⊛ Parametrik olarak verilen:

$$C_1 : \left\{ \begin{array}{l} x = t \\ y = 1-t \\ z = 1+t^2 \end{array} \right\} \quad \text{ve} \quad C_2 : \left\{ \begin{array}{l} x = 1+s \\ y = s-2 \\ z = 4+s^2 \end{array} \right\}$$

eğrileri arasındaki açıyı bulunuz.

* İki eğrinin kesiştiği noktadaki açı, teğet vektörleri arasındaki açıdır. Önce kesişim noktasını bulalım.

$$\left. \begin{array}{l} t = 1+s \\ 1-t = s-2 \\ 1+t^2 = 4+s^2 \end{array} \right\} \quad \begin{array}{r} t - s = 1 \\ -t - s = -3 \\ \hline \boxed{s = 1} \quad \boxed{t = 2} \end{array}$$

$$C_1 : \vec{r_1}(t) = t\,\vec{i} + (1-t)\,\vec{j} + (1+t^2)\,\vec{k} \xrightarrow{\text{Teğeti}} \vec{r_1'}(t) = \vec{i} - \vec{j} + 2t\,\vec{k} \overset{t=2}{\Longrightarrow}$$

$$\vec{r_1'}(2) = \vec{i} - \vec{j} + 4\vec{k}$$

$$C_2 : \vec{r_2}(t) = (1+s)\,\vec{i} + (s-2)\,\vec{j} + (4+s^2)\,\vec{k} \xrightarrow{\text{Teğeti}} \vec{r_2'}(t) = \vec{i} + \vec{j} + 2s\,\vec{k} \overset{s=1}{\Longrightarrow}$$

$$\vec{r_2'}(1) = \vec{i} + \vec{j} + 2\vec{k}$$

$$\cos\theta = \frac{\vec{r_1'} \cdot \vec{r_2'}}{|\vec{r_1'}| \cdot |\vec{r_2'}|} = \frac{\langle 1,-1,4 \rangle \cdot \langle 1,1,2 \rangle}{\sqrt{18} \cdot \sqrt{6}} = \frac{1-1+8}{3\sqrt{2} \cdot \sqrt{6}} = \frac{4}{3\sqrt{3}}$$

$$\theta = \operatorname{ArcCos}\left(\frac{4}{3\sqrt{3}}\right)$$

⊛ Aşağıdaki fonksiyonların tanım bölgelerini çiziniz.

a) $f(x,y)=\ln(1-x^2-y^2)+\sqrt{xy}$

b) $f(x,y)=\ln(x-y^2)+\sqrt{4-x^2-y^2}$

c) $f(x,y)=\ln\frac{y}{x^2+y^2-1}$

d) $f(x,y)=\sqrt{\frac{1-x^2-y^2}{xy}}$

Cevap:

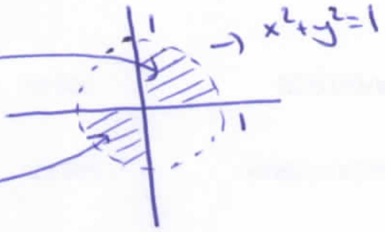
a) $1-x^2-y^2>0$ $\quad xy\geq 0$ olmalı

$\downarrow \quad\quad \downarrow$

$x^2+y^2<1 \quad xy\geq 0$

① $x^2+y^2<1$, $x\geq 0$, $y\geq 0$

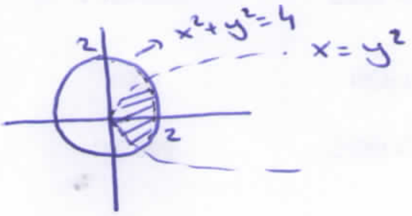
② $x^2+y^2<1$, $x\leq 0$, $y\leq 0$



b) $x-y^2>0$, $4-x^2-y^2\geq 0$

$\downarrow \quad\quad \downarrow$

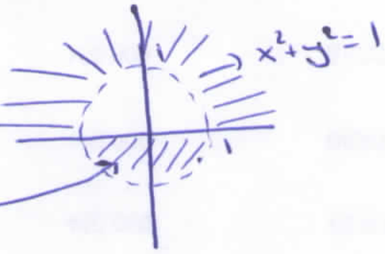
$x>y^2$, $4\geq x^2+y^2$ →



c) $\frac{y}{x^2+y^2-1}>0$, $x^2+y^2\neq 1$

① $y>0$, $x^2+y^2>1$

② $y<0$, $x^2+y^2<1$



d) $\frac{1-x^2-y^2}{xy}\geq 0$, $xy\neq 0$

① $1-x^2-y^2\geq 0$, $xy>0$

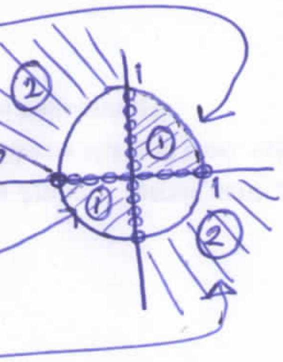
$\downarrow$

$1\geq x^2+y^2$ + $\begin{matrix}x>0\\y>0\end{matrix}$ $\quad \begin{matrix}x<0\\y<0\end{matrix}$

② $1-x^2-y^2\leq 0$ $\quad xy<0$

$\downarrow$

$1\leq x^2+y^2$ + $\begin{matrix}x>0\\y<0\end{matrix}$ $\quad \begin{matrix}x<0\\y>0\end{matrix}$



⊛ $z = x.f(ye^x) \Rightarrow \frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \frac{z}{x}$ old. gösteriniz.

$\frac{\partial z}{\partial x} = f.(ye^x) + x.ye^x f'(e^x y)$ →

$\frac{\partial z}{\partial y} = x.e^x f'(ye^x)$ → $\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = f + xy\cancel{e^x f'} - yx\cancel{e^x f'}$

$= f = \frac{z}{x}$

$(\frac{z}{x} = f(ye^x)$ olduğundan$)$

⊛ $\lim\limits_{(x,y)\to(1,1)} \frac{y\sin\pi x}{x+y-2}$ limitinin varlığını inceleyin.

$$\lim_{x\to1}\left[\lim_{y\to1}\frac{y\sin\pi x}{x+y-2}\right] = \lim_{x\to1}\left[\frac{\sin\pi x}{x-1}\right] = \lim_{x\to1}\frac{\pi\cos\pi x}{1} = -\pi$$

↓ 0/0 → L'H.

$$\lim_{y\to1}\left[\lim_{x\to1}\frac{y\sin\pi x}{x+y-2}\right] = \lim_{x\to1}\frac{0}{y-1} = 0$$

→ ≠ iki kat limit yok

⇓

Limit yok

⊛ $x - az = f(y-bz) \Rightarrow a.\frac{\partial z}{\partial x} + b.\frac{\partial z}{\partial y} = 1$ old. gösterin.

z: bağımlı x,y: bağımsız

⇓

x-az = f(y-bz) kapalı fonk.

F: x-az-f(y-bz)=0 olsun.

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-a+bf'(y-bz)}$ ↘

$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-f'(y-bz)}{-a+bf'(y-bz)}$ ↗

$a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} = \frac{-a}{-a+bf'} + \frac{bf'}{-a+bf'} = \underline{\underline{1}}$

⊛ $x \operatorname{Sin} z = 1 + z^2 y \Rightarrow \frac{\partial z}{\partial x} = ? \quad \frac{\partial^2 z}{\partial y \partial x} = ?$ $\quad f : x \operatorname{Sin} z - 1 - z^2 y = 0$

$$\frac{\partial z}{\partial x} = - \frac{\operatorname{Sin} z}{x \operatorname{Cos} z - 2zy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = - \frac{\operatorname{Cos} z . \frac{\partial z}{\partial y} (x \operatorname{Cos} z - 2zy) - \operatorname{Sin} z . \left[-x \operatorname{Sin} z \frac{\partial z}{\partial y} - 2z - 2y \frac{\partial z}{\partial y} \right]}{(x \operatorname{Cos} z - 2zy)^2}$$

$$\frac{\partial z}{\partial y} = + \frac{z^2}{x \operatorname{Cos} z - 2zy}$$

⊛ $x = u \operatorname{Cos} v$, $y = u \operatorname{Sin} v$, $z = f(x,y) \Rightarrow \left(\frac{\partial z}{\partial u} \right)^2 + \frac{1}{u^2} \left(\frac{\partial z}{\partial v} \right)^2 = \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2$

old. gösterin.

$z \rightarrow x, y \rightarrow u, v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = \operatorname{Cos} v \frac{\partial z}{\partial x} + \operatorname{Sin} v \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial v} = -u \operatorname{Sin} v \frac{\partial z}{\partial x} + u \operatorname{Cos} v \frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\partial u} \right)^2 = \operatorname{Cos}^2 v \left(\frac{\partial z}{\partial x} \right)^2 + 2 \operatorname{Sin} v \operatorname{Cos} v \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + \operatorname{Sin}^2 v \left(\frac{\partial z}{\partial y} \right)^2$$

$$\frac{1}{u^2} \left(\frac{\partial z}{\partial v} \right)^2 = \frac{1}{u^2} \left[u^2 \operatorname{Sin}^2 v \left(\frac{\partial z}{\partial x} \right)^2 - 2 \operatorname{Sin} v . \operatorname{Cos} v . u^2 \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} + u^2 \operatorname{Cos}^2 v \left(\frac{\partial z}{\partial y} \right)^2 \right]$$

$$= \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2$$

⊛ $\left.\begin{array}{l} u = z - x \\ v = y - z \\ w = x - y \end{array}\right\}$ olmak üzere $f(u,v,w)$ türevlenebilir ise $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$ olduğunu gösteriniz.

$$\begin{tikzcd} f \arrow[r, "\text{Ç.D.}"'] \arrow[rr, bend left, "\text{Ç.D.}"] & u,v,w \arrow[r, "\text{Ç.D.}"'] & x,y,z \end{tikzcd}$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial x}}_{-1} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial x}}_{0} + \frac{\partial f}{\partial w}\cdot\underbrace{\frac{\partial w}{\partial x}}_{1}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial y}}_{0} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial y}}_{1} + \frac{\partial f}{\partial w}\cdot\underbrace{\frac{\partial w}{\partial y}}_{-1}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial u}\cdot\underbrace{\frac{\partial u}{\partial z}}_{1} + \frac{\partial f}{\partial v}\cdot\underbrace{\frac{\partial v}{\partial z}}_{-1} + \frac{\partial f}{\partial w}\cdot\underbrace{\frac{\partial w}{\partial z}}_{0}$$

$+$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$$

⊛ $\lim\limits_{(x,y)\to(0,0)} \frac{2xy + x^{3/2}}{y^3 + xy}$ limitinin varlığını inceleyin.

$x = ky^2$ boyunca limit:

$$\lim_{y\to 0} \frac{2ky^3 + k^{3/2}y^3}{y^3 + ky^3} = \frac{2k + k^{3/2}}{1+k}$$ → Sonuç k'ya bağlı

↓

Limit yok

⊛ $f(x,y) = \dfrac{3x^2-y^2}{x^2+3y^3}$ fonksiyonunun $(0,0)$ da limiti var mıdır?

$y=mx$ için:

$$\lim_{x\to 0} \frac{3x^2-m^2x^2}{x^2+3m^3x^3} = \lim_{x\to 0} \frac{\cancel{x^2}(3-m^2)}{\cancel{x^2}(1+3m^3x)} = 3-m^2$$ → Sonuç m'ye bağlı

Limit yok!

⊛ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+xy+y^2} = ?$ $\quad x^3-y^3 = (x-y).(x^2+xy+y^2)$

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)(x^2+\cancel{y^2+xy})}{x^2+\cancel{xy+y^2}} = \lim_{(x,y)\to(0,0)} (x-y) = 0 \checkmark$$

⊛ $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} = ?$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2).(\sqrt{x^2+y^2+1}+1)}{(\sqrt{x^2+y^2+1}-1)(\sqrt{x^2+y^2+1}+1)} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x^2+y^2)}.(\overbrace{\sqrt{x^2+y^2+1}}^{1}+1)}{\underbrace{x^2+y^2+\cancel{1}-\cancel{1}}_{\cancel{(x^2+y^2)}}} = 2$$

⊛ $f(x,y) = \dfrac{x^2y^4}{x^4+y^4}$ fonksiyonunun $(0,0)$ noktasındaki limitinin 0 olduğunu gösteriniz.

Her $\varepsilon>0$ için $\sqrt{x^2+y^2}<\delta$ iken $\left|\dfrac{x^2y^4}{x^4+y^4}-0\right|<\varepsilon$ olacak şekilde $\delta=\delta(\varepsilon)>0$ var mı?

$$\left|\frac{x^2y^4}{x^4+y^4}\right| \le \frac{x^2(x^4+\cancel{y^4})}{x^4+\cancel{y^4}} = x^2 \le x^2+y^2 < \delta^2 = \varepsilon \Rightarrow \delta^2=\varepsilon \rightarrow \delta, \cancel{-\sqrt{\varepsilon}}$$

$\rightarrow \boxed{\delta=+\sqrt{\varepsilon}}$

$\delta=\sqrt{\varepsilon}$ bulunur.

Limit gerçekten 0'dır.

2. $f(x,y)=4x^3y^2+2y$ olmak üzere fonksiyonun $x=1.1$ ve $y=-1.9$ noktalarındaki yaklaşık değerini lineerizasyonu kullanarak bulunuz.

Çözüm: $x_0=1$ ve $y_0=-2$ alınırsa

$f_x(x,y)=12x^2y^2$ ve $f_x(x_0,y_0)=12.1.4=48$ (5)

$f_y(x,y)=8x^3y+2$ ve $f_y(x_0,y_0)=8.1.(-2)+2=-14$ olarak bulunur. (5)

Bu konu sınava dahil değil!

$$L(x,y)=f(x,y)+f_x(x_0,y_0).(x-x_0)+f_y(x_0,y_0).(y-y_0)$$

$$=12+48.(x-1)-14.(y+2)$$ (10)

$$=48x-14y-64$$

$f(1.1,-1.9)\approx 48(1.1)-14(-1.9)-64$ (5)

$f(1.1,-1.9)\approx 15.4$ olarak bulunur.

---

④ $f(x,y)=\dfrac{x^2+y^2-y}{x-3y^2+3y}$ fonksiyonunun

a) [redacted] $y=x+1$ doğrusu boyunca (0,1) deki limitini bulunuz.

b) (0,1) de limitinin mevcut olmadığını gösterin.

a) $y=x+1$ boyunca limit:

$$\lim_{x\to 0}\frac{x^2+(x+1)^2-(x+1)}{x-3(x+1)^2+3(x+1)}=\lim_{x\to 0}\frac{2x^2+x}{-3x^2-2x}=\lim_{x\to 0}\frac{\cancel{x}(2x+1)}{\cancel{x}(-3x-2)}=-\frac{1}{2}\,//$$

b) İki kat limite bakalım:

$$\lim_{x\to 0}\left[\lim_{y\to 1}\frac{x^2+y^2-y}{x-3y^2+3y}\right]=\lim_{x\to 0}\frac{x^2}{x}=\lim_{x\to 0}x=0$$

$$\lim_{y\to 1}\left[\lim_{x\to 0}\frac{x^2+y^2-y}{x-3y^2+3y}\right]=\lim_{y\to 1}\frac{\cancel{y^2-y}}{-3(\cancel{y^2-y})}=-\frac{1}{3}$$

$\neq$ İki kat limit yok

Dolayısıyla limit YOK