10}\sqrt{1 + \frac{\omega^2}{a^2}}$

Plotted against frequency ω,
the single-zero $|\mathbf{H}(j\omega)|$ looks like:

$$\omega << a, \quad H_{\text{dB}} \approx 0$$

$$\omega >> a, \quad H_{\text{dB}} \approx 20\log_{10}(\omega/a)$$



Term Types

Let another simple transfer function be the **single-pole** function given by

$$\mathbf{H}(j\omega) = \frac{1}{1 + j\omega/a}$$

The amplitude may be written as

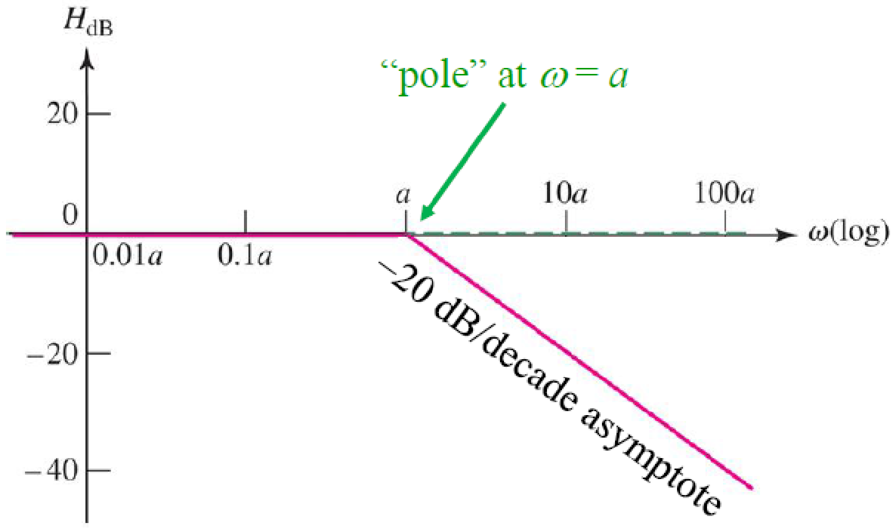
$$\left|\mathbf{H}(j\omega)\right| = \frac{1}{\sqrt{1 + \dfrac{\omega^2}{a^2}}}$$

$$H_{\mathrm{dB}} = -20\log_{10}\sqrt{1 + \frac{\omega^2}{a^2}}$$

Plotted against frequency ω,
the single-pole $|\mathbf{H}(j\omega)|$ looks like:

$$\omega << a, \quad H_{\mathrm{dB}} \approx 0$$

$$\omega >> a, \quad H_{\mathrm{dB}} \approx -20\log_{10}(\omega/a)$$



Term Types

Let another simple transfer function be the **single-pole** function given by

$$\mathbf{H}(j\omega) = \frac{1}{1 + j\omega/a}$$

a = "pole" of $\mathbf{H}(j\omega)$

The **phase** may be written as

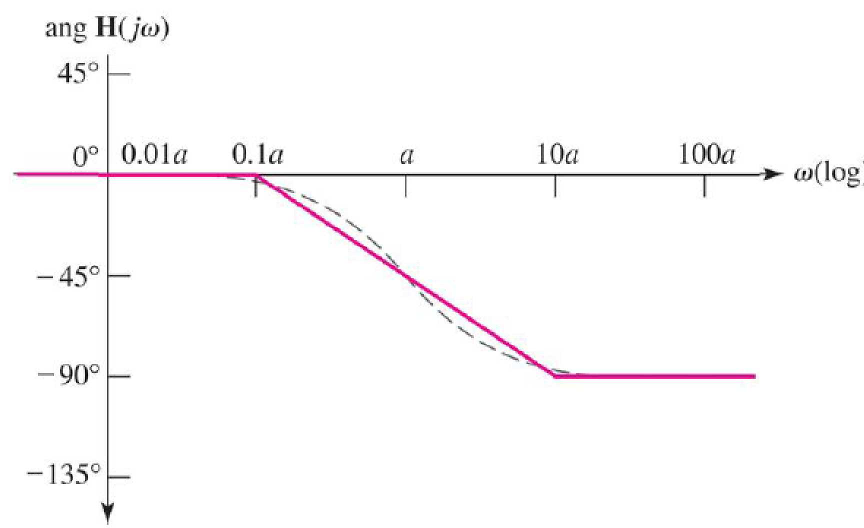
$$\text{ang}\{\mathbf{H}(j\omega)\} = -\tan^{-1}\frac{\omega}{a}$$

Plotted against frequency ω,
the single-pole ang$\{\mathbf{H}(j\omega)\}$ looks like:

$$\omega < a/10, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx 0$$

$$\omega \approx a, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx -45°$$

$$\omega > 10a, \quad \text{ang}\{\mathbf{H}(j\omega)\} \approx -90°$$



Example 9

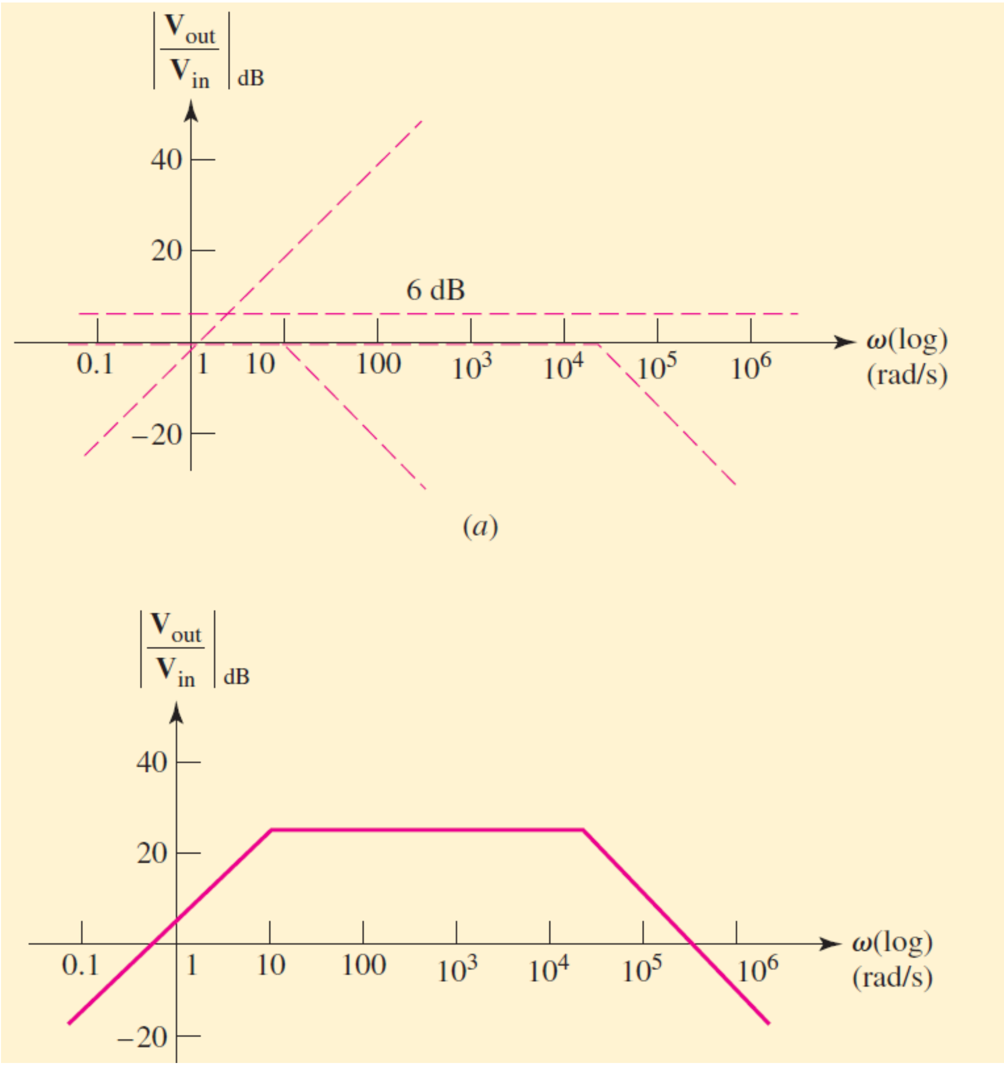
Construct a Bode plot for the amplitude and phase of the transfer function that is given.

$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)}$$

Solution 9 - Amplitude

Construct a Bode plot for the amplitude and phase of the transfer function that is given.

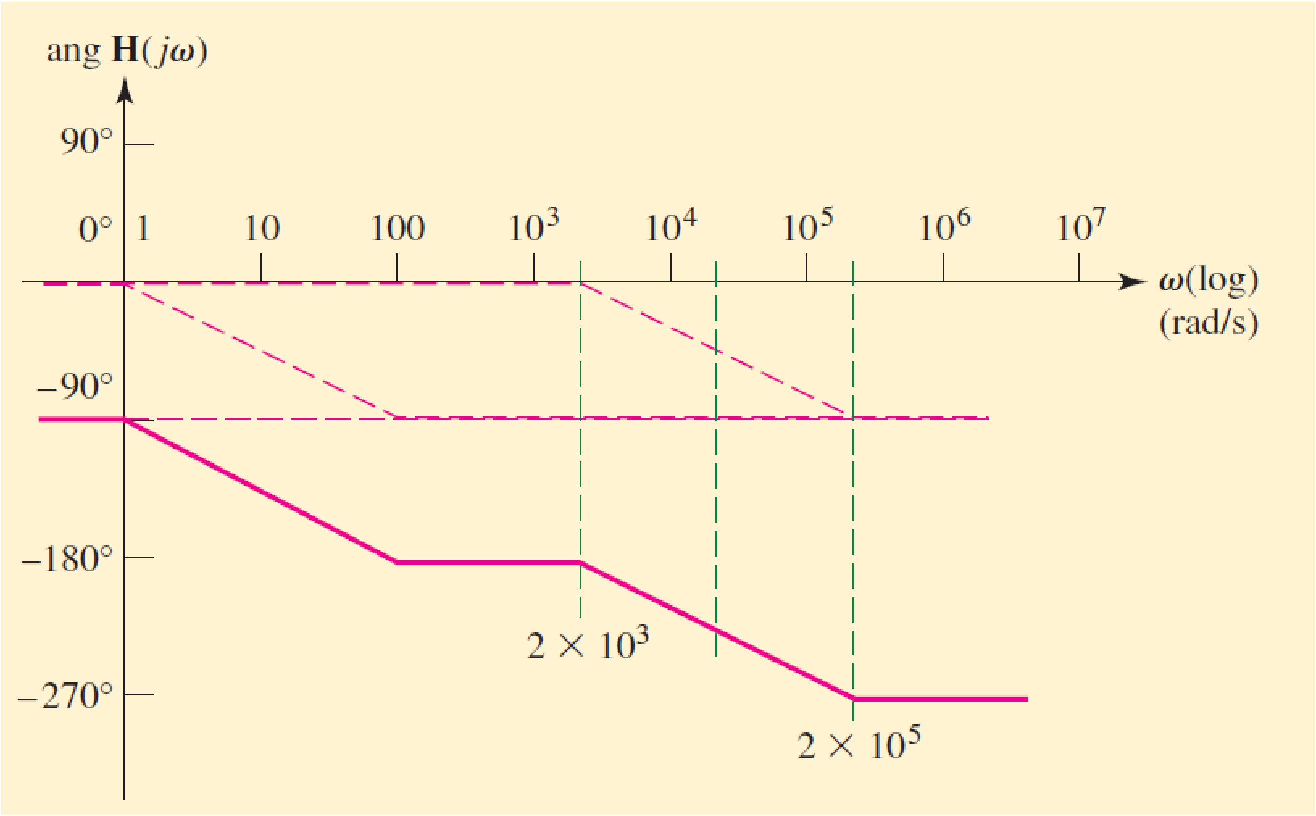
$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)}$$



Solution 9 - Phase

Construct a Bode plot for the amplitude and phase of the transfer function that is given.

$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)}$$



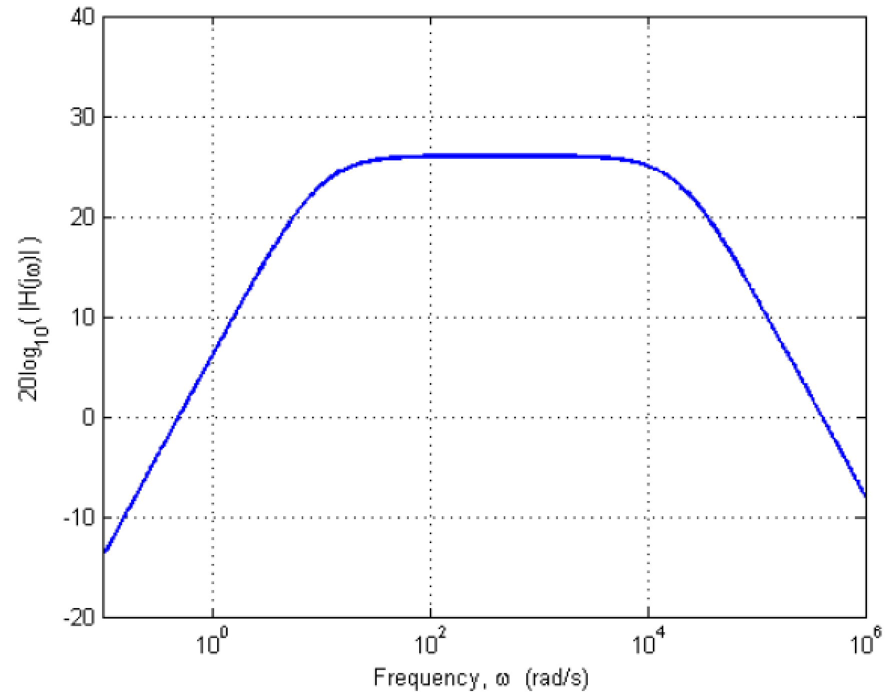
Solution 9 – Amplitude - Matlab

Construct a Bode plot for the amplitude and phase of the transfer function that is given.

$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20{,}000)}$$

The **exact** amplitude, plotted using Matlab:

```
w = logspace(-1,6,1001);

H = -2*j*w ./ ( (1 + j*w/10) ...
         .*(1 + j*w/20000) );

semilogx(w,20*log10(abs(H)),...
           'Linewidth',2)
axis([10^-1 10^6 -20 40]);
grid
ylabel('20log_{10}( |H(j\omega)| )');
xlabel('Frequency, \omega  (rad/s)')
```

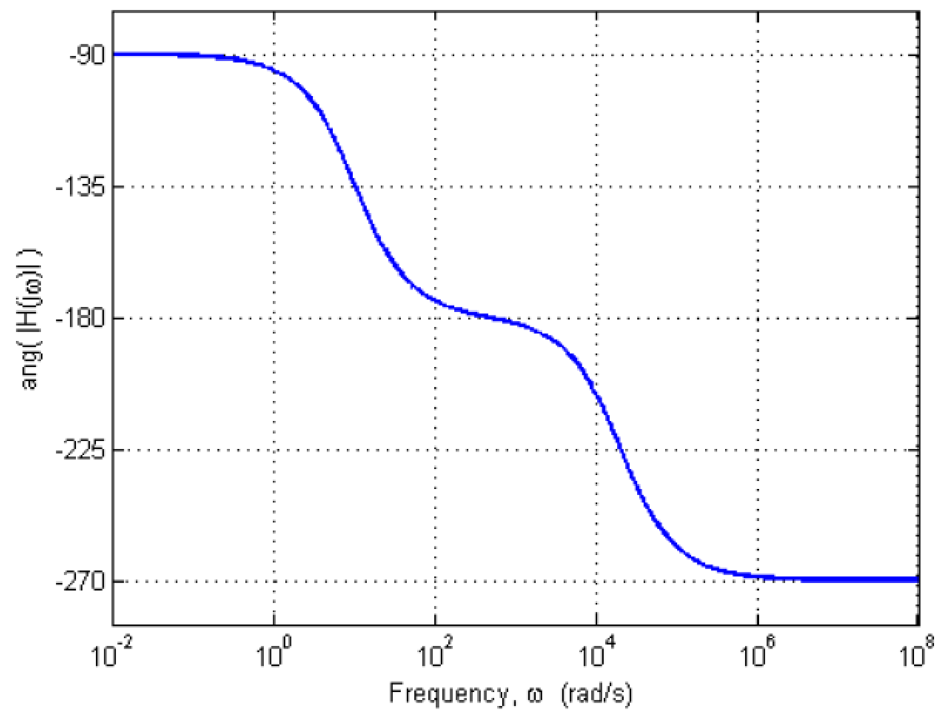


Solution 9 – Phase - Matlab

Construct a Bode plot for the amplitude and phase of the transfer function that is given.

$$\mathbf{H}(j\omega) = \frac{-2j\omega}{(1 + j\omega/10)(1 + j\omega/20,000)}$$

```
w = logspace(-2,8,1001);

H = -2*j*w ./ ( (1 + j*w/10) ...
                .*(1 + j*w/20000) );

semilogx(w,phase(H)*180/pi,...
        'Linewidth',2)
axis([10^-2 10^8 -285 -75]);
set(gca,'Ytick',[-360:45:360])
grid
ylabel('ang( |H(j\omega)| )');
xlabel('Frequency, \omega  (rad/s)')
```



Example 10

Construct Bode plots for

$$H(\omega)=\frac{200\,j\omega}{(j\omega+2)(j\omega+10)}$$

- Express transfer function in Standard form.

$$\text{STANDARD FORM}\quad H(\omega)=\frac{10\,j\omega}{(1+j\omega/2)(1+j\omega/10)}$$

- Express the magnitude and phase responses.

$$H_{db}=20\log_{10}10+20\log_{10}|j\omega|-20\log_{10}|1+j\omega/2|-20\log_{10}|1+j\omega/10|$$

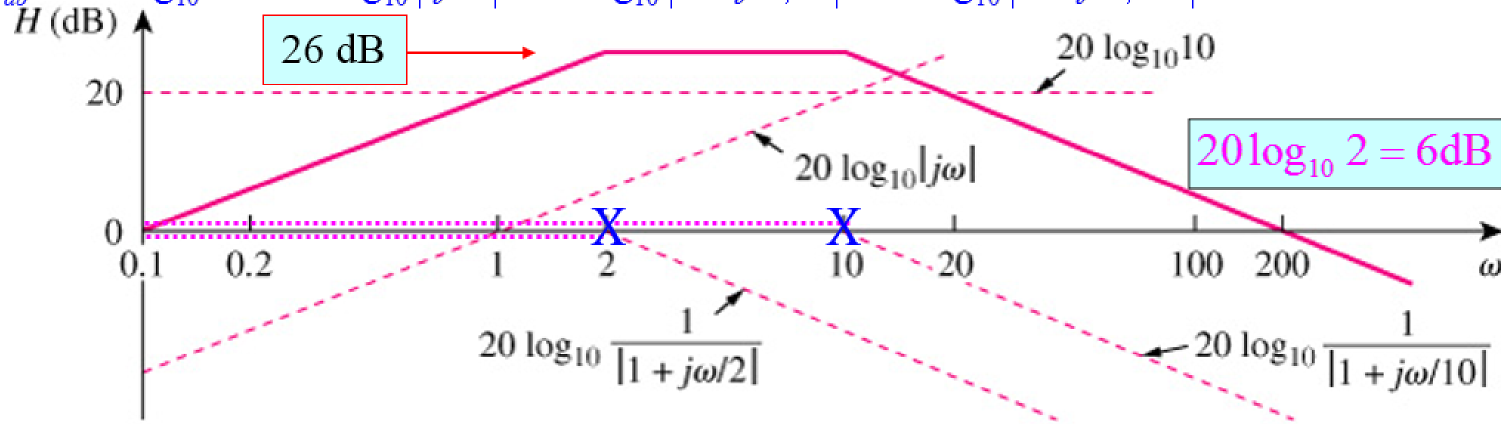
$$\phi=90°-\tan^{-1}(\omega/2)-\tan^{-1}(\omega/10)$$

- Two corner frequencies at ω=2, 10 and a zero at the origin ω=0.
- Sketch each term and add to find the total response.

Solution 10

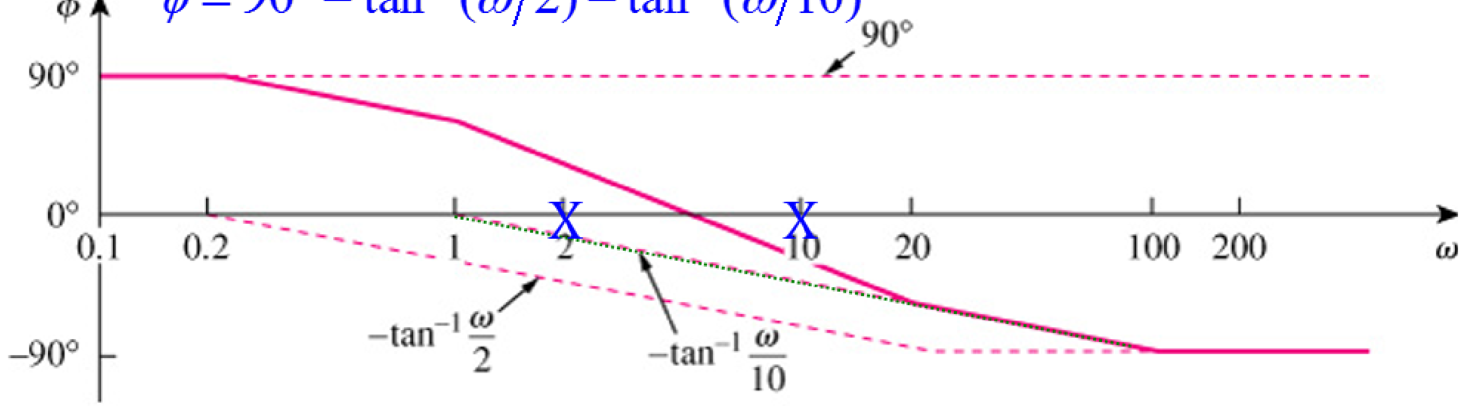
Construct Bode plots for $H(\omega) = \dfrac{200\,j\omega}{(j\omega+2)(j\omega+10)}$

$$H_{db} = 20\log_{10} 10 + 20\log_{10}|j\omega| - 20\log_{10}|1+j\omega/2| - 20\log_{10}|1+j\omega/10|$$



(a)

$$\phi = 90° - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/10)$$



(b)