System Dynamics;

Deals with the mathematical modeling of dynamic systems and response analyses of such systems.

To 1) understand the dynamic nature of the system

2) improve the system's performance

Frequently, computer simulations are used for response analysis of dynamic systems.

Dynamic systems may be;

mechanical, electrical, pneumatic, hydrolic systems, also nonengineering systems, such as; economic, biological systems.

System is a combination of components acting together to perform a specific objective.

Component is a single functioning unit of a system.

Systems are not limited with physical systems.

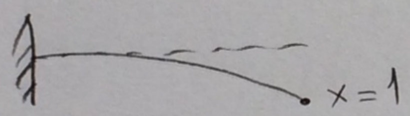
Non-physical dynamic systems may be about; economics, transportation, population growth and biology.

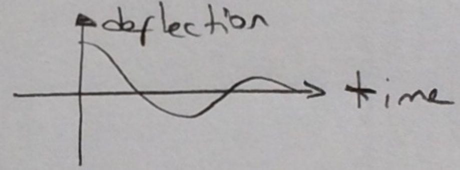
Dynamic;

its present output depends on past input

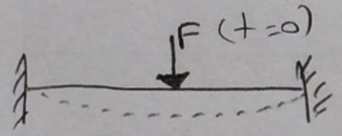
Static;

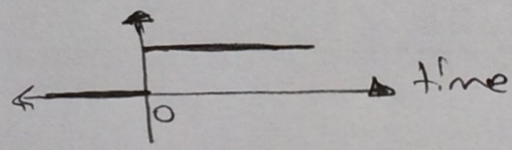
its current output depends only on current input





(damping)





→ The output of static system remains constant if the input does not change

→ The output of dynamic system changes with time if the system is not in a state of equilibrium.

Mathematical Model;

Before actually building a system we need a prediction of the system's performance.

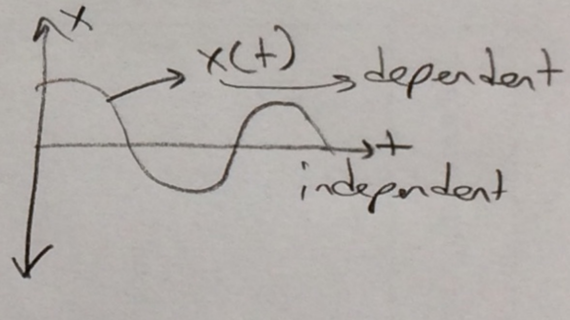
Such prediction is based on a; mathematical description of the system's dynamic characteristics (behavior).

This mathematical description is called a mathematical model.

Mathematical models are described in terms of differential equations.

Linear differential equation;

Dependent variable and its derivatives appear as linear combinations.



$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 10x = 0$$

Coefficients of all terms are constant.

This is also a linear, time-invariant differential equation. (LTI)

Linear, time-varying differential equation; (LTV)

The dependent variable and its derivatives appear as linear combinations. Also,

Coefficients of terms may involve the independent variable.

$$\frac{d^2x}{dt^2} + (1-\cos 2t)\,x = 0$$

Nonlinear Differential Equations;

$$\frac{d^2x}{dt^2} + (x^2-1)\frac{dx}{dt} + x = 0$$

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^3 = \sin\omega t$$

Contains powers, other functions, products of the dependent variables or its derivatives.

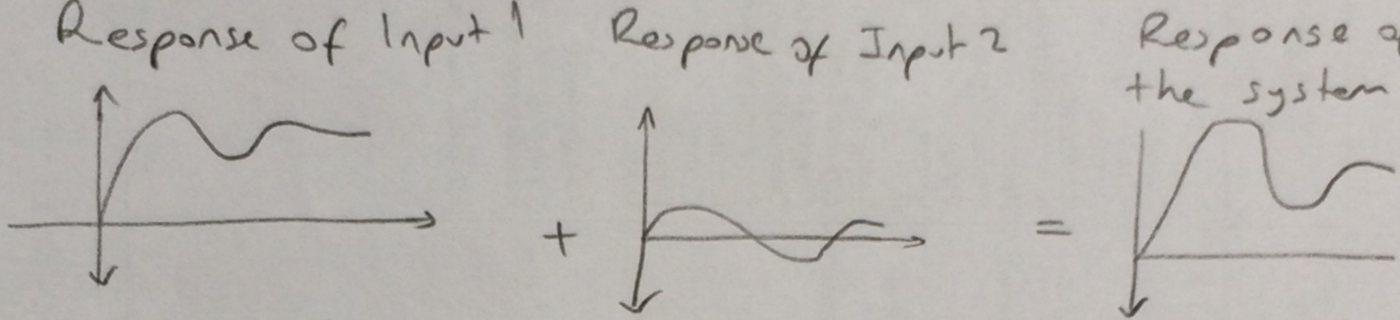
Linear Systems;

The equations that constitute the modal are linear.

The most important property of linear systems is that the principal of superposition is applicable.

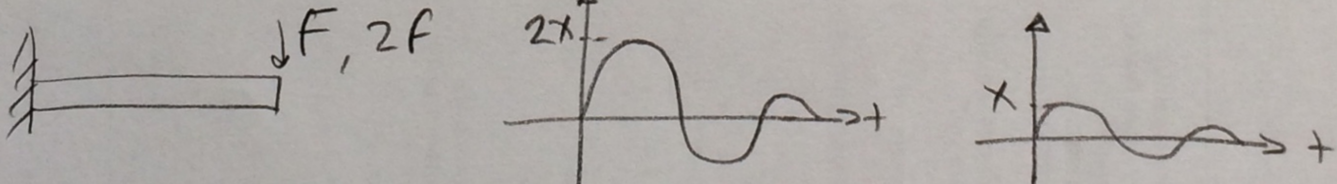
Principle of Superposition;

The response produced by simultaneous applications of two different inputs is the sum of two individual responses.



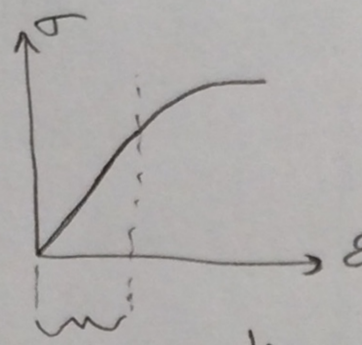
Complicated solutions to linear differential equations can be derived as a sum of simple solutions.

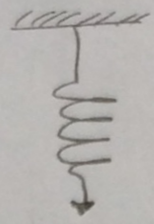
In an experimental investigation of a system; if input and output are <u>proportional</u>, superposition holds. And system; can be considered as <u>linear</u>.



In reality, most systems are not linear.

Linear systems are actually,
linear only within limited operating ranges.





softening spring
hardening spring

linear spring within that operating range.

<u>for nonliner systems</u>;

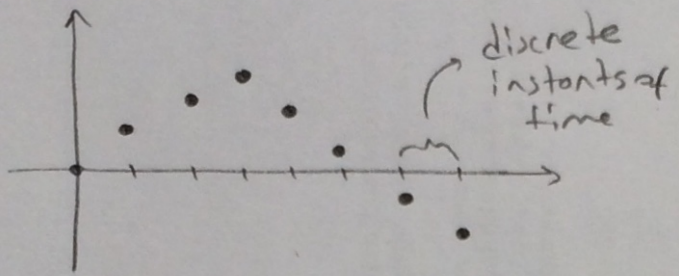
Principle of superposition <u>is not</u> applicable.
Such systems are extremely complicated, because of the mathematical difficulty of solving nonlinear equations.
So, generally, we need to linearize the nonlinear system near the operating range.

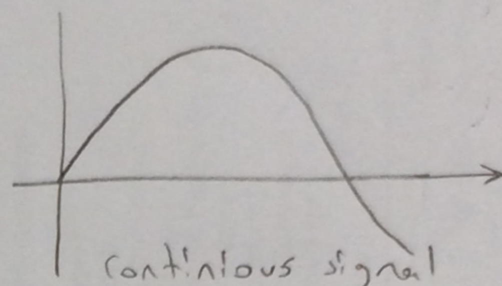
<u>Continuos-time Systems</u>;

The signals involved are continious in time.
These systems described by <u>differential equations</u>.

Discrete-time Systems ;

One or more variables can change only at discrete instants of time.





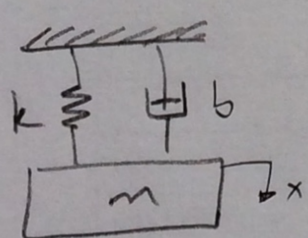
Discrete-time systems are described by difference equations.

Continuous-time signals may also be discretized.

Mathematical Modeling ;

involves description of important system characteristics by sets of equations.

By applying physical laws to a specific system, it may be possible to develop a mathematical model that describes the dynamics of the system.



- inertial force
- spring force
- damping force

$$m\ddot{x} + b\dot{x} + kx = 0$$

Experimental modeling tegniques can also be used to obtain a mathematical model by evaluating input-output relationships from the data measured.

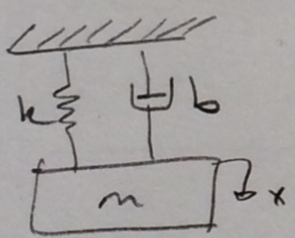
System analysis ;

means the investigation of the system's performance using its mathematical model.

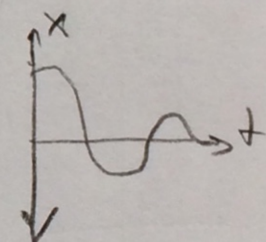
1) develop a mathematical model for each component

2) combine them to build a model of complete system.

3) use analytical or computer techniques to analyze.



$$\left.\begin{matrix} m\ddot{x} \\ kx \\ b\dot{x} \end{matrix}\right\} \rightarrow m\ddot{x}+b\dot{x}+kx=0 \rightarrow$$



LAPLACE TRANSFORMATION

One of the most important mathematical tools available for modeling and analyzing linear systems. (linear time-invariant differential equations)

With this method;

differential equations in time (t) $\longrightarrow$ algebraic equations in s-domain

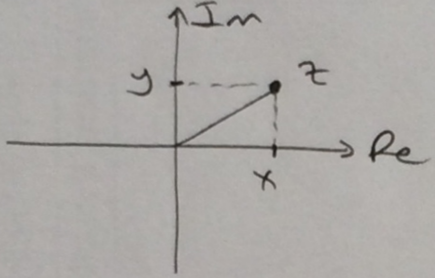
Laplace Transformation;

$f(t)$ is a time function

Laplace transform of $f(t)$ is given by

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

s is a complex variable



$j = \sqrt{-1}$ (unit imaginary number)

$z = x + jy$ (a complex number)

$F(s)$ is the Laplace transform of $f(t)$

Inverse Laplace Transformation;

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$.

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

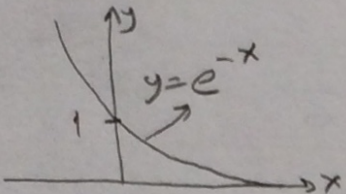
Exponential function;

$$f(t) = A e^{-\alpha t}$$

where A and α are constants.

Laplace transform of this exponential function;

$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty A e^{-\alpha t} e^{-st}\, dt = A \int_0^\infty e^{-(\alpha+s)t}\, dt$$

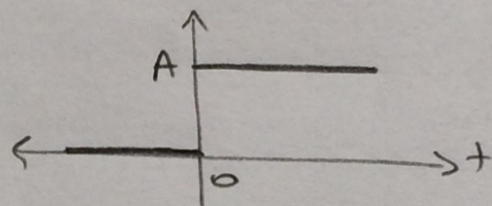


$$= A \cdot \left[\frac{e^{-(\alpha+s)t}}{-(\alpha+s)} \right]_0^\infty = A \cdot \left[0 - \frac{1}{-(\alpha+s)} \right]$$

$$\boxed{\mathcal{L}[Ae^{-\alpha t}] = \frac{A}{s+\alpha}}$$

Step function:

$f(t) = A$



A is constant.

Laplace transform of this step function;

$$\mathcal{L}[A] = \int_0^\infty A e^{-st}\, dt = A \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{A}{s}$$

Step function is a special case of exponential function $Ae^{-\alpha t}$, where $\alpha = 0$.

Unit step function : $A = 1$

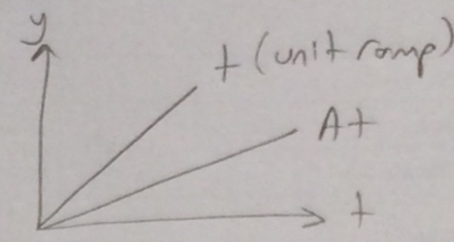
$$\mathcal{L}[1(t)] = \frac{1}{s}$$

$1(t) = 1$ for $t > 0$
$\quad = 0$ for $t < 0$

Ramp function;

$f(t) = 0 \quad \text{for} \quad t < 0$

$ = At \quad \text{for} \quad t \geq 0$



A is contant. Laplace transform of this ramp function;

$$\mathcal{L}[At] = A\int_0^{\infty} t\, e^{-st}\, dt$$

To evaluate this integral we use the

Integration by parts formula.

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$u = t \;,\; dv = e^{-st}\,dt \;\rightarrow\; du = dt \;,\; v = \frac{e^{-st}}{-s}$$

$$\mathcal{L}[At] = A\left(t \cdot \frac{e^{-st}}{-s}\Big|_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s}\, dt \right)$$

$$= \frac{A}{s}\underbrace{\int_0^{\infty} e^{-st}\, dt}_{\text{Laplace transform of unit step function}} = \boxed{\frac{A}{s^2}}$$

$$\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \left[0 - \frac{1}{-s}\right] = \frac{1}{s}$$

Unit Ramp;

$$\boxed{\mathcal{L}[t] = \frac{1}{s^2}} \quad (A=1)$$

Euler's Theorem (formula)

The power series expansion of $\cos\theta$ and $\sin\theta$;

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

$$(j^2 = -1,\ j^3 = -j,\ j^4 = 1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{\cos\theta + j\sin\theta = e^{j\theta}}$$

$$\boxed{\cos\theta - j\sin\theta = e^{-j\theta}}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Sinusoidal function;

$$f(t) = A\sin\omega t$$

A and ω are constants. And

$$\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

Laplace transform of this sinusoidal function;

$$\mathcal{L}[A \sin wt] = \frac{A}{2j}\int_0^{\infty}\left(e^{jwt} - e^{-jwt}\right)e^{-st}\,dt$$

$$= \frac{A}{2j}\int_0^{\infty}\left(e^{(jw-s)t} - e^{(-jw-s)t}\right)dt$$

$$= \frac{A}{2j}\left[\frac{e^{(jw-s)t}}{jw-s} - \frac{e^{(-jw-s)t}}{-jw-s}\right]_0^{\infty}$$

$$= \frac{A}{2j}\left(\frac{1}{s-jw} - \frac{1}{s+jw}\right) = \frac{A}{2j}\left(\frac{s+jw-(s-jw)}{s^2-(jw)^2}\right)$$

$$\mathcal{L}[A \sin wt] = \boxed{\frac{Aw}{s^2+w^2}}$$

Similarly, $\mathcal{L}[A \cos wt] = \boxed{\frac{As}{s^2+w^2}}$

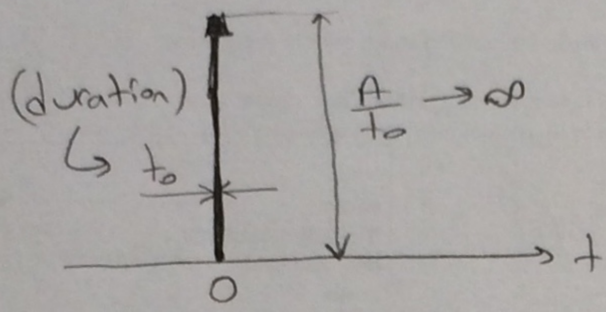
It is not necessary to derive the Laplace transform of f(t) each time. Laplace transform tables can be used to find the transform of any given function f(t).

Table 2-1, pages 18-20.

Impulse function;

$$f(t) = \lim_{t_0 \to 0} \frac{A}{t_0} \qquad \text{for } 0 < t < t_0$$

$$= 0 \qquad \text{for } t < 0,\ t_0 < t$$



Area under the impulse = A

The magnitude of impulse is measured by its area.

From Laplace transformation tables;

$$\mathcal{L}[f(t)] = A$$

Unit Impulse function; (Dirac delta function)

when the area under the impulse function is equal to unity.

$$\mathcal{L}[\delta(t)] = 1$$

Multiplication of $f(t)$ by $e^{-\alpha t}$;

If Laplace transform of $f(t)$ is $F(s)$

Then Laplace transform of $e^{-\alpha t} \cdot f(t)$ is

$$\mathcal{L}[e^{-\alpha t} f(t)] = \int_0^\infty e^{-\alpha t} f(t) e^{-st} dt = \int_0^\infty f(t) e^{-(\alpha + s)t} dt$$

$$= F(s+\alpha)$$

Let's find the Laplace transform of

$e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$.

Since; $\mathcal{L}[\sin\omega t] = \dfrac{\omega}{s^2+\omega^2}$, $\mathcal{L}[\cos\omega t] = \dfrac{s}{s^2+\omega^2}$

$$\mathcal{L}[e^{-\alpha t}\sin\omega t] = F(s+\alpha) = \frac{\omega}{(s+\alpha)^2+\omega^2}$$

$$\mathcal{L}[e^{-\alpha t}\cos\omega t] = F(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

<u>Differentiation Theorem;</u>

The Laplace transform of the derivative of the function $f(t)$ is

$$\mathcal{L}\left[\frac{d\,f(t)}{dt}\right] = sF(s) - f(0)$$

$f(0)$ is the initial value of $f(t)$ for $t=0$.

<u>Proof:</u> Laplace transform of $f(t)$;

$$\int_0^\infty f(t)e^{-st}dt = f(t)\frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty \frac{d\,f(t)}{dt}\frac{e^{-st}}{-s}dt$$

(Int. by parts. form.) $\rightarrow \int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du \quad \begin{bmatrix} u = f(t) \\ dv = e^{-st}dt \end{bmatrix}$

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathcal{L}\left[\frac{df(t)}{dt}\right]$$

$$\boxed{\mathcal{L}[\dot{f}(t)] = sF(s) - f(0)}$$

Similarly, for the second derivative of $f(t)$;

$$\boxed{\mathcal{L}[\ddot{f}(t)] = s^2 F(s) - s f(0) - \dot{f}(0)}$$

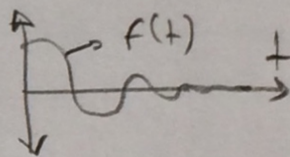
$\dot{f}(0)$ is the value of $\frac{df(t)}{dt}$ for $t=0$.

for the nth derivative of $f(t)$;

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \dot{f}(0) - \dots - \overset{(n-1)}{f}(0)$$

<u>Final Value Theorem</u>;

If $f(t)$ and $\frac{df(t)}{dt}$ are Laplace transformable, Laplace transform of $f(t)$ is $F(s)$, and if $\lim_{t\to\infty} f(t)$ exists ($f(t)$ settles down to a definite value as $t\to\infty$), then;



$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

<u>Initial Value Theorem</u>;

Using this theorem, we are able to find the value of $f(t)$ at $t=0+$ directly from the Laplace transform of $f(t)$

$$\boxed{f(0+) = \lim_{s\to\infty} sF(s)}$$

<u>Laplace transform of $\int f(t)dt$</u>; $\rightarrow$ value of $\int f(t)dt$ at $t=0$

$$\mathcal{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s}$$