STABILITY

(Textbook Ch.6)

Definition of Stability

- Stability is the most important system specification. If a system is unstable, transient response and steady-state errors are moot points.

- An unstable system cannot be designed for a specific transient response or steady-state error requirement.

- There are many definitions for stability, depending upon the kind of system or the point of view. In this section, we limit ourselves to linear, time-invariant systems.

Definition of Stability

The total response of a system is the sum of the forced and natural responses. Using these concepts, we present the following definitions of stability, instability, and marginal stability:

- **A linear, time-invariant system is <u>stable</u> if the natural response approaches zero as time approaches infinity.**

- **A linear, time-invariant system is <u>unstable</u> if the natural response grows without bound as time approaches infinity.**

- **A linear, time-invariant system is <u>marginally stable</u> if the natural response neither decays nor grows but remains constant or oscillates as time approaches infinity.**

Definition of Stability

The alternate definition of stability, one that regards the total response and implies the first definition based upon the natural response, is this:

- **A system is stable if every bounded input yields a bounded output. We call this statement the <u>bounded-input, bounded-output (BIBO)</u> definition of stability.**

The alternate definition of instability, one that regards the total response, is this:

- **A system is unstable if any bounded input yields an unbounded output.**

The Stability of a Closed-Loop System

If the closed-loop system poles are in the left half of the plane and hence have a negative real part, the system is stable.

- **Stable systems have closed-loop transfer functions with poles only in the left half-plane.**

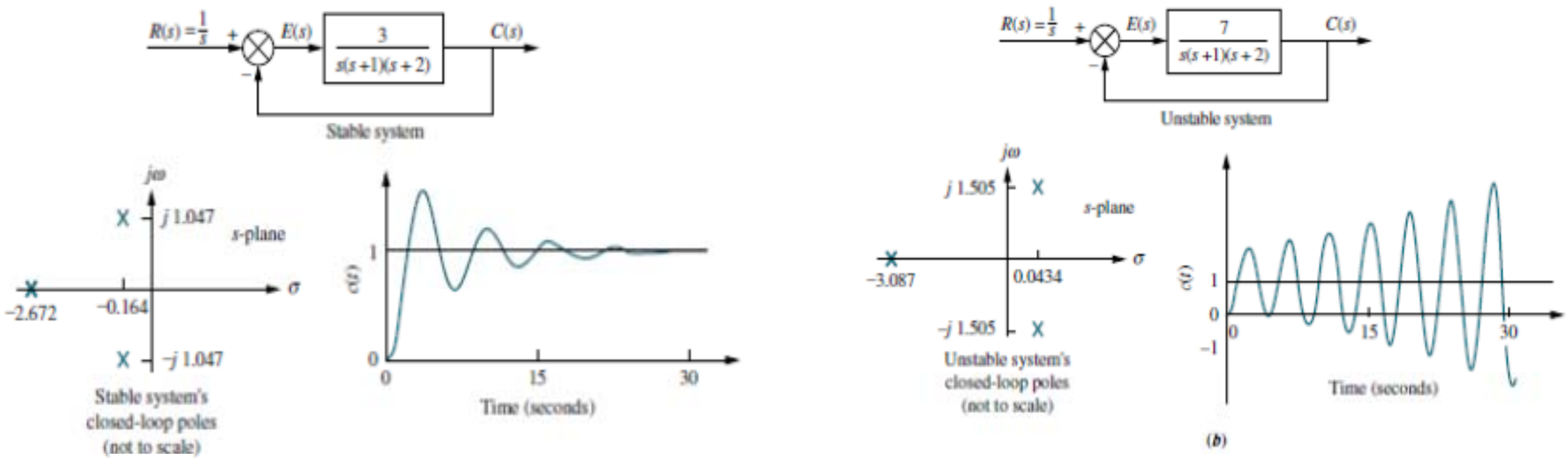
- **Unstable systems have closed-loop transfer functions with at least one pole in the right half-plane and/or poles of multiplicity greater than 1 on the imaginary axis.**

- **Marginally stable systems have closed-loop transfer functions with only imaginary axis poles of multiplicity 1 and poles in the left half-plane.**

The Stability of a Closed-Loop System

➢ It is not always a simple matter to determine if a feedback control system is stable. Unfortunately, a typical problem that arises is shown in the figures below.

➢ Although we know the poles of the forward transfer function, we do not know the location of the poles of the equivalent closed-loop system without factoring or otherwise solving for the roots.



Routh-Hurwitz Criterion

- Using this method, we can tell how many closed-loop system poles are in the left half-plane, in the right half-plane, and on the $j\omega$ -axis.

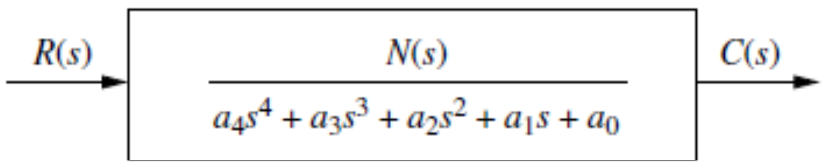
- Notice that we say how many, not where.

- We can find the number of poles in each section of the s-plane, but we cannot find their coordinates.

<u>The method requires two steps:</u>

- Generate a data table called a Routh table
- Interpret the Routh table to tell how many closed-loop system poles are in the left half-plane, the right half-plane, and on the $j\omega$ -axis.

Generating Basic Routh Table



s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

- Begin by labeling the rows with powers of s from the highest power of the denominator of the closed-loop transfer function to s^0.

- Start with the coefficient of the highest power of s in the denominator and list, horizontally in the first row, every other coefficient.

- In the second row, list horizontally, starting with the next highest power of s, every coefficient that was skipped in the first row.

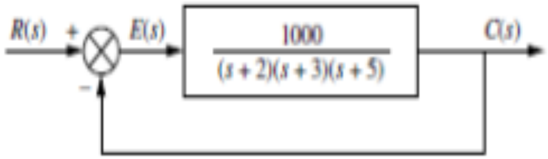
Generating Basic Routh Table

- The remaining entries are filled in as follows.
  - Each entry is a negative determinant of entries in the previous two rows divided by the entry in the first column directly above the calculated row.
  - The left-hand column of the determinant is always the first column of the previous two rows, and the right-hand column is the elements of the column above and to the right.
  - The table is complete when all of the rows are completed down to s^0.

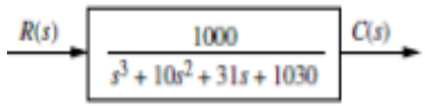
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Example 6.1

Make the Routh table for the system shown in



We need to find the equivalent closed-loop system because we want to test the denominator of this function, not the given forward transfer function, for pole location.



The Routh-Hurwitz criterion will be applied to this denominator. First label the rows with powers of s from s^3 down to s^0 in a vertical column, as shown in

s^3	1	31	0
s^2	$\cancel{10}\ \ 1$	$\cancel{1030}\ \ 103$	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

Example 6.1

- Form the first row of the table, using the coefficients of the denominator of the closed-loop transfer function.

- Start with the coefficient of the highest power and skip every other power of s.

- Form the second row with the coefficients of the denominator skipped in the previous step.

- For convenience, any row of the Routh table can be multiplied by a positive constant without changing the values of the rows below.

- This can be proved by examining the expressions for the entries and verifying that any multiplicative constant from a previous row cancels out. In the second row, for example, the row was multiplied by 1/10.

Interpreting Basic Routh Table

- The Routh-Hurwitz criterion declares that the number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column.

- If the closed-loop transfer function has all poles in the left half of the s-plane, the system is stable. Thus, a system is stable if there are no sign changes in the first column of the Routh table.

Routh-Hurwitz Criterion: Special Cases:

Two special cases can occur:

- The Routh table sometimes will have a zero only in the first column of a row.

- The Routh table sometimes will have an entire row that consists of zeros.

Zero Only in the First Column

➢ If the first element of a row is zero, division by zero would be required to form the next row.

➢ To avoid this phenomenon, an epsilon, ε, is assigned to replace the zero in the first column.

➢ The value ε is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the first column can be determined.

Example 6.2:

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Example 6.2

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\ \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

➢ Begin by assembling the Routh table down to the row where a zero appears only in the first column (the s^3 row).

➢ Next replace the zero by a small number, ε, and complete the table.

➢ To begin the interpretation, we must first assume a sign, positive or negative, for the quantity ε.

Example 6.2

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	$+$	$+$
s^4	2	$+$	$+$
s^3	$\cancel{0}\ \epsilon$	$+$	$-$
s^2	$\dfrac{6\epsilon - 7}{\epsilon}$	$-$	$+$
s^1	$\dfrac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	$+$	$+$
s^0	3	$+$	$+$

The table above shows the first column of the previous table along with the resulting signs for choices of ε positive and ε negative.

➢ If ε is chosen positive, the Routh table will show a sign change from the s^3 row to the s^2 row, and there will be another sign change from the s^2 row to the s^1 row.

➢ Hence, the system is unstable and has two poles in the right half-plane.

Entire Row is Zero

- Sometimes while making a Routh table, we find that an entire row consists of zeros because there is an even polynomial that is a factor of the original polynomial.

Example 6.4:

Determine the number of right-half-plane poles in the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Example 6.4

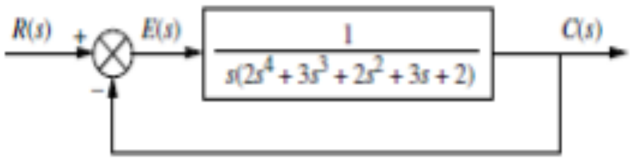
s^5			1			6			8
s^4		$\cancel{7}$	1		$\cancel{42}$	6		$\cancel{56}$	8
s^3	$\cancel{0}$	$\cancel{4}$	1	$\cancel{0}$	$\cancel{12}$	3	$\cancel{0}$	$\cancel{0}$	0
s^2			3			8			0
s^1			$\frac{1}{3}$			0			0
s^0			8			0			0

- At the second row we multiply through by 1/7 for convenience.
- We stop at the third row, since the entire row consists of zeros, and use the following procedure.
  - We return to the row immediately above the row of zeros and form an auxiliary polynomial, using the entries in that row as coefficients.
  - The polynomial will start with the power of s in the label column and continue by skipping every other power of s.
  - The polynomial formed for this example is $P(s) = s^4 + 6s^2 + 8$
  - We differentiate the polynomial with respect to s and obtain $\frac{dP(s)}{ds} = 4s^3 + 12s$

Finally, we use the coefficients to replace the row of zeros. For convenience, the third row is multiplied by 1/4 after replacing the zeros.

Example 6.7

Find the number of poles in the left half-plane, the right half-plane, and on the jw-axis for the system of



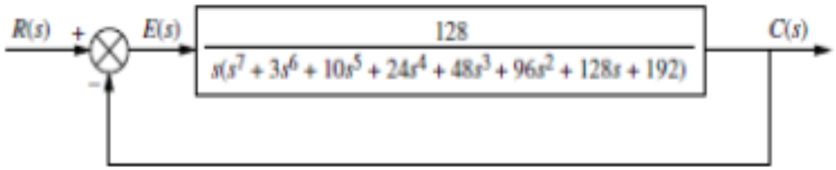
The closed-loop transfer function is $T(s) = \dfrac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}\quad \epsilon$	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

- A zero appears in the first column of the s^3 row. Replace the zero with a small quantity, ε, and continue the table.
- Permitting ε to be a small, positive quantity, we find that the first term of the s^2 row is negative.
- There are two sign changes, and the system is unstable, with two poles in the right half-plane. The remaining poles are in the left half-plane.

Example 6.8

Find the number of poles in the left half-plane, the right half-plane, and on the jw-axis for the system



The closed-loop transfer function for the system is

$$T(s) = \frac{128}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}$$

s^8	1	10	48	128	128
s^7	$\cancel{3}\ \ 1$	$\cancel{24}\ \ 8$	$\cancel{96}\ \ 32$	$\cancel{192}\ \ 64$	
s^6	$\cancel{2}\ \ 1$	$\cancel{16}\ \ 8$	$\cancel{64}\ \ 32$	$\cancel{128}\ \ 64$	
s^5	$\cancel{0}\ \ \cancel{6}\ \ 3$	$\cancel{0}\ \ \cancel{32}\ \ 16$	$\cancel{0}\ \ \cancel{64}\ \ 32$	$\cancel{0}\ \ \cancel{0}\ \ 0$	
s^4	$\cancel{\frac{8}{3}}\ \ 1$	$\cancel{\frac{64}{3}}\ \ 8$	$\cancel{64}\ \ 24$		
s^3	$\cancel{-8}\ \ -1$	$\cancel{-40}\ \ -5$			
s^2	$\cancel{3}\ \ 1$	$\cancel{24}\ \ 8$			
s^1	3				
s^0	8				

Example 6.8

- A row of zeros appears in the s^5 row. Thus, the closed-loop transfer function denominator must have an even polynomial as a factor. Return to the s^6 row and form the even polynomial:

$$P(s) = s^6 + 8s^4 + 32s^2 + 64$$

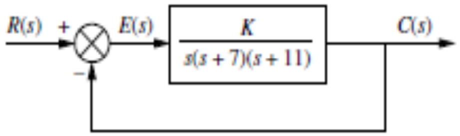
- Differentiate this polynomial with respect to s to form the coefficients that will replace the row of zeros:

$$\frac{dP(s)}{ds} = 6s^5 + 32s^3 + 64s$$

- Replace the row of zeros at the s^5 row by the coefficients and multiply through by 1/2 for convenience.
- There are two sign changes from the even polynomial at the s^6 row down to the end of the table. Hence, the even polynomial has two right–half- plane poles.
- Because of the symmetry about the origin, the even polynomial must have an equal number of left–half-plane poles. Therefore, the even polynomial has two left–half-plane poles.
- Since the even polynomial is of sixth order, the two remaining poles must be on the jω-axis.
- There are no sign changes from the beginning of the table down to the even polynomial at the s^6 row, the rest of the polynomial has no right–half plane poles.

Example 6.9

- Find the range of gain, K, for the system of



that will cause the system to be stable, unstable, and marginally stable. Assume K > 0.

- First find the closed-loop transfer function as

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

- Next form the Routh table shown as

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

- Since K is assumed positive, we see that all elements in the first column are always positive except the s^1 row.
- This entry can be positive, zero, or negative, depending upon the value of K.

Example 6.9

- If K < 1386, all terms in the first column will be positive, and since there are no sign changes, the system will have three poles in the left half-plane and be stable.

- If K > 1386, the s^1 term in the first column is negative. There are two sign changes, indicating that the system has two right–half-plane poles and one left–half-plane pole, which makes the system unstable.

- If K = 1386, we have an entire row of zeros, which could signify jw poles. Returning to the s^2 row and replacing K with 1386, we form the even polynomial $P(s) = 18s^2 + 1386$

- Differentiating with respect to s, we have $\frac{dP(s)}{ds} = 36s$

Example 6.9

➢ Replacing the row of zeros with the coefficients, we obtain the Routh-Hurwitz table shown as

s^3	1	77
s^2	18	1386
s^1	$\cancel{0}$ 36	
s^0	1386	

for the case of K = 1386.

➢ Since there are no sign changes from the even polynomial (s^2 row) down to the bottom of the table, the even polynomial has its two roots on the jω-axis of unit multiplicity.

➢ Since there are no sign changes above the even polynomial, the remaining root is in the left half-plane. Therefore the system is marginally stable.