BME2301 - Circuit Theory

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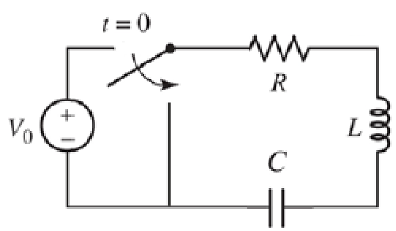
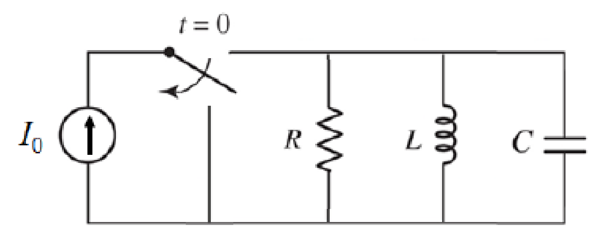
Why do we need to use Complex Numbers?

If we know that the only independent sources in our circuit are **sinusoidal**, and we know that all transients are gone (steps, switches, pulses)…



$\alpha = \frac{R}{2L}, \quad \omega_0 =$



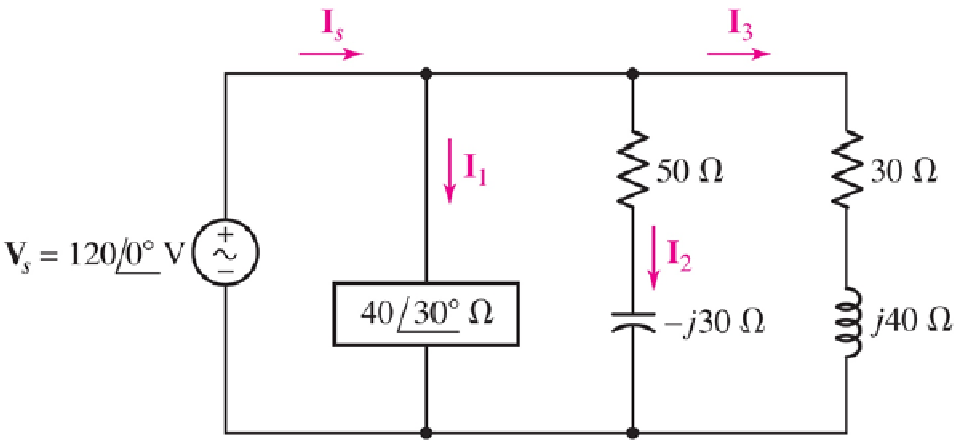
$\alpha =$, $\quad \omega_0 = \frac{1}{\sqrt{LC}}$

NOT NECESSARY

$\alpha > \omega_0$		$\alpha < \omega_0$
OVERDAMPED	**CRITICALLY** DAMPED	**UNDER**DAMPED
$x(t) = X_1 e^{c_1 t} + X_2 e^{c_2 t} + X_3$	$x(t) = e^{-\alpha t}(X_1 t + X_2) + X_3$	$x(t) = e^{-\alpha t}\begin{bmatrix} X_1 \cos(\omega_d t) \\ +X_2 \sin(\omega_d t) \end{bmatrix} + X_3$

…we may instead solve the circuit **algebraically** (e.g. nodal, mesh) without determining initial conditions, final conditions, etc.

Why do we need to use Complex Numbers?



…because most analog signals consist of one or more sinusoids, by design.

60-Hz power, 900-MHz cellular telephones, 2.4-GHz wireless internet

…because **all** signals (analog *or* digital) may be **analyzed** as a sum of sinusoids.

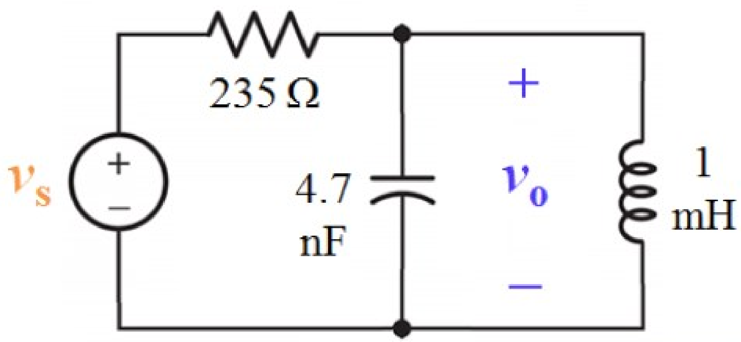
(You will see this in your Signals & Systems and Engineering Math courses.)

…because the **differential equations** governing **practical** systems are nearly impossible to derive and are **time-consuming** to solve.

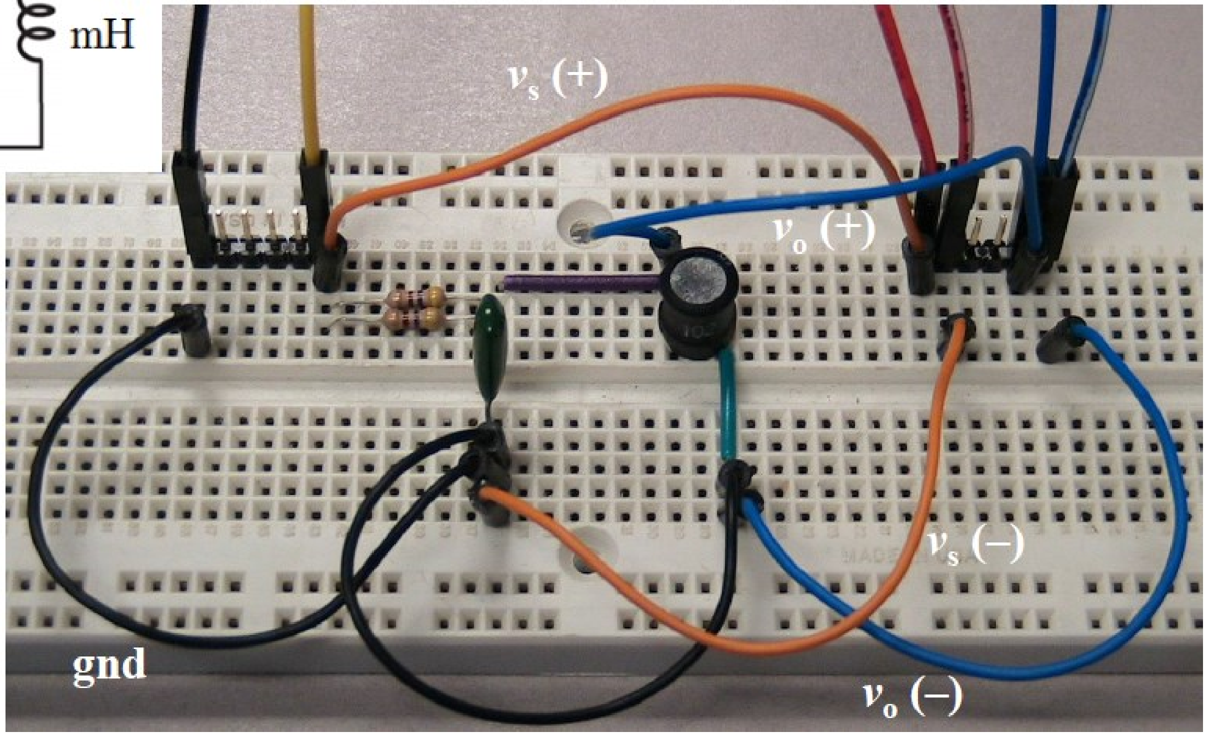
Software tools (e.g. Matlab) can perform complex algebra *very quickly*.

A Simple Circuit

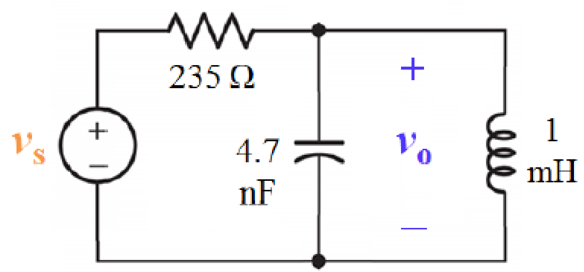
Compare $v_o(t)$ to $v_s(t)$ for $f = 5$ kHz and $f = 50$ kHz .



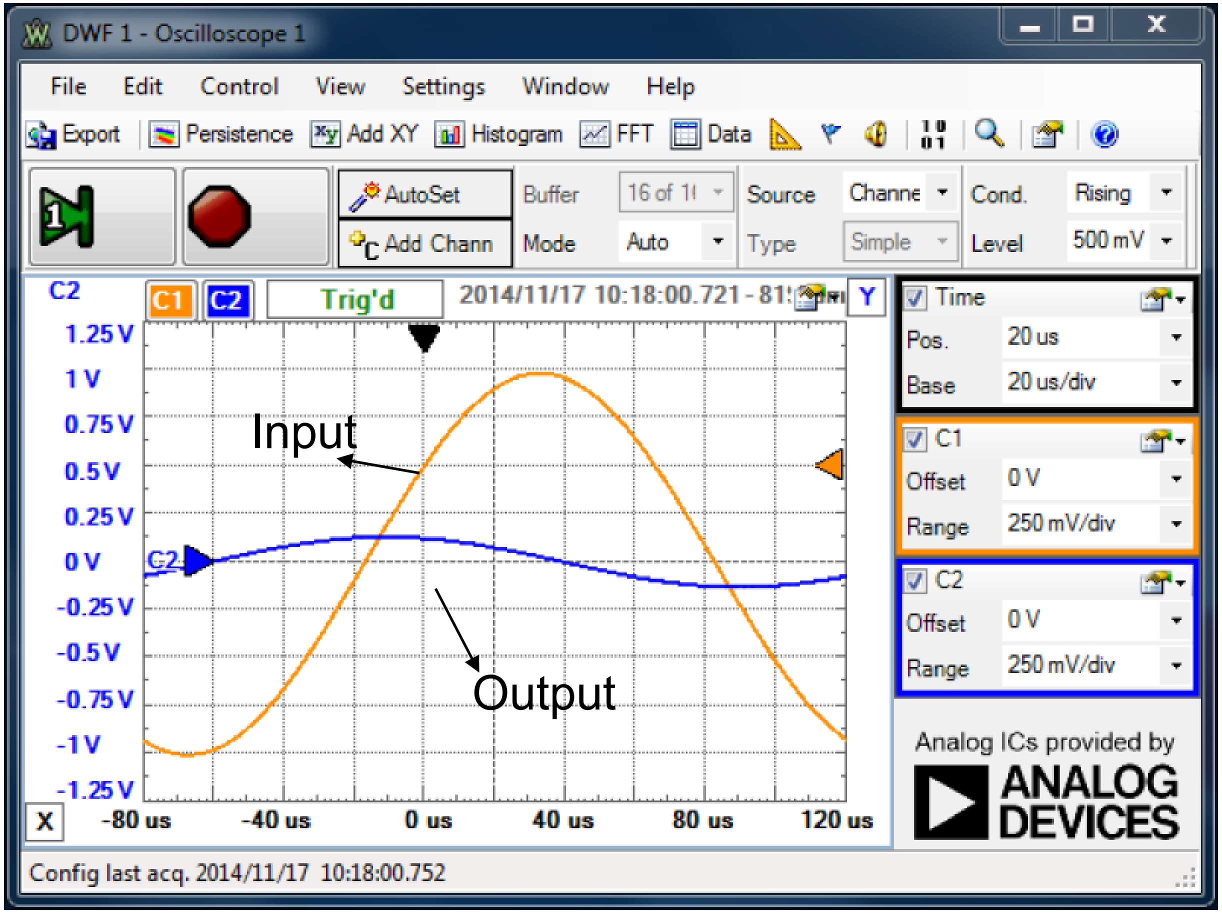
$$v_s = \sin\left(2\pi \cdot f \cdot t\right) \mathrm{V}$$



Input Voltage vs Output Voltage

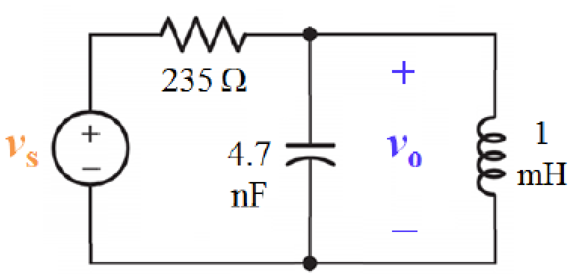


$$v_s = \sin\left(2\pi \cdot f \cdot t\right)\text{ V}$$

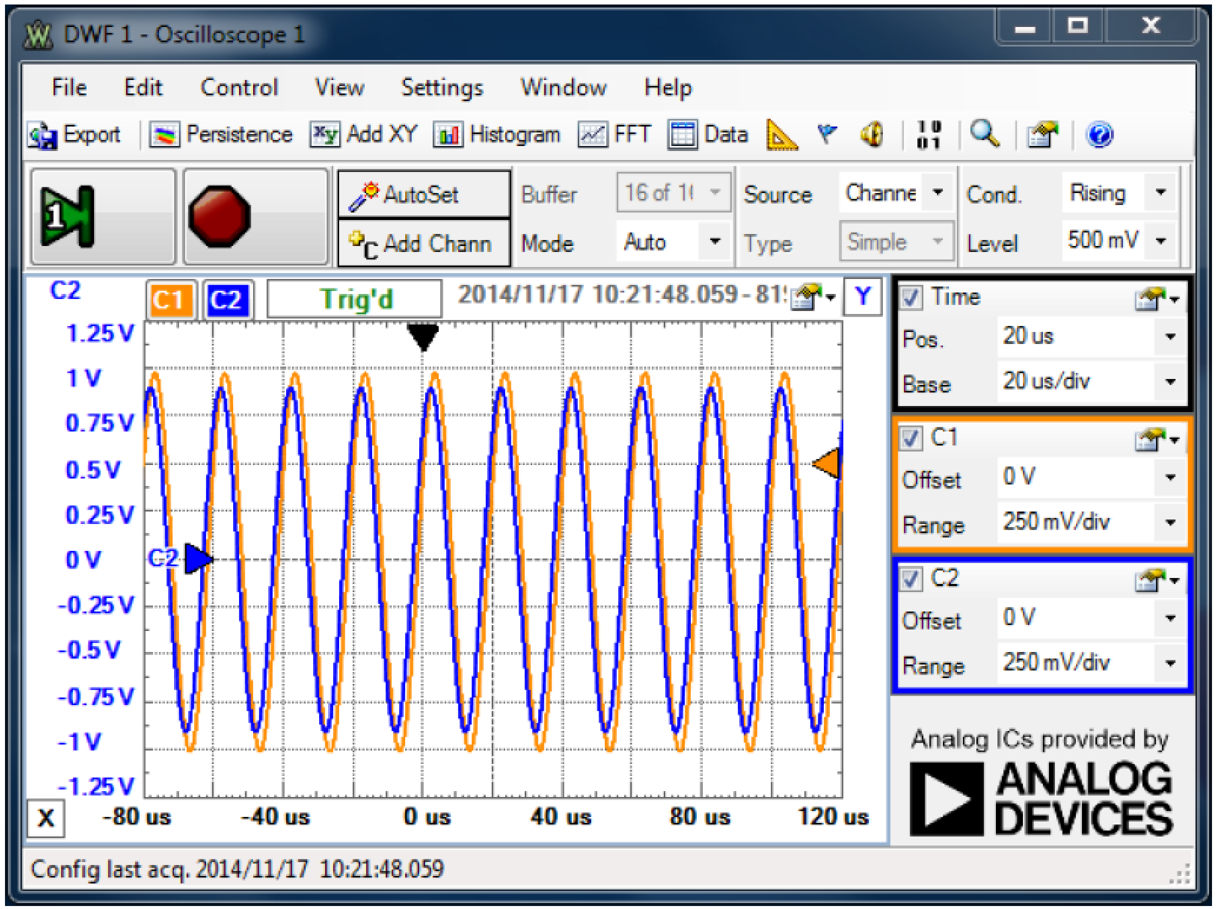


$f = 5$ kHz

Input Voltage vs Output Voltage

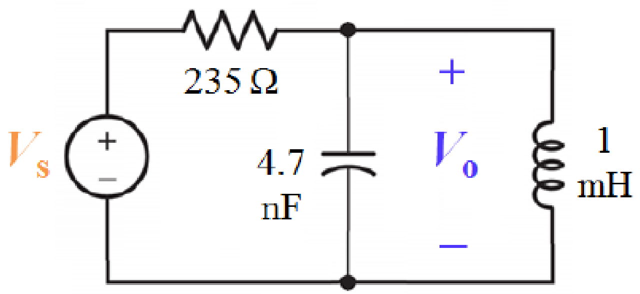


$$v_s = \sin\left(2\pi \cdot f \cdot t\right)\ \mathrm{V}$$



$f = 50$ kHz

Why do we need to use Complex Numbers?



$$v_s = \sin\left(2\pi \cdot 10^4 \cdot t\right)\ \mathrm{V}$$

Complex algebra is the math that electrical engineers use to analyze AC circuits.

- sinusoids become phasors:

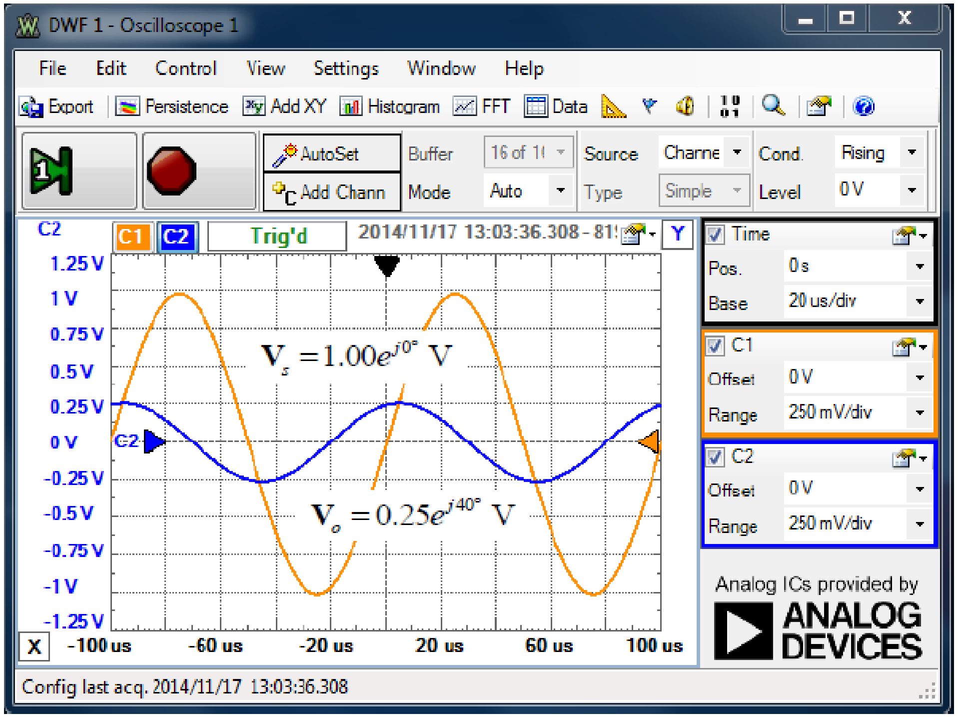
$$\mathbf{V}_s = 1.00 e^{j0^\circ}\ \mathrm{V}$$

- KCL/KVL/Ohm's Law are solved with phasors:

$$\mathbf{V}_o = 0.25 e^{j40^\circ}\ \mathrm{V}$$

- phasors turn back into sinusoids:

$$v_o = 250 \sin\left(2\pi \cdot 10^4 \cdot t + 40^\circ\right)\ \mathrm{mV}$$



Complex Numbers?

Real numbers (3, –2/7, π) are a <u>subset</u> of complex numbers.

Real numbers contain the roots of <u>some</u> algebraic equations.

$$s^2 = 4 \longrightarrow s_1 = -2,\ s_2 = +2$$

Complex numbers contain the roots of <u>all</u> algebraic equations.

$$s^2 = -4 \longrightarrow s_1 = -2j,\ s_2 = +2j$$

The imaginary operator ***j*** (or ***i*** in mathematics & physics literature) is defined as

$$j^2 = -1$$

Real/Complex Numbers?

$$j = \sqrt{-1}$$

The product of a real number & the operator ***j*** is an **imaginary number**.

$$3j,\ -\frac{2}{7}j,\ \pi j,\ 5.1j$$

The sum of a real number & an imaginary number is a **complex number**, ***z*** .

$$2+4j,\ 1+\pi j$$

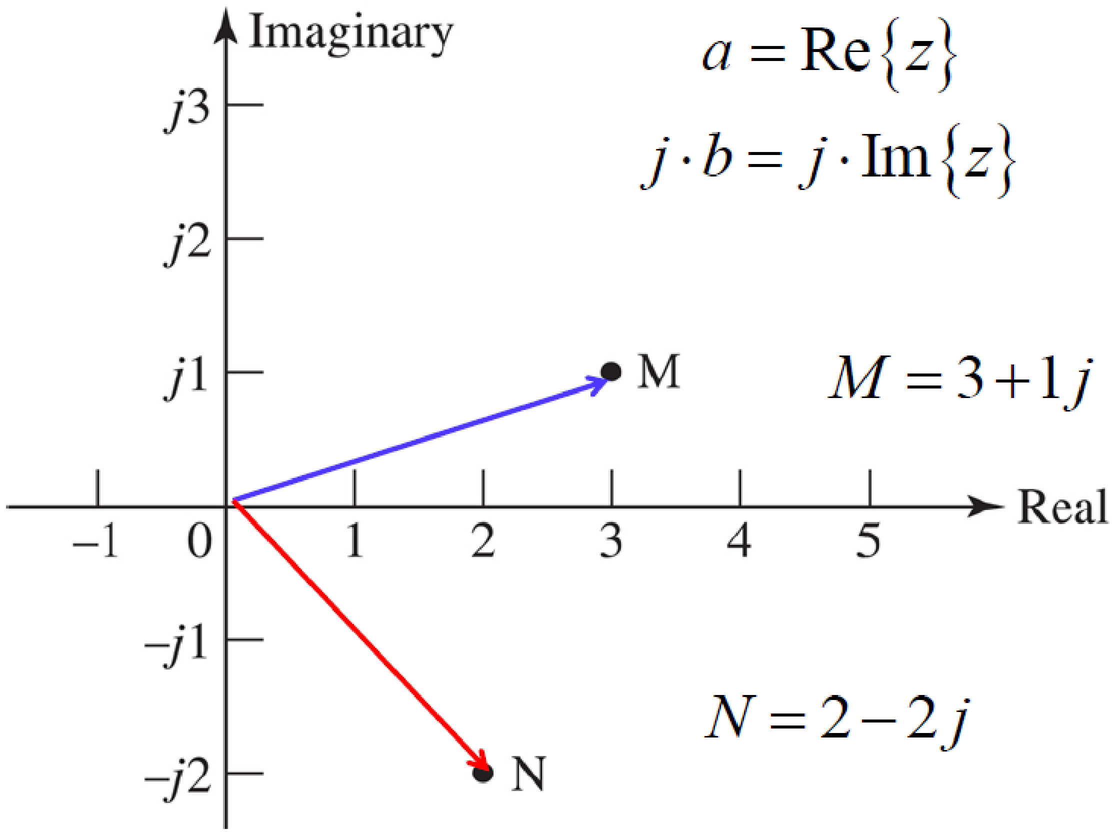
…where the *real part* is denoted $a = \mathrm{Re}\{z\}$

…and the *imaginary part* is denoted $b = \mathrm{Im}\{z\}$

rectangular form

$$\mathrm{Re}\{2+4j\} = 2,\ \mathrm{Im}\{2+4j\} = 4$$

Complex Plane



$$a = \mathrm{Re}\{z\}$$

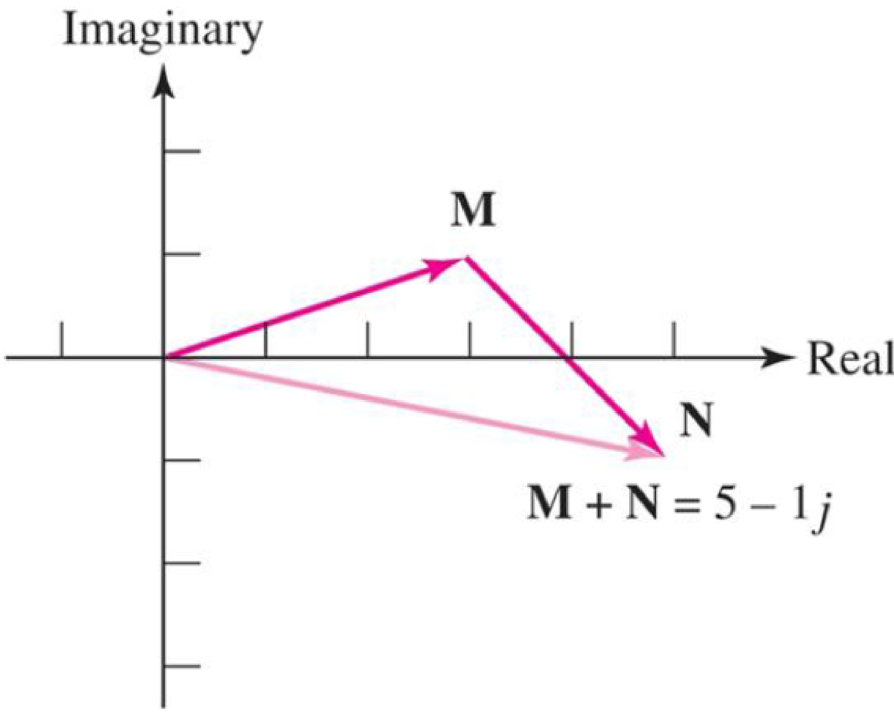
$$j \cdot b = j \cdot \mathrm{Im}\{z\}$$

$$M = 3 + 1j$$

$$N = 2 - 2j$$

Complex numbers may be visualized as *vectors* in the *complex plane*.

Addition and Subtraction



$$M = 3 + 1j$$

$$N = 2 - 2j$$

$$M + N = 5 - 1j$$

Graphical addition & subtraction are performed like vector addition ("tip-to-tail").

Algebraic addition & subtraction are performed piece-wise:

$$\begin{aligned} M &= a_1 + b_1 \cdot j \\ N &= a_2 + b_2 \cdot j \\ M + N &= (a_1 + a_2) + (b_1 + b_2) \cdot j \end{aligned}$$

Multiplication

Multiplication may be accomplished in rectangular form…

$$z_1 = a_1 + b_1 j$$

$$z_1 = a_2 + b_2 j$$

$$\begin{aligned} z_1 \cdot z_2 &= \left(a_1 + b_1 j\right)\left(a_2 + b_2 j\right) \\ &= a_1 a_2 + a_1 b_2 j + a_2 b_1 j + b_1 b_2 j^2 \\ &= a_1 a_2 + \left(a_1 b_2 + a_2 b_1\right) j - b_1 b_2 \\ &= \left(a_1 a_2 - b_1 b_2\right) + \left(a_1 b_2 + a_2 b_1\right) j \end{aligned}$$

$$M = 5 + 3j$$

$$N = 2 - 4j$$

$$\begin{aligned} M \cdot N &= \left(5 + 3j\right)\left(2 - 4j\right) \\ &= 10 - 20j + 6j - 12j^2 \\ &= 22 - 14j \end{aligned}$$

…but it is more easily accomplished in *polar* form.

Example: Complex Power

Find $v \times i$ in rectangular form:

$$v = 7 + 3j \ \text{mV}$$
$$i = -5 + 4j \ \text{mA}$$

$$v \cdot i = (7 + 3j)(-5 + 4j)$$
$$= -35 - 15j + 28j + 12j^2$$

$$v = 2 + 9j \ \text{V}, \ i = -3 + 5j \ \text{A}$$

$$v \cdot i = (2 + 9j)(-3 + 5j)$$
$$= -6 - 27j + 10j - 45$$

```
>> V = 2 + 9*j;
>> I = -3 + 5*j;
>> p = V * I

p = -51.0000 -17.0000i
```

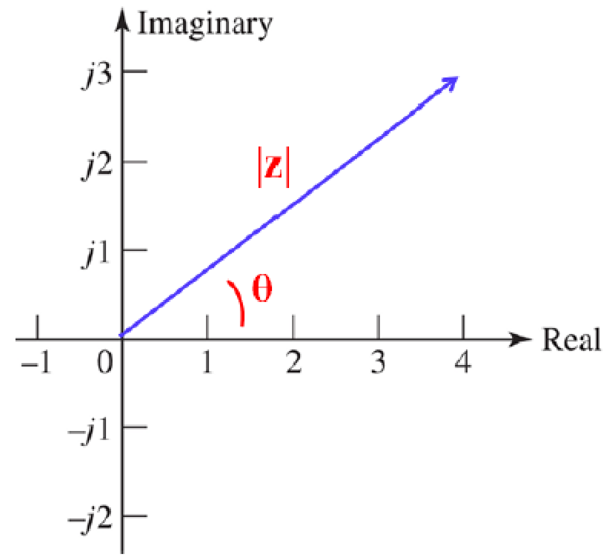
Exponential Form

$$
\begin{aligned}
e^{j\theta} &= \cos(\theta) + j\cdot\sin(\theta) \\
|z|\cdot e^{j\theta} &= |z|\cdot\cos(\theta) + j\cdot|z|\cdot\sin(\theta) \\
|z|\cdot e^{j\theta} &= a + b\cdot j
\end{aligned}
$$

assume $|z|$ is positive, real

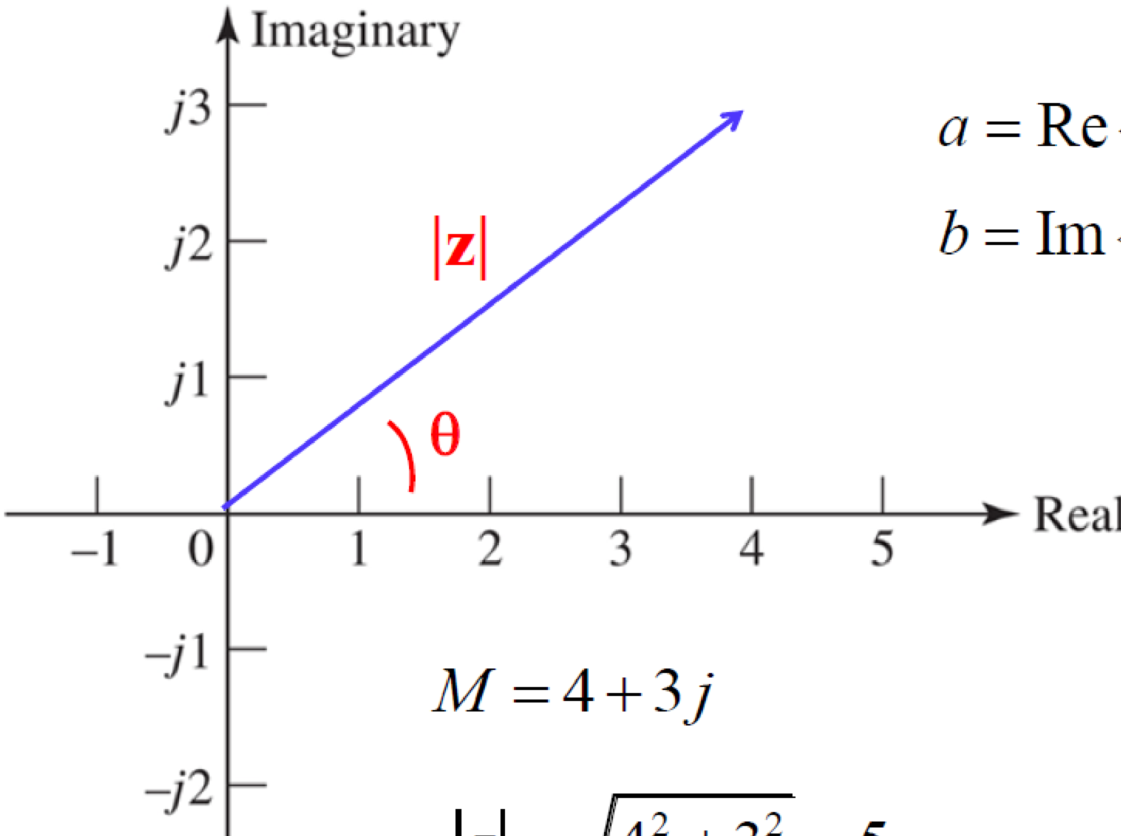
$$a = |z|\cdot\cos(\theta)$$
$$b = |z|\cdot\sin(\theta)$$

$$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) = \frac{b}{a}$$



$$a^2 + b^2 = |z|^2\cdot\cos^2(\theta) + |z|^2\cdot\sin^2(\theta)$$
$$a^2 + b^2 = |z|^2\cdot\left\{\cos^2(\theta) + \sin^2(\theta)\right\} = |z|^2$$
$$\sqrt{a^2 + b^2} = |z|$$

Rectangular to Exponential Form



$$a = \mathrm{Re}\{z\}$$

$$b = \mathrm{Im}\{z\}$$

$$\tan(\theta) = \frac{b}{a}$$

$$|z| = \sqrt{a^2 + b^2}$$

$$M = 4 + 3j$$

$|z|$ = **magnitude** of z

θ = **phase/angle** of z

$$|z| = \sqrt{4^2 + 3^2} = 5$$

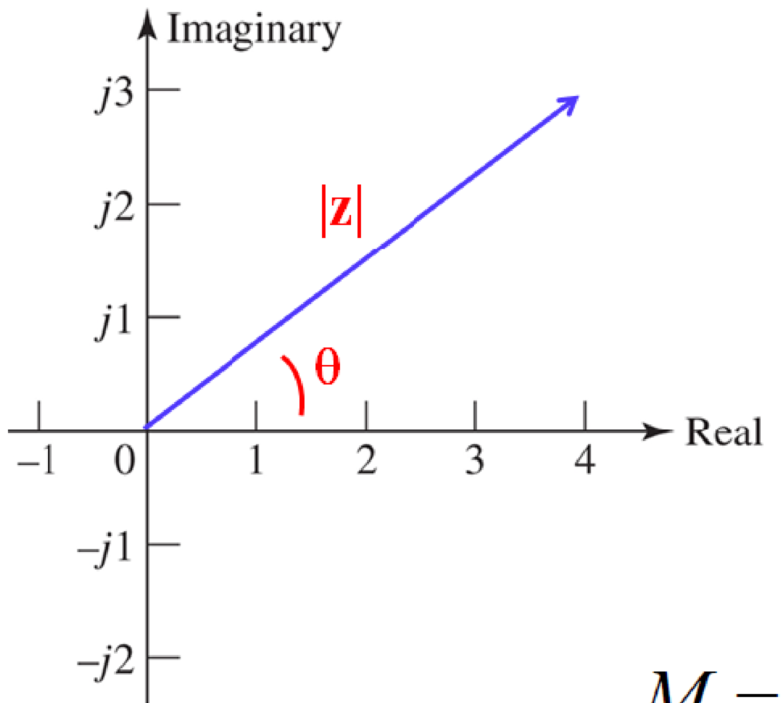
$$\theta = \tan^{-1}\{3/4\} = 37°$$

$$M = 5e^{j37°}$$

Polar Form

$$z = a + b \cdot j = |z| \cdot e^{j\theta} = \boxed{|z| \angle \theta}$$

rectangular exponential polar



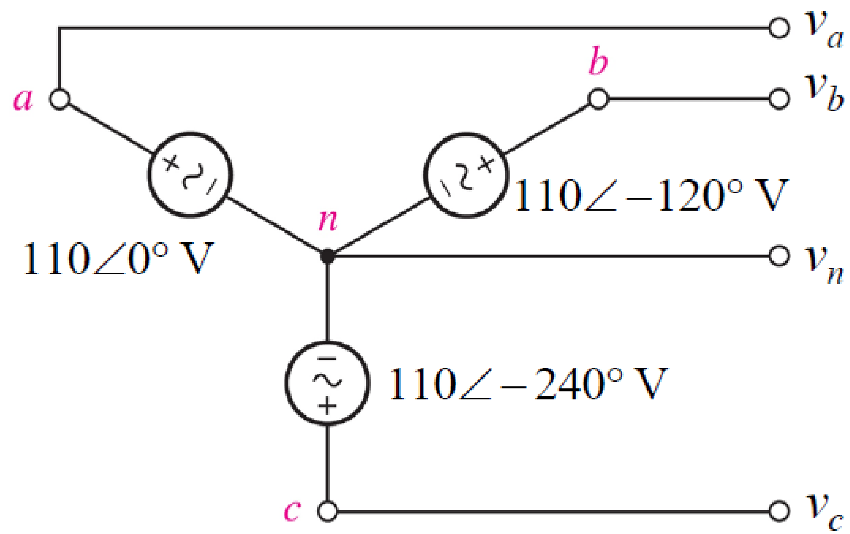
Polar form is a *shorthand* for the exponential form.

$|z|$ = **magnitude** of z
θ = **phase/angle** of z

$$M = 4 + 3j = 5e^{j37^\circ} = 5\angle 37^\circ$$

Polar Form

Determine the quantity $v_a - v_b$ in polar form if $v_n = 0$.



$$
\begin{aligned}
v_a - v_b &= 110\angle 0° - 110\angle -120° \\
&= \left[110\cos 0° + j\cdot 110\sin 0°\right] \\
&\quad - \left[110\cos(-120°) + j110\sin(-120°)\right] \\
&= [110 + j0] - \left[110\cdot(-1/2) + j\cdot 110\cdot\left(-\sqrt{3}/2\right)\right] \\
&= 165 + j\cdot 55\sqrt{3} \\
&= \sqrt{(165)^2 + \left(55\sqrt{3}\right)^2}\ \tan^{-1}\left\{55\sqrt{3}/165\right\}
\end{aligned}
$$

```
>> v_a = 110*exp(j*0);
>> v_b = 110*exp(j*-2*pi/3);
>> v = v_a - v_b;
>> abs(v)
              ans =  190.5256
>> angle(v)*180/pi
              ans =  30.0000
```

Multiplication in Polar Form

Multiplication in polar form is carried out using exponentials…

$$z_1 = a_1 + b_1 j \Rightarrow |z_1| e^{j\theta_1}$$

$$z_2 = a_2 + b_2 j \Rightarrow |z_2| e^{j\theta_2}$$

$$\begin{aligned} z_1 \cdot z_2 &= |z_1| e^{j\theta_1} \cdot |z_2| e^{j\theta_2} \\ &= |z_1||z_2| e^{j\theta_1 + j\theta_2} \\ &= |z_1||z_2| e^{j(\theta_1 + \theta_2)} \end{aligned}$$

$$z_1 = |z_1| \angle \theta_1 , \quad z_2 = |z_2| \angle \theta_1$$

$$z_1 \cdot z_2 = |z_1||z_2| \angle (\theta_1 + \theta_2)$$

$$M = 3 + 4j = 5 \angle 53^\circ$$

$$N = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} j = 3 \angle 45^\circ$$

$$\begin{aligned} M \cdot N &= (5 \angle 53^\circ)(3 \angle 45^\circ) \\ &= 15 \angle 98^\circ \end{aligned}$$

Division in Polar Form

$$z_1 = |z_1| \angle \theta_1 \,, \quad z_2 = |z_2| \angle \theta_1$$

$$\frac{|z_1| e^{j\theta_1}}{|z_2| e^{j\theta_2}} = \frac{|z_1|}{|z_2|} \frac{e^{j\theta_2}}{e^{j\theta_2}} = \frac{|z_1|}{|z_2|} e^{j\theta_1 - j\theta_2}$$

$$\frac{z_1 \angle \theta_1}{z_2 \angle \theta_2} = \frac{|z_1|}{|z_2|} \angle (\theta_1 - \theta_2)$$

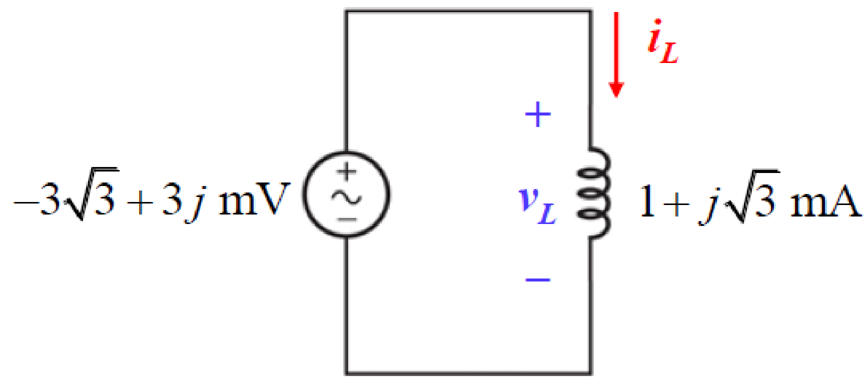
$$M = 6 + 8j = 10 \angle 53°$$

$$N = \frac{5}{\sqrt{2}} + \frac{5}{\sqrt{2}} j = 5 \angle 45°$$

$$M/N = (10 \angle 53°)/(5 \angle 45°) = 2 \angle 8°$$

Example : Ohm's Law

Determine the ratio of v_L to i_L:



$$\frac{v_L}{i_L}=\frac{|v_L|}{|i_L|}\angle\left(\theta_{VL}-\theta_{IL}\right)$$

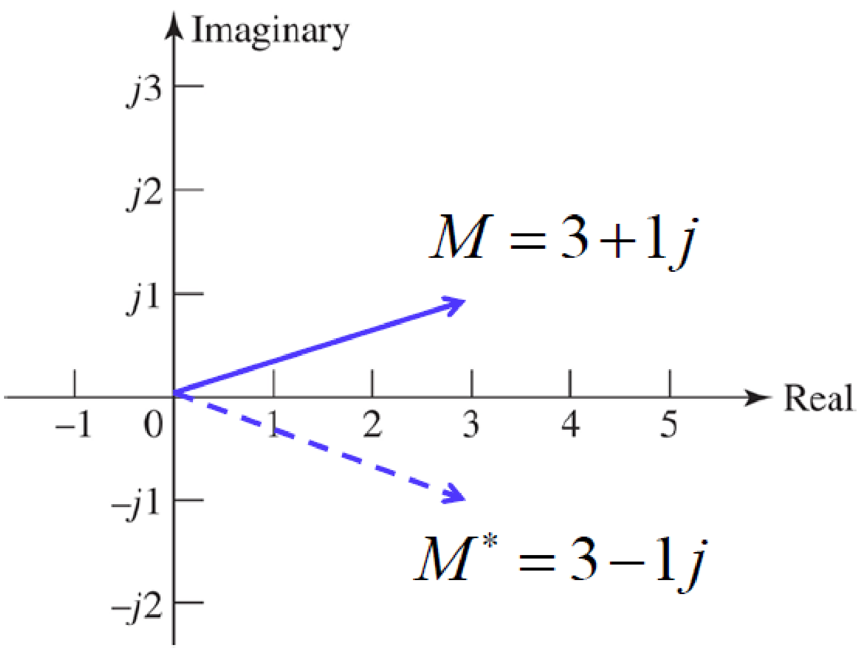
$$\frac{v_L}{i_L}=\frac{-3\sqrt{3}+3j\ \ \text{mV}}{1+j\sqrt{3}\ \ \text{mA}}$$

$$=\frac{\sqrt{\left(3\sqrt{3}\right)^2+(3)^2}\angle\tan^{-1}\left\{1/-\sqrt{3}\right\}}{\sqrt{(1)^2+\left(\sqrt{3}\right)^2}\angle\tan^{-1}\left\{\sqrt{3}\right\}}=\frac{6\angle150°}{2\angle60°}$$

Complex Conjugate

The **complex conjugate** of z is denoted z^*

and if $z = a + b \cdot j$ then $z^* = a - b \cdot j$



The conjugate of z is the same number, except that the imaginary part is negated.

Graphically, the complex conjugate of z is the mirror image of z across the *Real* axis.

Example : Power Absorbed

Write the quantity $V \times I^*$ in polar form, given

$$V = 3 - 5j \text{ V}$$
$$I = 6 + 7j \text{ mA}$$

$$V \cdot I^* = (3-5j)(6-7j)$$
$$= 18 - 30j - 21j - 35$$
$$= -17 - 51j \text{ mW}$$

```
>> V = 3 - 5*j;
>> I = 6 + 7*j;
>> S = V * conj(I)

S = -17.0000 -51.0000i
```

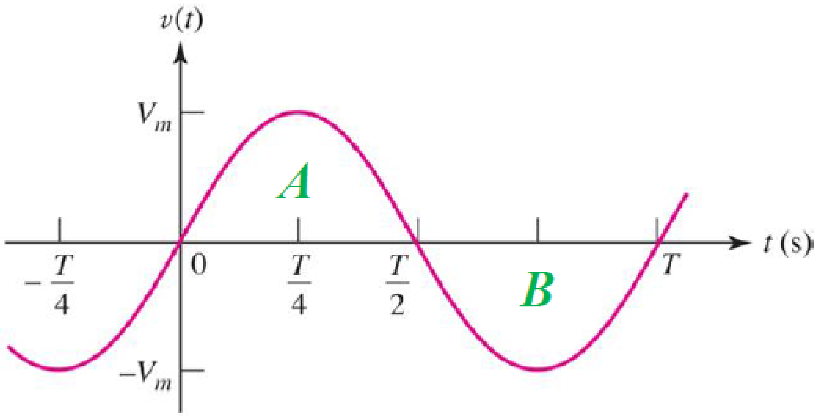
$$V \cdot I^* = \sqrt{17^2 + 51^2} \tan^{-1}\{-51/-17\}$$
$$= 53.8\angle -108° \text{ mW}$$

```
>> abs(S)

   ans = 53.7587

>> angle(S)*180/pi

   ans = -108.4349
```

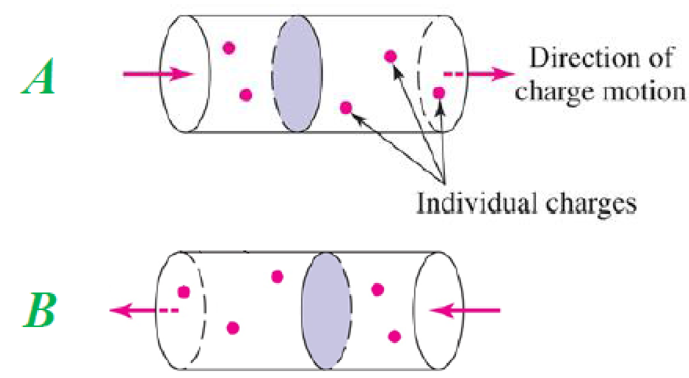
$$V \cdot I^* = \left\{\sqrt{34}\angle -59°\right\}\left\{\sqrt{85}\angle -49°\right\}$$
$$= \sqrt{34 \cdot 85}\angle -59° - 49°$$
$$= 53.8\angle -108° \text{ mW}$$

Alternating Current (AC) - Sinusoidal

$$v(t) = V_m \cdot \sin(\omega t + \phi_0) \ , \quad \phi_0 = 0$$



$+ \quad v(t) \quad -$



$$V_m = \text{amplitude (in Volts)}, \ \phi_0 = \text{phase (in radians)}$$

$$\omega = \text{frequency (in radians/second)}$$

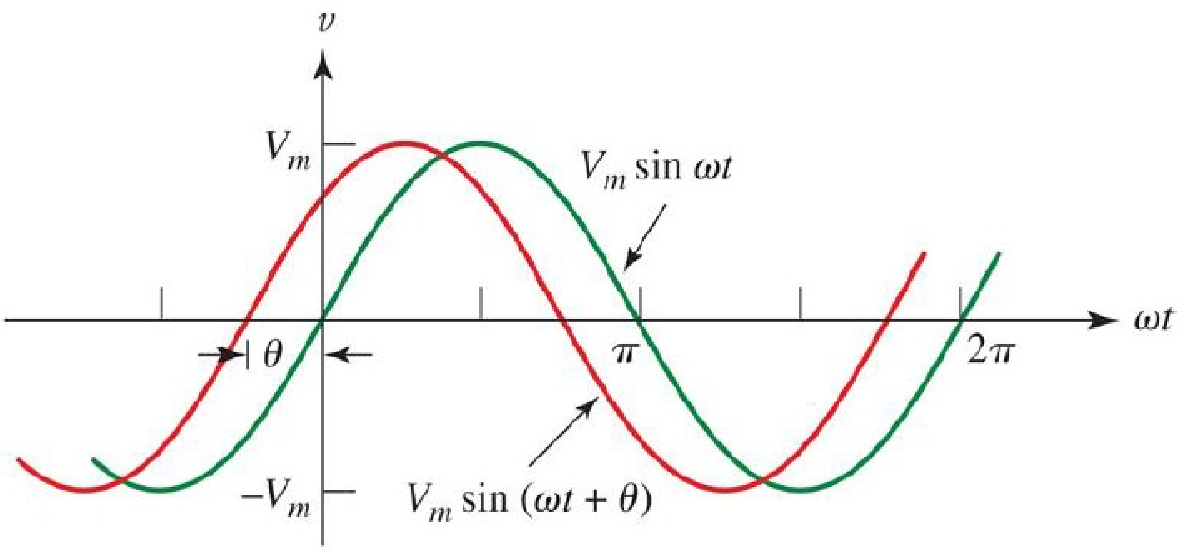
$$T = \text{period (in seconds)}$$

$$f = \text{frequency (in cycles/second)} = 1/T = \omega / 2\pi$$

$$V_m \cdot \sin(\omega t + \phi_0) = V_m \cdot \cos(\omega t + \phi_0 - \pi/2)$$

Sinusoids

$$v_1(t) = V_m \cdot \sin(\omega t), \quad v_2(t) = V_m \cdot \sin(\omega t + \theta)$$

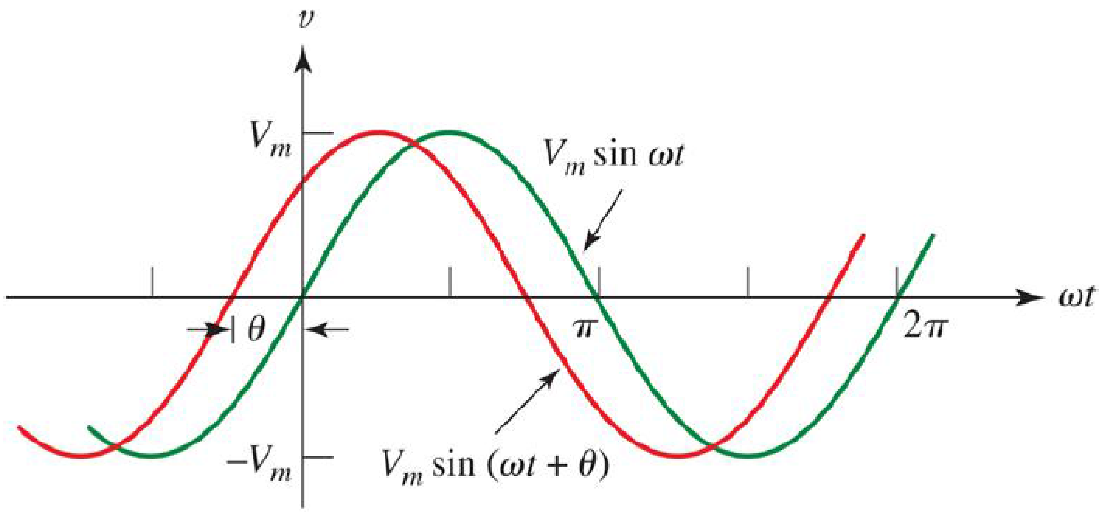


v_2 "leads" v_1 by θ

v_1 "lags" v_2 by θ

Sinusoids and Exponential Form

$$v_1(t) = V_m \cdot \sin(\omega t), \quad v_2(t) = V_m \cdot \sin(\omega t + \theta)$$

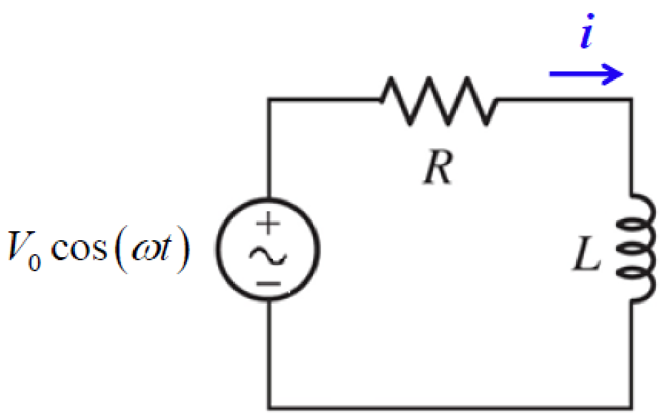


$$e^{j(\theta)} = \cos(\theta) + j \cdot \sin(\theta)$$

$$V_m e^{j(\omega t+\theta)} = V_m \cos(\omega t + \theta) + j \cdot V_m \sin(\omega t + \theta)$$

$$\mathrm{Re}\left\{V_m e^{j(\omega t+\theta)}\right\} = V_m \cos(\omega t + \theta) \qquad \mathrm{Im}\left\{V_m e^{j(\omega t+\theta)}\right\} = V_m \sin(\omega t + \theta)$$

RL Circuit with a Sinusoidal Source



$$\frac{d}{dt} i(t) + \frac{R}{L} \cdot i(t) = \frac{V_0}{L} \cos(\omega t)$$

-- oscillates forever

-- never settles to a DC value (e.g. zero)

It's possible that the solution is of the form $i(t) = I_0 \cos(\omega t + \theta)$

Substituting $i(t)$ into the differential equation…

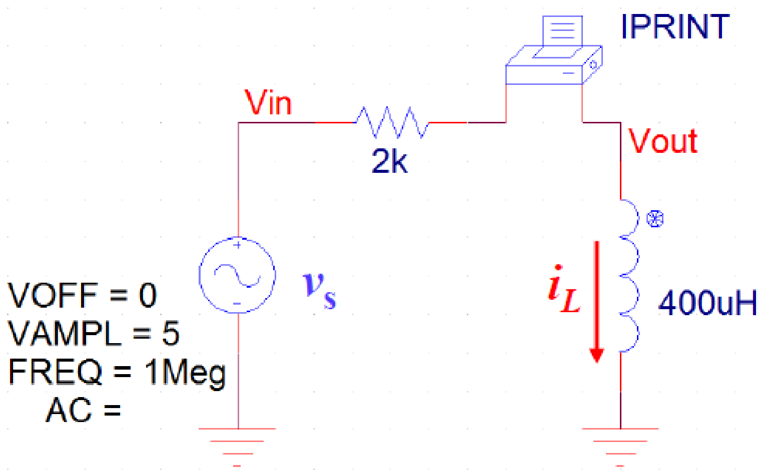
$$-\omega I_0 \sin(\omega t + \theta) + \frac{R}{L} I_0 \cos(\omega t + \theta) = \frac{V_0}{L} \cos(\omega t)$$

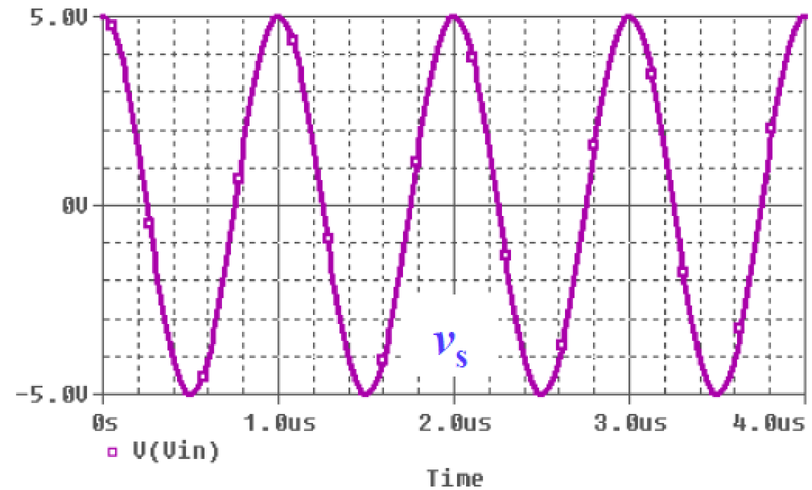
Solving for I_0 and substituting back into $i(t)$ yields

$$i(t) = \boxed{\frac{V_0}{\sqrt{R^2 + \omega^2 L^2}}} \cos\left\{\omega t - \boxed{\tan^{-1}\left(\frac{\omega L}{R}\right)}\right\}$$

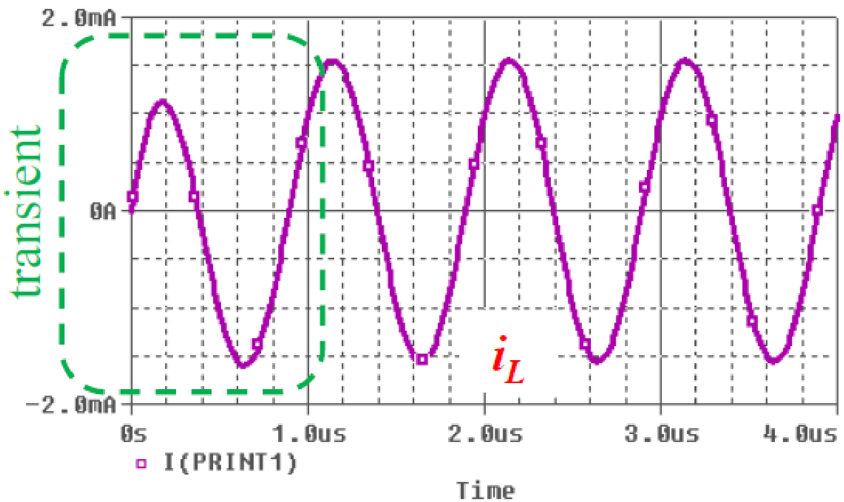
amplitude scaling,
phase shift

RL Circuit with a Sinusoidal Source





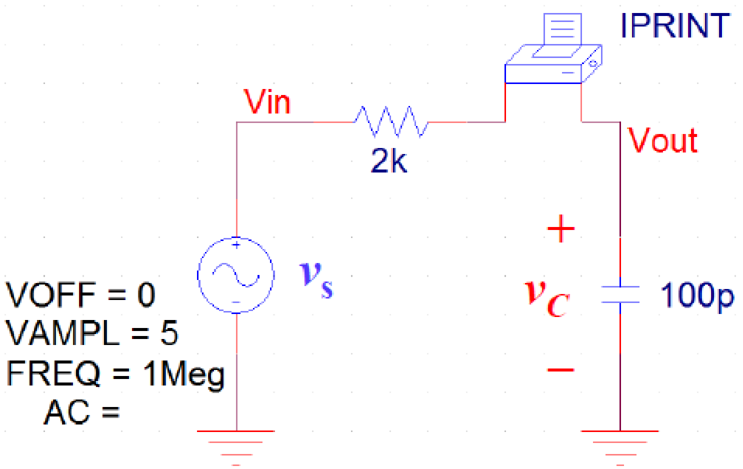
$$5 \cdot \tau = 5 \cdot \frac{L}{R} = (5)\frac{400\ \mu\text{H}}{2\ \text{k}\Omega} = 1\ \mu\text{s}$$

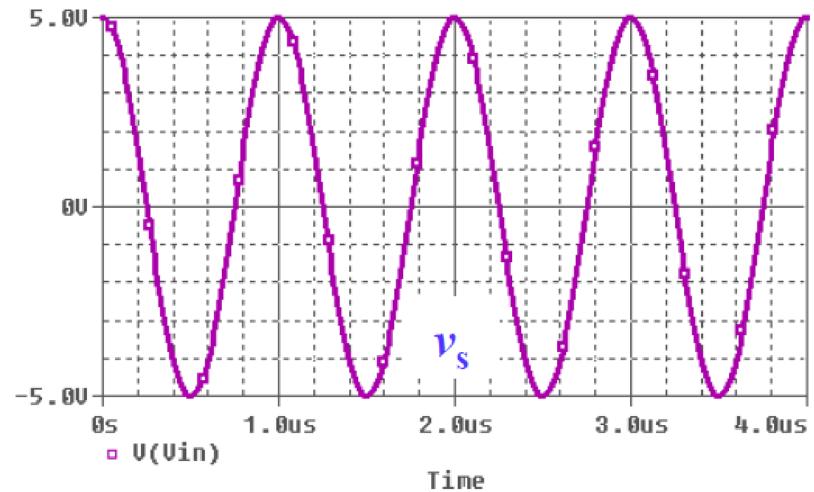


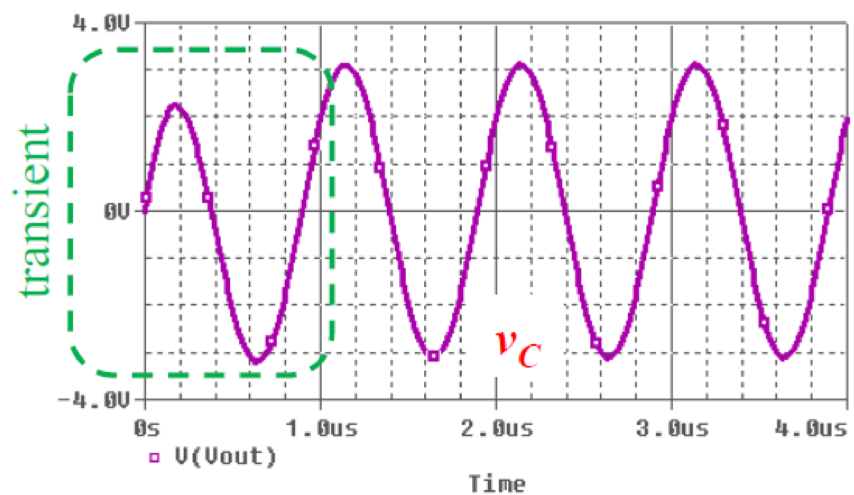
The *RL* circuit's *transient* response is negligible after $\approx 5\tau$.

The remaining response is sinusoidal.

RC Circuit with a Sinusoidal Source





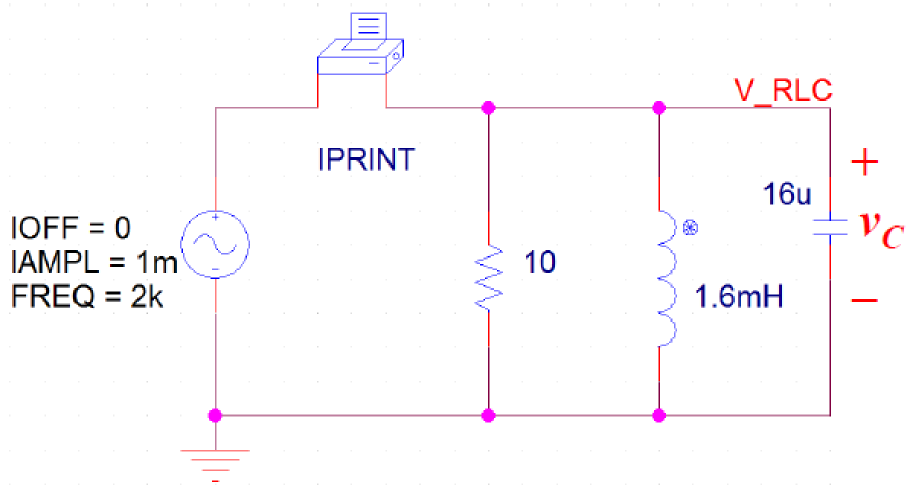


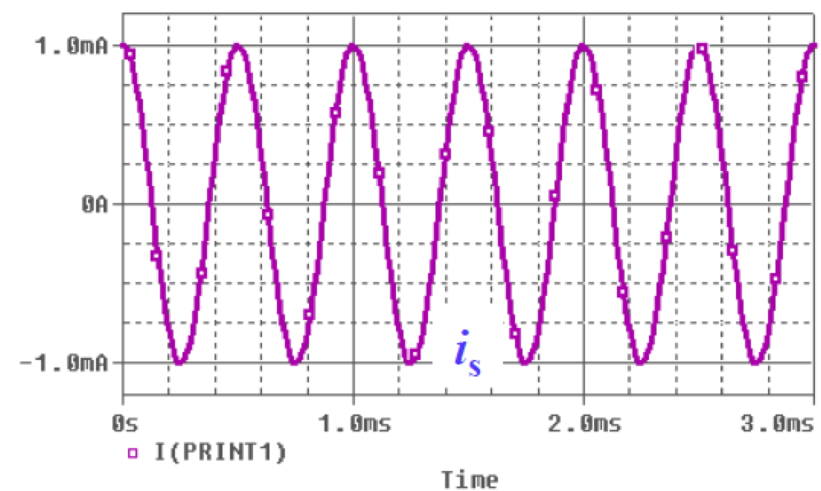
$$5 \cdot \tau = 5 \cdot RC$$
$$= (5)(2\ \text{k}\Omega)(100\ \text{pF}) = 1\ \mu\text{s}$$

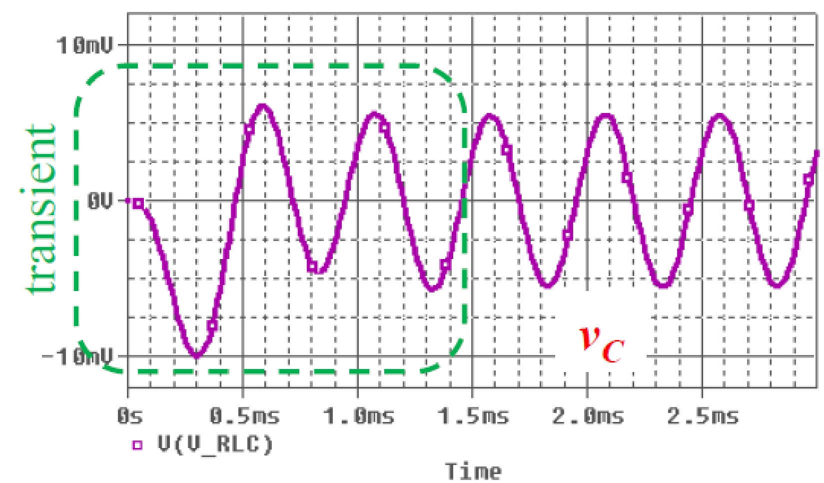
The *RC* circuit's *transient* response is negligible after $\approx 5\tau$.

The remaining response is sinusoidal.

RLC Circuit with a Sinusoidal Source







$$v_c(t) = e^{-\alpha t}\left[V_1 \cos(\omega_d t) + V_2 \sin(\omega_d t)\right] + V_3$$

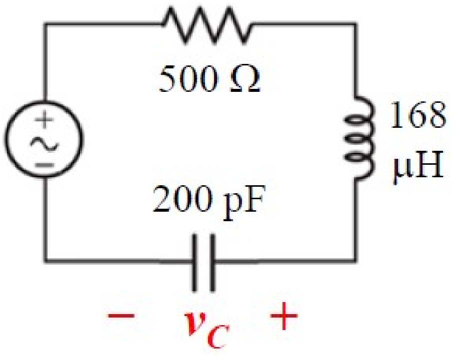
$$\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot 10\,\Omega \cdot 16\,\mu\text{F}} = 3125\,\frac{\text{rad}}{\text{s}}$$

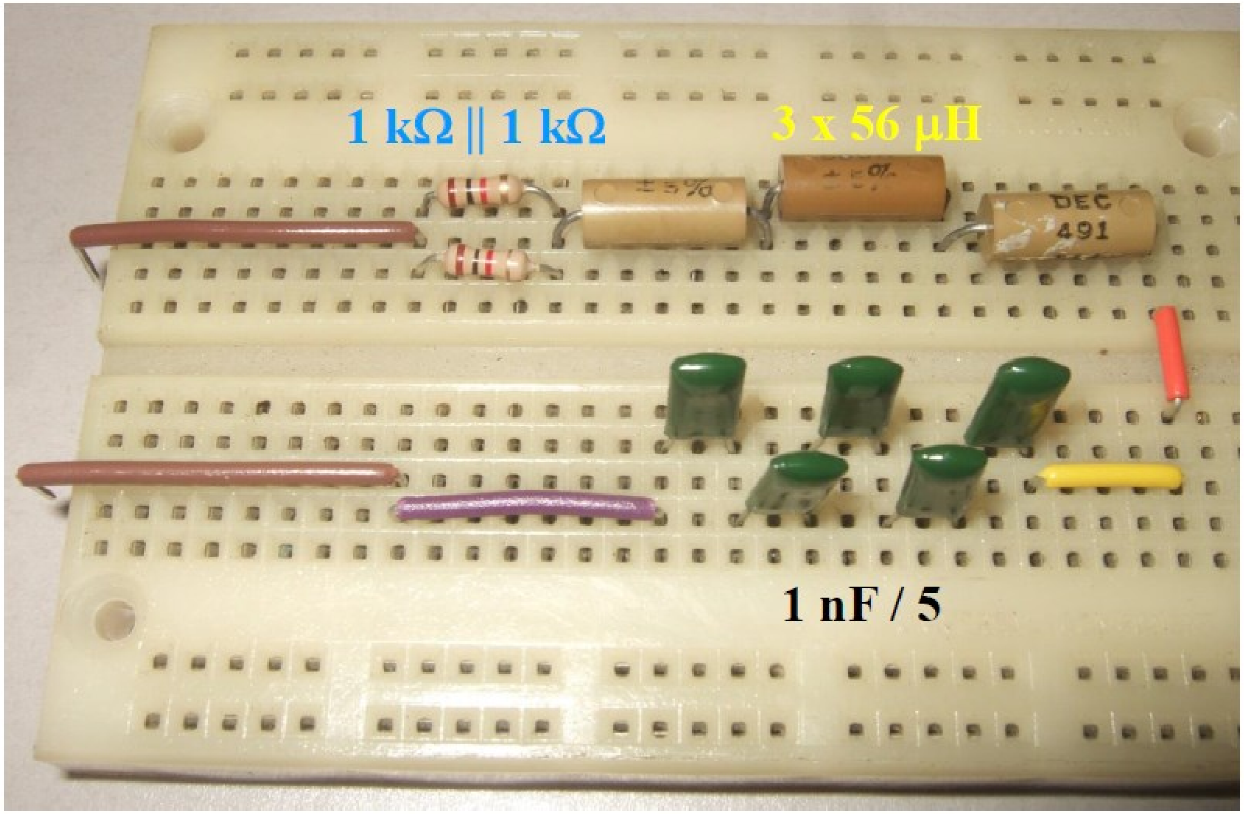
$$t_s \approx {}^{5}\!/_{\alpha} \approx 1.6\text{ ms}$$

The *RLC* circuit's *transient* response is negligible after $\approx t_S$.

The remaining response is sinusoidal.

RLC Sinusoidal Steady State Dema

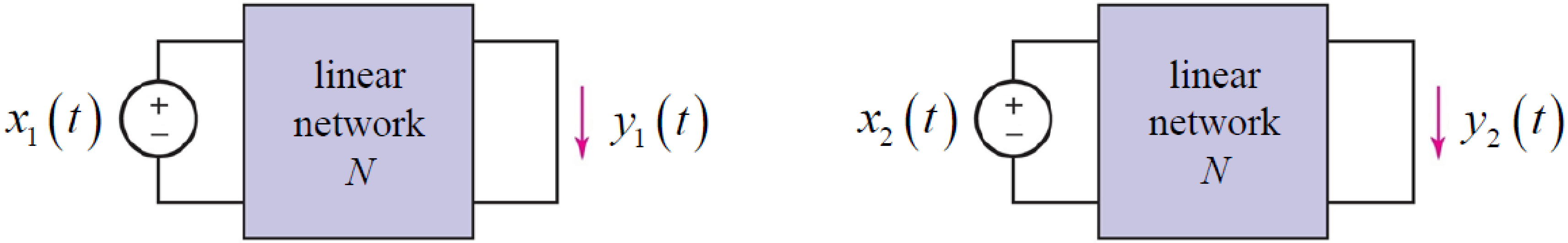




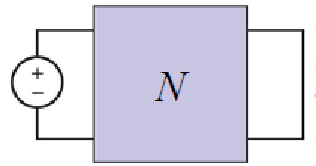
For a circuit that contains at least one sinusoidal source…

After the transients have died out ($t > 5\tau$ or $t > t_s$), the remaining response (i.e. the voltage or current *anywhere* in the circuit) is sinusoidal.

Consequences of Linearity

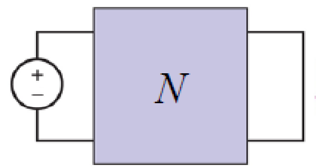


$$A_1 x_1(t-t_1) + j \cdot A_2 x_2(t-t_2)$$



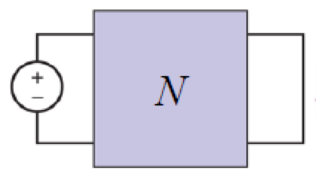
$$A_1 y_1(t-t_1) + j \cdot A_2 y_2(t-t_2)$$

$$\mathrm{Re}\{A_1 x_1(t-t_1) + j \cdot A_2 x_2(t-t_2)\} = A_1 x_1(t-t_1)$$



$$\mathrm{Re}\{A_1 y_1(t-t_1) + j \cdot A_2 y_2(t-t_2)\} = A_1 y_1(t-t_1)$$

$$\mathrm{Im}\{A_1 x_1(t-t_1) + j \cdot A_2 x_2(t-t_2)\} = A_2 x_2(t-t_2)$$

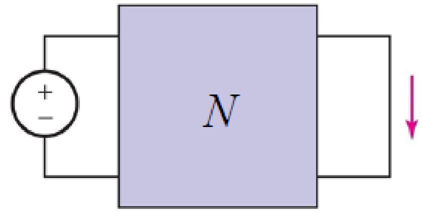


$$\mathrm{Im}\{A_1 y_1(t-t_1) + j \cdot A_2 y_2(t-t_2)\} = A_2 y_2(t-t_2)$$

The <u>real</u> part of the response is caused by the <u>real</u> part of the source(s).
The <u>imaginary</u> part of the response is caused by the <u>imaginary</u> part of the source(s).

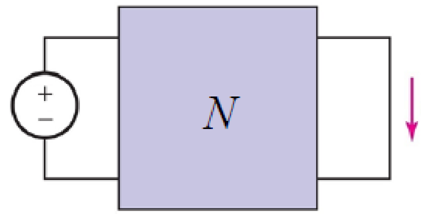
Consequences of Linearity

$$V_m \cos\left(\omega t - \theta_1\right)$$



$$I_m \cos\left(\omega t - \theta_2\right)$$

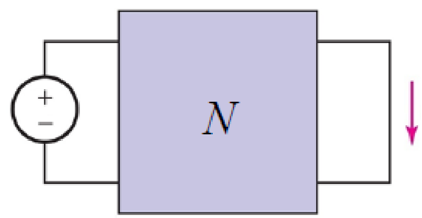
$$j \cdot V_m \sin\left(\omega t - \theta_1\right)$$



$$j \cdot I_m \sin\left(\omega t - \theta_2\right)$$

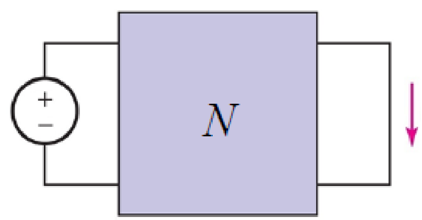
$$e^{j\theta} = \cos\left(\theta\right) + j \cdot \sin\left(\theta\right)$$

$$V_m e^{j(\omega t - \theta_1)}$$



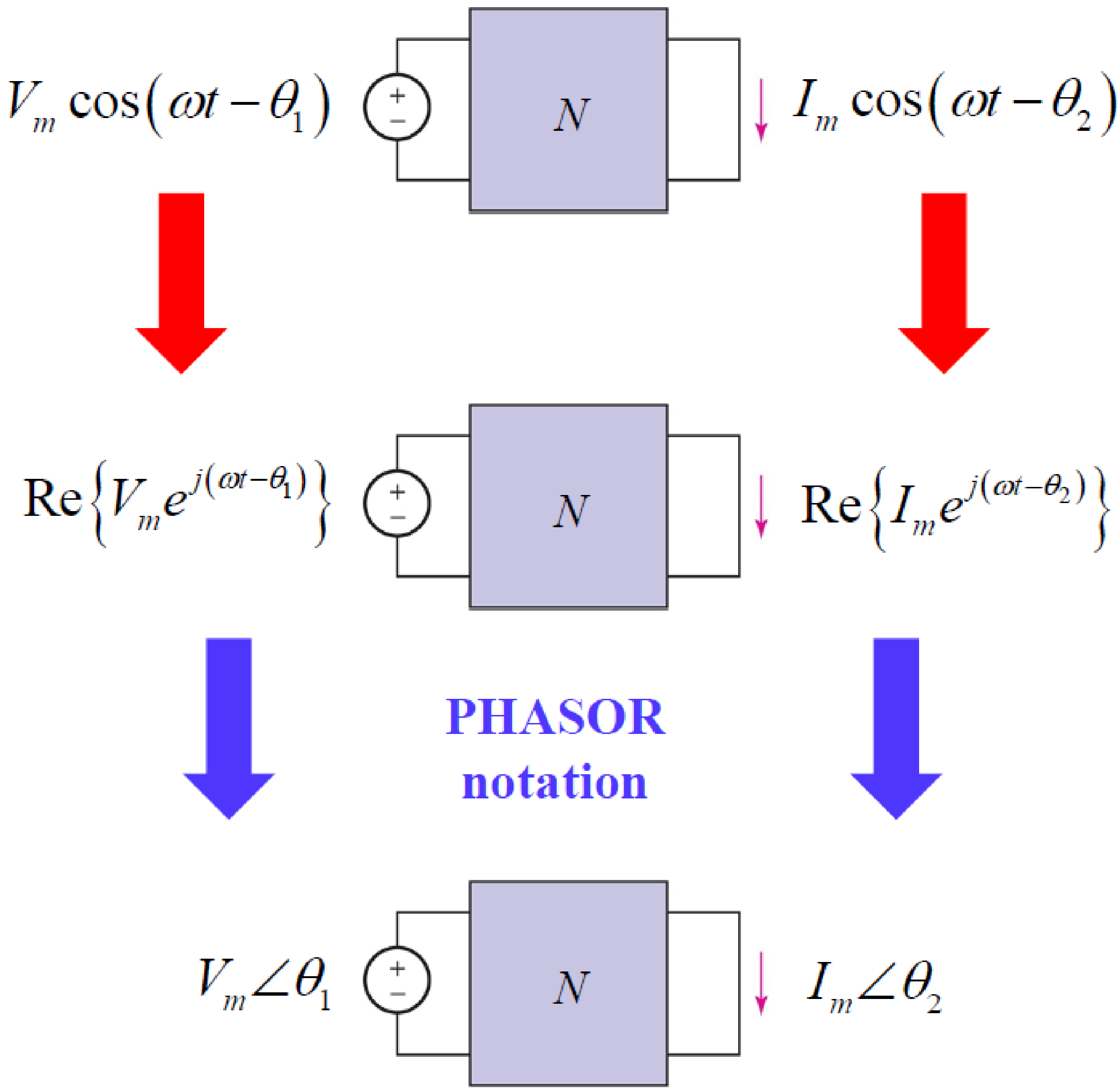
$$I_m e^{j(\omega t - \theta_2)}$$

$$\mathrm{Re}\left\{V_m e^{j(\omega t - \theta_1)}\right\}$$

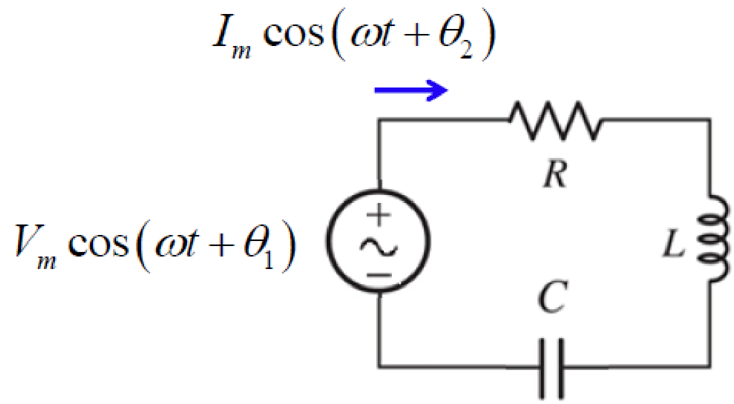


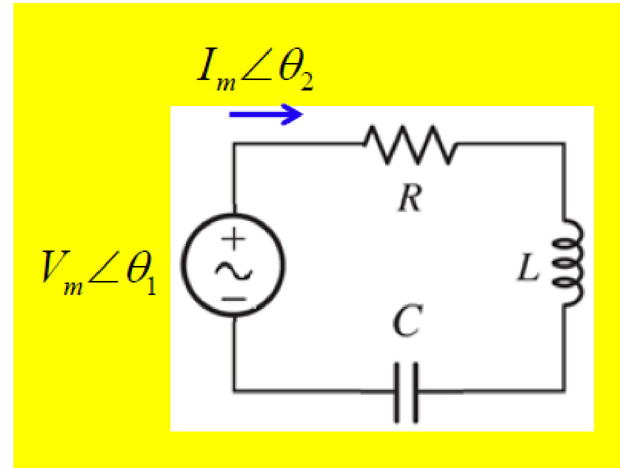
$$\mathrm{Re}\left\{I_m e^{j(\omega t - \theta_2)}\right\}$$

Sinusoidal vs. Complex Representation



Phasor Notation





$$V_m \cos(\omega t + \theta_1) = \mathrm{Re}\left\{V_m e^{j(\omega t + \theta_1)}\right\} = \mathrm{Re}\left\{V_m e^{j\omega t} e^{j\theta_1}\right\} \Rightarrow V_m e^{j\theta_1} = V_m \angle \theta_1$$

$$I_m \cos(\omega t + \theta_2) = \mathrm{Re}\left\{I_m e^{j(\omega t + \theta_2)}\right\} = \mathrm{Re}\left\{I_m e^{j\omega t} e^{j\theta_2}\right\} \Rightarrow I_m e^{j\theta_2} = I_m \angle \theta_2$$

Assume all voltages & currents oscillate with frequency $\omega = 2\pi f$ …

Pick off the amplitude & phase for each *v/i* ; write each in polar form.

phasor notation

Phasor Voltage vs. Current: *R, L, C*

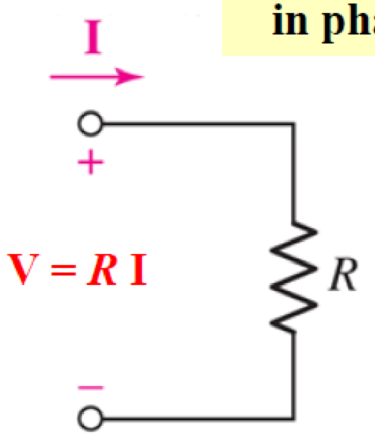
$$v(t) = V_m \cos(\omega t + \theta_1) \qquad i(t) = I_m \cos(\omega t + \theta_2)$$

$$v(t) = R \cdot i(t)$$

For this equation to be true,

$V_m = I_m \cdot R$ and $\theta_1 = \theta_2$

v and i are **in phase**

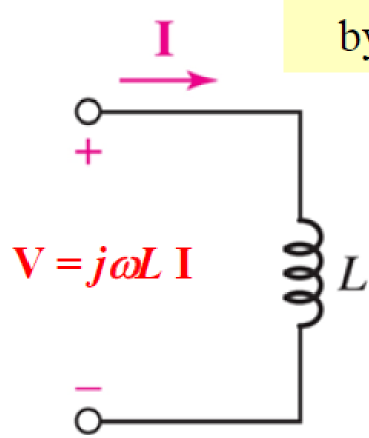


$$v_L(t) = L \cdot \frac{d}{dt} i_L(t)$$

For this equation to be true,

$$\theta_1 = \theta_2 + 90^\circ \ , \quad \frac{V_m}{I_m} = \omega L$$

v* leads *i
by 90°

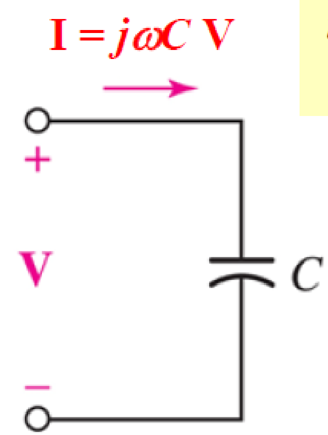


$$i_C(t) = C \cdot \frac{d}{dt} v_C(t)$$

For this equation to be true,

$$\theta_2 = \theta_1 + 90^\circ \ , \quad \frac{I_m}{V_m} = \omega C$$

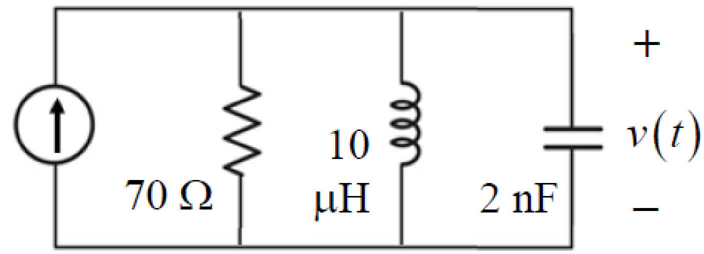
i* leads *v
by 90°



Example

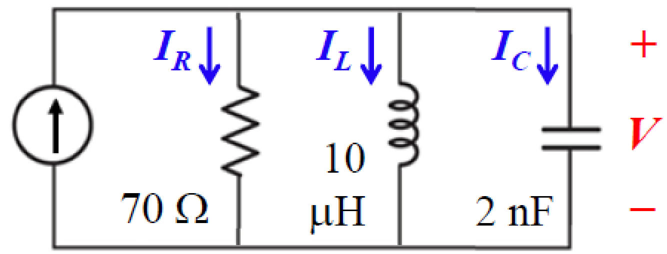
Determine $v(t)$.

$$i_s(t) = 8\cos\left(2\pi \cdot 10^6 t + 30^\circ\right) \text{ mA}$$



- Convert to phasor form…

$$\mathbf{I}_s = 8\angle 30^\circ \text{ mA}$$



- Employ the appropriate Kirchhoff Law(s)…

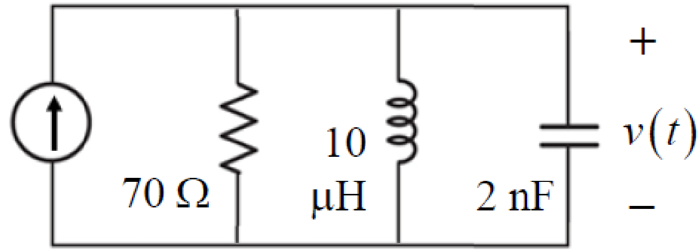
$$\mathbf{I}_s - \frac{\mathbf{V}}{R} - \frac{\mathbf{V}}{j\omega L} - j\omega C \cdot \mathbf{V} = 0$$

$$8\angle 30^\circ - \frac{\mathbf{V}}{70} - \frac{\mathbf{V}}{j\left(2\pi \cdot 10^6\right)\left(10 \cdot 10^{-6}\right)} - j\left(2\pi \cdot 10^6\right)\left(2 \cdot 10^{-9}\right) \cdot \mathbf{V} = 0$$

Example

Determine $v(t)$.

$$i_s(t) = 8\cos\left(2\pi\cdot 10^6 t + 30^\circ\right)\text{ mA}$$



- Convert between rectangular & polar forms as necessary…

$$8\angle 30^\circ - \frac{\mathbf{V}}{70} - \frac{\mathbf{V}}{j\left(2\pi\cdot 10^6\right)\left(10\cdot 10^{-6}\right)} - j\left(2\pi\cdot 10^6\right)\left(2\cdot 10^{-9}\right)\cdot\mathbf{V} = 0$$

$$\mathbf{V}\left\{\frac{1}{70} + \frac{1}{j(62.8)} + j(0.0126)\right\} = 8\angle 30^\circ$$

$$\mathbf{V}\cdot\left(0.0143 - 0.0159j + 0.0126j\right) = 8\angle 30^\circ$$

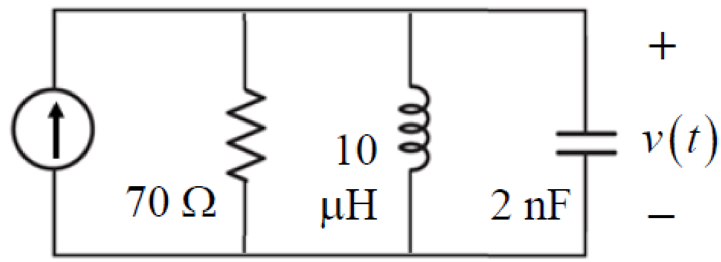
$$\mathbf{V}\cdot\left(0.0143 - 0.0033j\right) = 8\angle 30^\circ$$

$$\mathbf{V}\cdot\left\{0.0147\angle -13^\circ\right\} = 8\angle 30^\circ$$

Example

Determine $v(t)$.

$$i_s(t) = 8\cos\left(2\pi \cdot 10^6 t + 30^\circ\right) \text{mA}$$



- Convert between rectangular & polar forms as necessary…

$$\mathbf{V} \cdot \{0.0147\angle -13^\circ\} = 8\angle 30^\circ$$

$$\mathbf{V} = \frac{8\angle 30^\circ \text{ mA}}{0.0147\angle -13^\circ \ \Omega} = 544\angle 43^\circ \text{ mV}$$

- Convert from phasors to time domain…

```
omega = 2*pi*10^6;
I = 8*exp(j*30*pi/180);

R = 70;
L = 10e-6;
C = 2e-9;

Y = (1/R + 1/(j*omega*L) + j*omega*C);
V = I / Y;

abs(V)
                        ans = 545.2174
angle(V)*180/pi
                        ans = 43.1941
```

Impedance

Impedance, Z is the ratio of phasor voltage to phasor current for an electrical element or network. → like resistance, but it is *complex*

-- It is a measure of an element/network's ability to *impede* current flow.

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}}$$

For a resistor, $\mathbf{V} = R \cdot \mathbf{I} \quad \Rightarrow \quad \mathbf{Z}_R = R$

-- current and voltage are always in-phase

-- there is *no frequency dependence*

For an inductor, $\mathbf{V} = j\omega L \cdot \mathbf{I} \quad \Rightarrow \quad \mathbf{Z}_L = j\omega L$

-- voltage always *leads* current by 90°

-- at higher frequencies, *less* current is passed (for constant V)

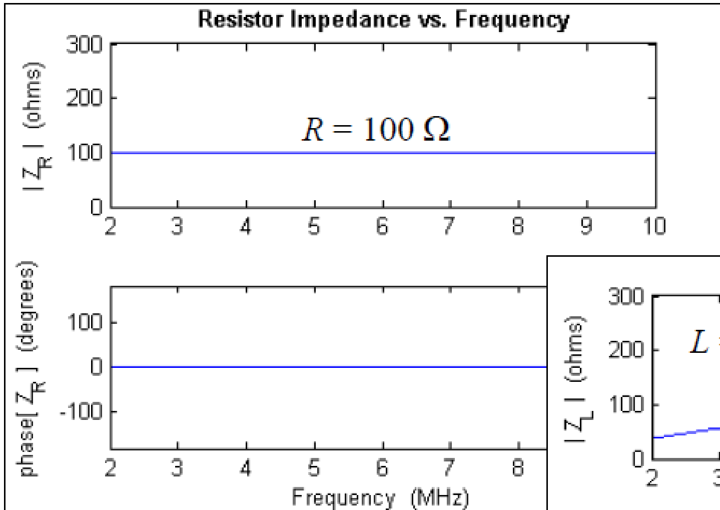
For a capacitor, $\mathbf{I} = j\omega C \cdot \mathbf{V} \quad \Rightarrow \quad \mathbf{Z}_C = 1/j\omega C = -j/\omega C$

-- current always *leads* voltage by 90°

-- at higher frequencies, *more* current is passed (for constant V)

Impedance vs. Frequency

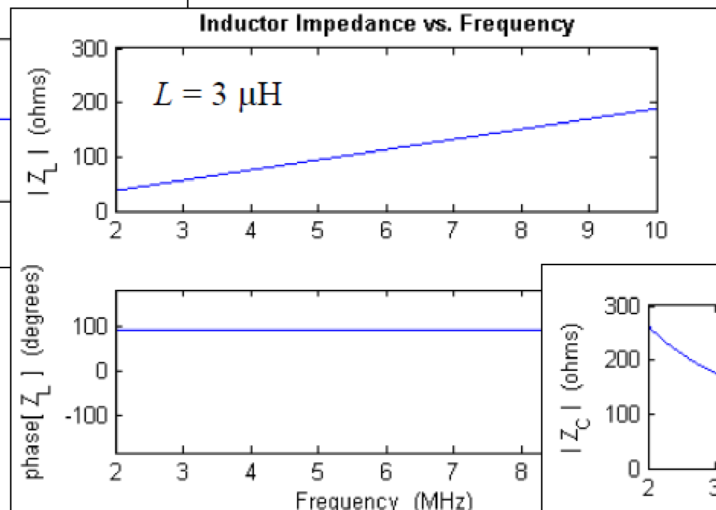
$$\omega = 2\pi f$$



$$\mathbf{Z}_R = R$$

$$|\mathbf{Z}_R| = R$$

$$\theta(\mathbf{Z}_R) = 0$$



$$\mathbf{Z}_L = j\omega L$$

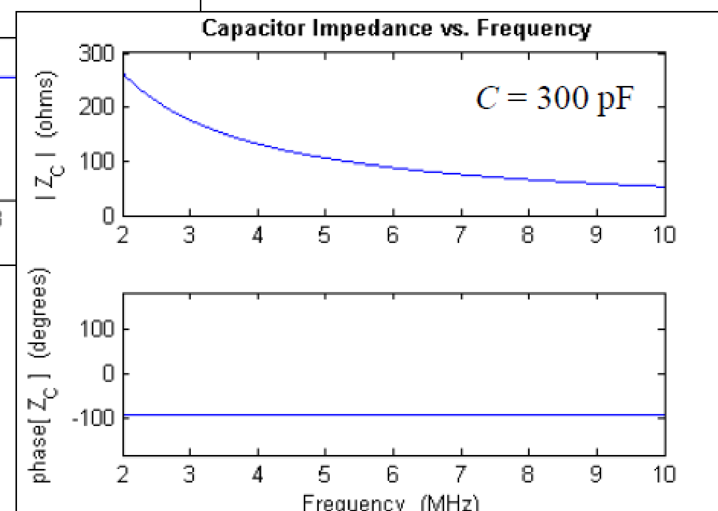
$$|\mathbf{Z}_L| = \omega L$$

$$\theta(\mathbf{Z}_L) = 90^\circ$$

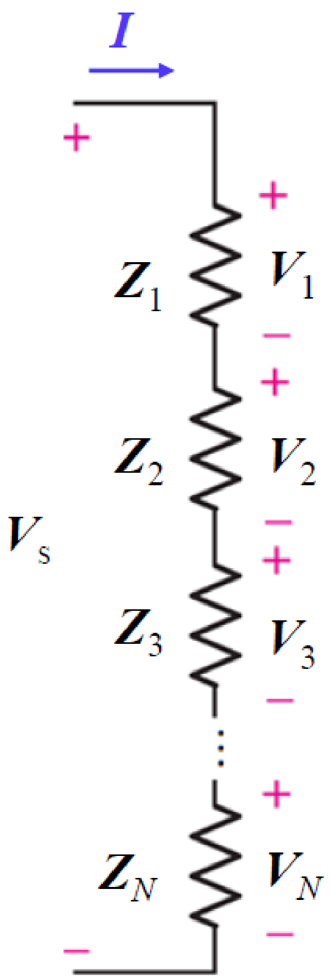
$$\mathbf{Z}_C = 1/j\omega C$$

$$|\mathbf{Z}_C| = 1/\omega C$$

$$\theta(\mathbf{Z}_C) = -90^\circ$$



KVL, Impedances in Series



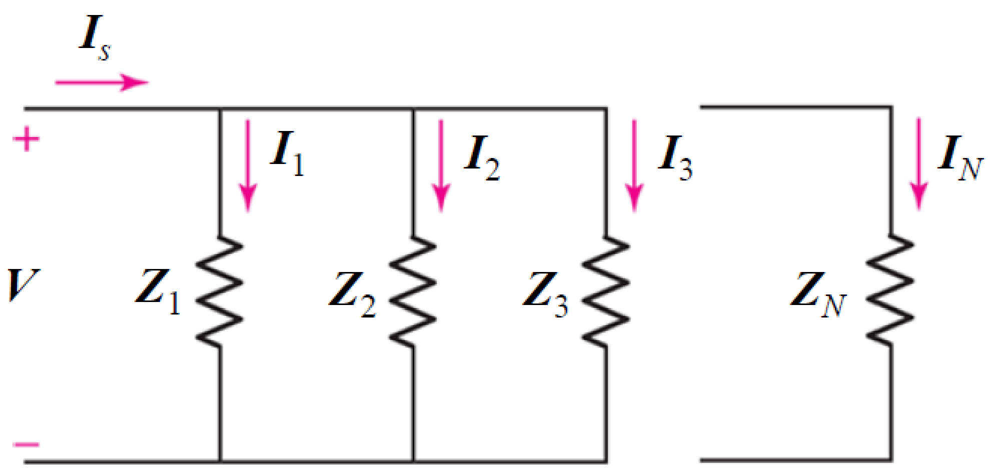
$$\mathbf{V}_s = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 + \ldots + \mathbf{V}_N$$
$$= \mathbf{I}\left[\mathbf{Z}_1 + \mathbf{Z}_2 + \ldots + \mathbf{Z}_N\right]$$
$$= \mathbf{I} \cdot \sum_{n=1}^{N} \mathbf{Z}_n$$

$$\mathbf{I} = \frac{\mathbf{V}_s}{\sum_{n=1}^{N} \mathbf{Z}_n}$$

Impedances in **series** are combined *like resistors in series*.

$$R_s = \sum_{n=1}^{N} R_n \quad \Rightarrow \quad \boxed{\mathbf{Z}_s = \sum_{n=1}^{N} \mathbf{Z}_n}$$

KCL, Impedances in Parallel



$$\begin{aligned}\mathbf{I}_s &= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \ldots + \mathbf{I}_N \\ &= \mathbf{V}\left[1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + \ldots + 1/\mathbf{Z}_N\right] \\ &= \mathbf{V} \cdot \sum_{n=1}^{N} 1/\mathbf{Z}_n\end{aligned}$$

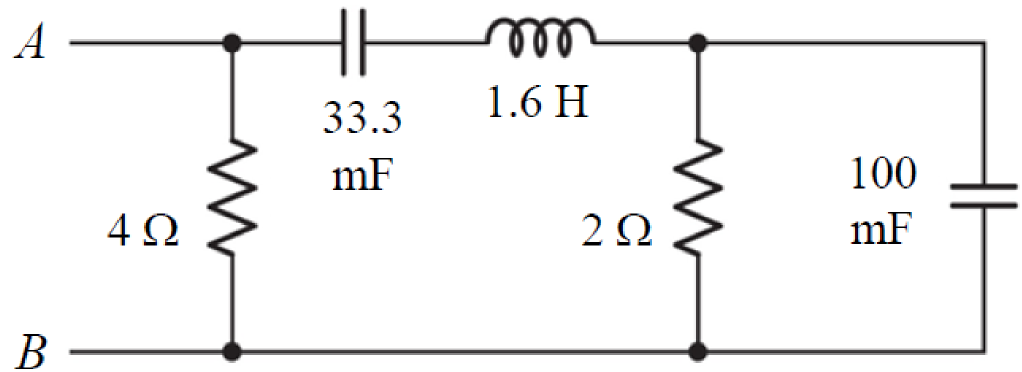
$$\mathbf{V} = \frac{\mathbf{I}_s}{\sum_{n=1}^{N} 1/\mathbf{Z}_n}$$

Impedances in **parallel** are combined *like resistors in parallel*.

$$1/R_p = \sum_{n=1}^{N} 1/R_n \quad \Rightarrow \quad 1/\mathbf{Z}_p = \sum_{n=1}^{N} 1/\mathbf{Z}_n \qquad \mathbf{Z}_p = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

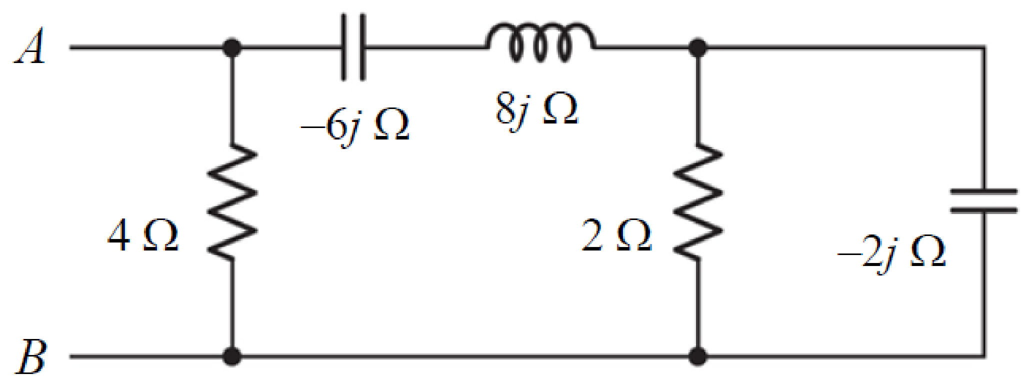
Example

Determine the equivalent impedance of the network at terminals A–B if $\omega = 5$ rad/s.



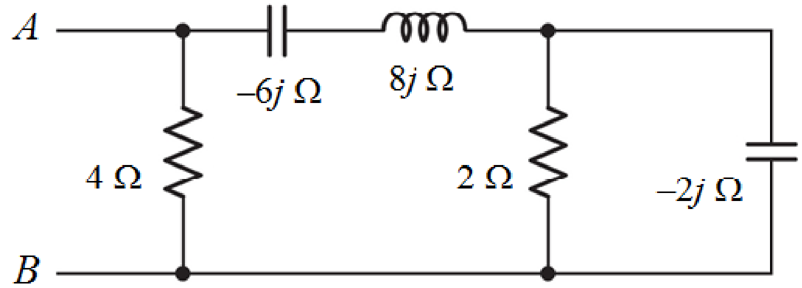
- Convert all resistances, inductances, capacitances into *impedances*…

$$\mathbf{Z}_R = R$$
$$\mathbf{Z}_L = j\omega L$$
$$\mathbf{Z}_C = -j/\omega C$$

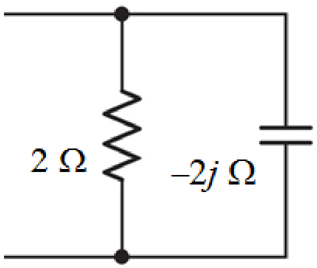


Example

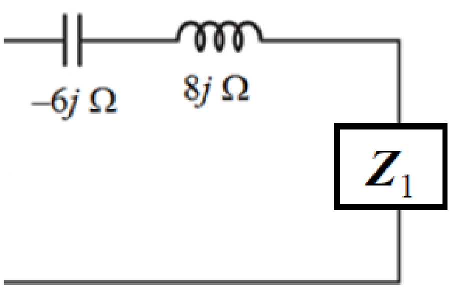
- Combine impedances in series & parallel, starting away from *A–B* and working towards *A–B*…



$$\mathbf{Z}_1 = \frac{(2)(-2j)}{(2-2j)} = \frac{4\angle -90^\circ}{\sqrt{8}\angle -45^\circ} = \sqrt{2}\angle -45^\circ = 1 - j\ \Omega$$

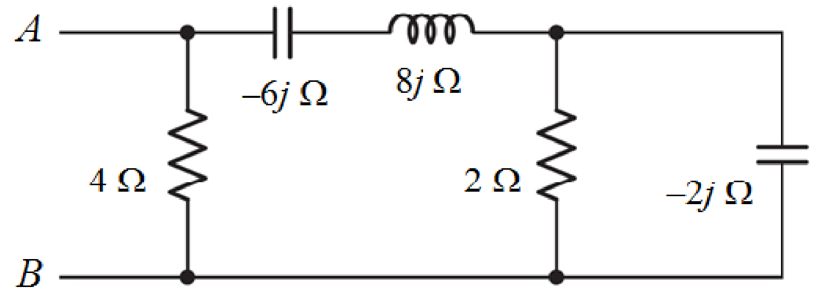


$$\begin{aligned}\mathbf{Z}_2 &= \mathbf{Z}_1 + 8j - 6j \\ &= 1 - j + 8j - 6j = 1 + j\ \Omega\end{aligned}$$



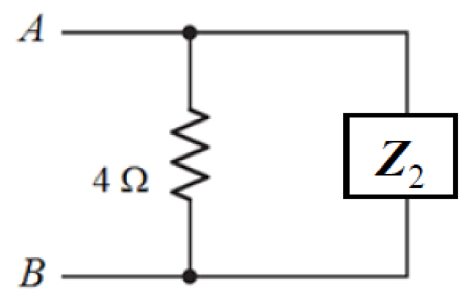
Example

- Combine impedances in series & parallel, starting away from *A–B* and working towards *A–B*…



$$\mathbf{Z}_{A-B} = \frac{\mathbf{Z}_2(4)}{(\mathbf{Z}_2 + 4)} = \frac{(1+j)(4)}{1+j+4}$$

$$= \frac{4+4j}{5+j} = \frac{\sqrt{32}\angle 45.0^\circ}{\sqrt{26}\angle 11.3^\circ} =$$



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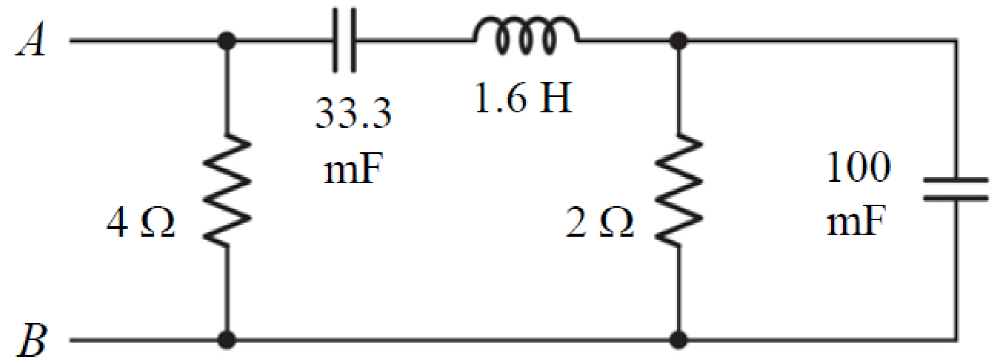



```
omega = 5;
C1 = 100e-3;
C2 = 33.3e-3;
R1 = 2;
R2 = 4;
L1 = 1.6;
```

```
Z_C1 = 1/(j*omega*C1);
Z_C2 = 1/(j*omega*C2);
Z_R1 = R1;
Z_R2 = R2;
Z_L1 = j*omega*L1;
```

```
Z_eq1 = Z_C1*Z_R1/(Z_C1 + Z_R1);
Z_eq2 = Z_eq1 + Z_L1 + Z_C2;
Z_ab = Z_R2*Z_eq2 / (Z_R2 + Z_eq2)
Z_ab =
                  0.9217 + 0.6120i
abs(Z_ab)
                     ans = 1.1063
phase(Z_ab)*180/pi
                     ans = 33.5837
```

Reactance

When impedance is expressed in rectangular form, $\mathbf{Z} = \mathbf{V}/\mathbf{I} = R + j \cdot X$

R is the resistance and X is the **reactance**.

Reactance is a measure of the *energy-storage* capability of an electrical network.

For a resistor, $\mathbf{Z}_R = R \quad \Rightarrow \quad X = 0$

-- zero reactance (cannot store electromagnetic energy)

For an inductor, $\mathbf{Z}_L = j\omega L \quad \Rightarrow \quad X = \omega L$

-- at higher frequencies, reactance is higher (stores *more* electromagnetic energy)

For a capacitor, $\mathbf{Z}_C = 1/j\omega C \quad \Rightarrow \quad X = 1/\omega C$

-- at higher frequencies, reactance is lower (stores *less* electromagnetic energy)

Admittance

Admittance, Y is the ratio of phasor current to phasor voltage for an electrical element or network. → like *conductance*, but it is *complex*

$$\mathbf{Y} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{1}{\mathbf{Z}}$$

For a resistor, $\mathbf{V} = R \cdot \mathbf{I} \quad \Rightarrow \quad \mathbf{Y}_R = 1/R = G$

-- current and voltage are always in-phase

-- there is no frequency dependence

For an inductor, $\mathbf{Z}_L = j\omega L \quad \Rightarrow \quad \mathbf{Y}_L = 1/j\omega L = -j/\omega L$

-- voltage always *leads* current by 90°

-- at higher frequencies, *less* current is passed

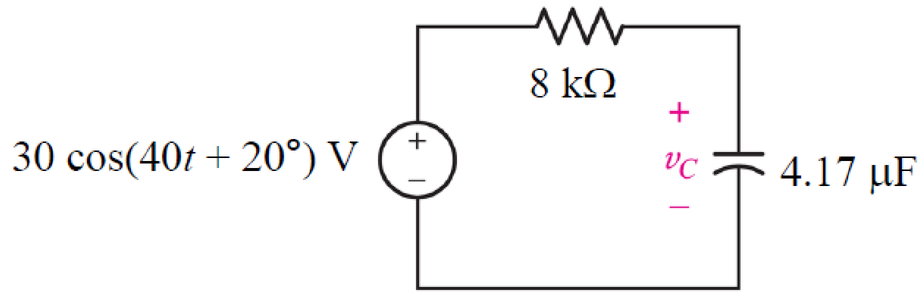
For a capacitor, $\mathbf{Z}_C = 1/j\omega C \quad \Rightarrow \quad \mathbf{Y}_C = j\omega C$

-- current always *leads* voltage by 90°

-- at higher frequencies, *more* current is passed

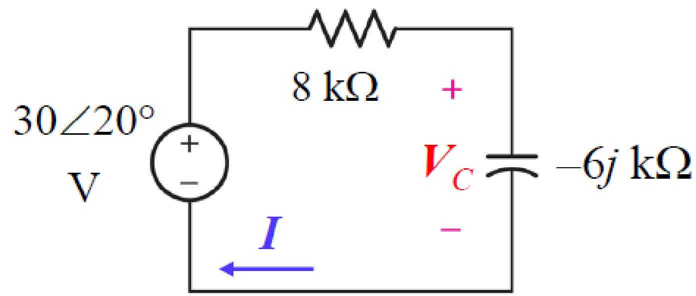
Example

Determine $v_C(t)$.



- Convert the circuit from the *time domain* to the *phasor domain*.

$$\mathbf{Z}_C = -j/\omega C = -j/\left(40 \cdot 4.17 \cdot 10^{-6}\right) = -6j \text{ k}\Omega$$



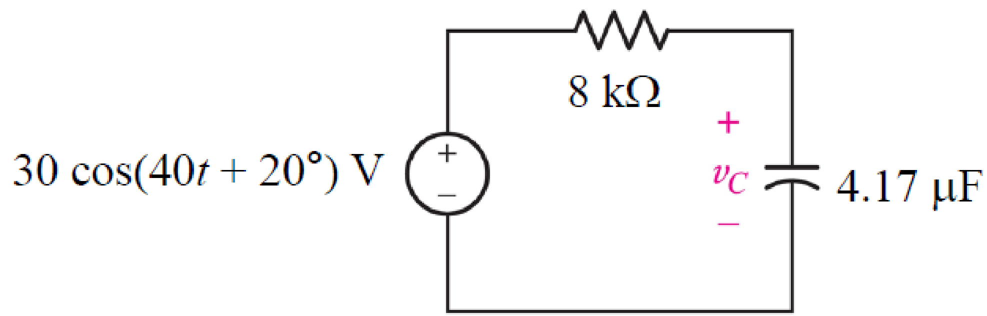
- Use KVL/KCL to solve for *V*/*I* in the phasor domain.

$$-30\angle 20° + \mathbf{I}(8) + \mathbf{I}(-6j) = 0$$

$$\mathbf{V}_C = (-6j)(\mathbf{I})$$

Example

Determine $v_C(t)$.



- Perform complex algebra to find *V*/*I*…

$$-30\angle 20° + \mathbf{I}(8) + \mathbf{I}(-6j) = 0 \qquad \mathbf{I}(8-6j) = 30\angle 20°$$

$$\mathbf{I} = \frac{30\angle 20°\ \text{V}}{8-6j\ \text{k}\Omega} = \frac{30\angle 20°}{\sqrt{8^2+6^2}\angle \tan^{-1}\{-6/8\}} = \frac{30\angle 20°}{10\angle -37°} = 3\angle 57°\ \text{mA}$$

$$\begin{aligned}\mathbf{V}_C &= (-6j\ \text{k}\Omega)(3\angle 57°\ \text{mA}) \\ &= (6\angle -90°\ \text{k}\Omega)(3\angle 57°\ \text{mA}) \\ &= 18\angle -33°\ \text{V}\end{aligned}$$

- Convert back to the time domain…