Boundedness and Ultimate Boundedness

Lyapunov analysis can be used to show boundedness of the solution of the state equation, even when there is no equilibrium point at the origin.

That is we can show, through Lyapunov type arguments, that a state eventually becomes bounded around origin even when there is no e.p. at the origin.

Consider

$$\dot{x} = -x + \varepsilon, \; x(0) = x_0 \Rightarrow \quad x = 0 \text{ is not an e.p of this system}$$

the solution is given by

$$x(t) = x(0) \cdot e^{-t} + \int_0^t e^{-(t-\tau)} \varepsilon \, d\tau$$

Lecture

$$x(t) = \underbrace{x(0) \cdot e^{-t}}_{a} + \varepsilon \, [1 - \underbrace{e^{-t}}_{b}] \qquad \forall t \geq 0$$

a & b will eventually approach to zero {exponentially fast}

further more If $\exists$ $a \geq \varepsilon$ then we can conclude that

$|x(t)| \leq a$ after some time (let it be 't_0')

So $|x(t)|$ will eventually enter into a Ball B_a and will stay inside this ball after that time.(1) If the size of this ball (a) does not depend on (t_0) then $x(t)$ approaches this bound uniformly(2)

1 is refered as ultimately boundedness

2 is referred as uniformly ultimately boundedness.

Definition

for the system $\dot{x} = f(x,t)$, with $f: D \times [0,\infty) \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on the interval $[0,\infty) \times D$, $D \subset \mathbb{R}^n$ is the domain containing the origin,

we say the solutions $x(t)$ of $\dot{x} = f(x,t)$ are

- Uniformly Bounded, if $\exists$ a positive constant c, independent of $t_0 \geq 0$ and $\forall$ $a \in (0,c)$ $\exists$ $\beta(a) > 0$ also independent of t_0 such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta \quad \forall\, t \geq t_0$$

If this is satisfied for arbitrary large values of 'a' then $x(t)$ is globally uniformly bounded

- Uniformly ultimately bounded with an ultimate bound 'b' if $\exists$ positive constants, b and c independent of $t_0 \geq 0$ and $\forall$ $a \in (0,c)$ $\exists$ $T(a,b) \geq 0$ also independent of $t_0 \geq 0$, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b \quad \forall\, t \geq t_0 + T$$

If this is satisfied for arbitrary large values of 'a' then $x(t)$ is globally uniformly ultimately bounded.

Lyapunov type analysis can be used to prove boundedness and ultimate boundedness, Consider a continuously differentiable positive definite function $V(x)$ and suppose that for a $c > 0$ the set $\{V(x) \leq c\}$ is compact. And let

$$\Lambda = \{ \varepsilon \leq V(x) \leq c \} \text{ for some } c > \varepsilon > 0$$

Suppose the derivative of V along the trajectories of the system $\dot{x} = f(x, t)$ satisfies

$$\dot{V}(x, t) \leq - W_3(x) \quad \forall x \in \Lambda \quad \forall t \geq t_0$$

where $W_3(x)$ being a continuous positive definite function.

This implies that the sets $\Omega_c = \{V(x) \leq c\}$ and $\Omega_\varepsilon = \{V(x) \leq \varepsilon\}$ are positively invariant since on the boundries of these sets $\dot{V}$ is negative.

Since $\dot{V}$ is negative in Λ, a trajectory starting Λ must move in the direction of decreasing $V(x(t))$

The trajectories behaves as if the origin was uniformly asymptotically stable and satisfies an inequality of the form

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) \quad \text{for some}$$

class $\mathcal{KL}$ function β.

$\{$ $V(x)$ will decrease until the trajectories enters Ω_ε in finite time and the trajectories will stay in there after that $\}$

The transient response (the magnitude of the finite time) required to enter the ultimate bound is also important when evaluating the stability properties of a system.

For example consider a positive function of time on $[0,\infty)$ $V \in \mathbb{R}^1_+$ that satisfies

$$\dot V \leq -\gamma V + \varepsilon \qquad \text{where } \gamma, \varepsilon \in \mathbb{R}^1 \text{ are positive constants.}$$

Then $V(t)$ satisfies

$$V(t) \leq V(0)\exp\{-\gamma t\} + \frac{\varepsilon}{\gamma}\cdot(1-\exp\{-\gamma t\}) \qquad t \in [0,\infty)$$

Proof rewrite the inequality

$$\dot V + \gamma V - \varepsilon \leq 0$$

and select a positive function $s(t) \in \mathbb{R}^1$

to define

$$\dot V + \gamma V - \varepsilon = -s$$

$$\Rightarrow \quad \dot V = -\gamma V + \varepsilon - s$$

diff equations theory

the solution of this equation (from linear control theory)

$$V(t) = V(0)\cdot\exp\{-\gamma t\} + \varepsilon\cdot\exp\{-\gamma t\}\cdot\int_0^t \exp(+\gamma z)\,dz$$
$$-\exp\{-\gamma t\}\cdot\int_0^t s(t)\cdot\exp\{+\gamma z\}\,dz$$

from the assumption that $s(t)$ is always positive

$$V(t) \leq V(0)\exp\{-\gamma t\} + \varepsilon\exp(-\gamma t)\int_0^t \exp(+\gamma z)\,dz$$

which brings us to the result

$$V(t) \leq V(0) \cdot \exp\{-\gamma t\} + \frac{\varepsilon}{\gamma} \cdot \left(1 - \exp\{-\gamma t\}\right) \qquad t \geq 0$$

That is $V(t)$ decreases (exponentially) until the ultimate bound $V \leq \frac{\varepsilon}{\gamma}$ is reached. and stays there fore future time.

Input-to-State Stability (ISS stability)

Consider the system

$$\dot{x} = f(t, x, u)$$

where $f: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ is piecewise continuous in t and locally Lipschitz in x and u.

Suppose that the input $u(t)$ is piecewise continuous, bounded function for $\forall\, t \geq 0$ and the unforced system

$$\dot{x} = f(t, x, 0)$$

has a globally uniformly stable e.p. at $x = 0$.

Can you comment on the behaviour of the 'forced system' in the presence of a bounded input u?

- First consider linear time-invariant systems.

$$\dot{x} = Ax + Bu$$

and assume that A is stable (a Hurwitz matrix) the solution for this L.T.I. system is then

$$x(t) = x(t_0) \cdot e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)} \cdot B \cdot u(\tau) \cdot d\tau$$

where $e \triangleq \exp\{\cdot\}$

applying $\| e^{A(t-t_0)} \| \leq k \cdot e^{-\lambda(t-t_0)}$ then

$x(t)$ can be bounded as

$$\|x(t)\| \leq k \cdot e^{-\lambda(t-t_0)} \cdot \|x(t_0)\| + \int_{t_0}^{t} k \cdot e^{-\lambda(t-\tau)} \|B\| \cdot \|u(\tau)\| \, d\tau$$

$$\|x(t)\| \leq \|x(t_0)\| \cdot k \cdot e^{-\lambda(t-t_0)} + \frac{k}{\lambda} \cdot \|B\| \cdot \left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \right)$$

- when $u(t) = 0$ (zero input response)
  $x(t)$ decays exponentially to zero
- when $u(t)$ is bounded $\Longrightarrow$ $\|x(t)\|$ is also bounded. (By some ultimate bound)

However this is different for non-linear systems!

Consider

$$\dot{x} = -x + (x + x^3)\, u$$

when $u = 0$ system becomes

$$\dot{x} = -x \Rightarrow x(t) = x(0) \cdot e^{-t}$$

Exponentially stable//

However when $u = 1$ (bounded input) system becomes

$\dot{x} = x^3$ which is unbounded as $t \to \infty$//.

Therefore the closed loop might become unbounded even when the unforced system is asymptotically stable (Here it is E.S)

We need new tools for the boundedness of the state when the input is bounded.

<u>Definition</u>: the system $\dot{x} = f(t, x, u)$ with $f: [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$ being piecewise continuous in t and locally Lipschitz in x and u, with bounded input $u(t)$. Then the system is said to be input-to-state stable (ISS) if $\exists$ a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \underbrace{\beta(\|x(t_0)\|, t - t_0)}_{\substack{\text{approaches to zero} \\ \text{as } t \to \infty}} + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$$

Therefore when $u(t)$ is bounded $\|x(t)\|$ approaches to an ultimate bound defined by the class $\mathcal{K}$ function $\gamma(\cdot)$ and the supremum of $u(t)$. For the unforced input the system becomes

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$$

which converges to zero as time approaches to infinity.

$\Rightarrow$ the ISS implies the origin of the unforced system is U.A.S. and depending on $x(t_0)$ it can become G.U.A.S.

Lyapunov-like approach to achieve a sufficient condition for ISS is as follows (Sontag)

Theorem

Suppose that for the system $\dot{x} = f(t,x,u)$ (*) there exists a C^1 function $V: \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$\gamma_1(\|x\|) \leq V(t,x) \leq \gamma_2(\|x\|)$$

$$\|x\| \geq \rho(\|u\|) > 0 \Longrightarrow \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x,u) \leq -\gamma_3(\|x\|)$$

where γ_1, γ_2 and ρ are class K_∞ functions and γ_3 is a class K function. Then the system (*) is ISS with $\gamma = \gamma_1^{-1} \circ \gamma_2 \circ \rho$.

Proof If $x(t_0)$ is in the set

$$R_{t_0} = \left\{ x \in \mathbb{R}^n : \|x\| < \rho\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right) \right\}$$

then $x(t)$ remains within the set

$$S_{t_0} = \left\{ x \in \mathbb{R}^n : \|x\| \leq \gamma_1^{-1} \circ \gamma_2 \circ \rho\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right) \right\}$$

for all $t \geq t_0$. Define $B = \{t_0, T\}$ as the time interval before $x(t)$ enters R_{t_0} for the first time. In view of the definition of R_{t_0} we have

$$\dot{V} \leq -\gamma_3 \circ \gamma_2^{-1}(V) \qquad \forall\ t \in B \qquad \text{(P1)}$$

Then {by Sontag's lemma} there exists a class KL function β_V such that

$$V(t) \leq \beta_V(V(t_0), t-t_0) \qquad \forall\ t \in B \qquad \text{(P2)}$$

which implies

$$\|x(t)\| \leq \underbrace{\gamma_1^{-1}\left(\beta_V\left(\gamma_2(\|x(t_0)\|), t-t_0\right)\right)}_{\triangleq \beta(\|x(t_0)\|,\, t-t_0)} \qquad \forall\ t \in B \qquad (P3)$$

By the definition of S_{t_0} we conclude

$$\|x(t)\| \leq \underbrace{\gamma_1^{-1} \circ \gamma_2 \circ \rho\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right)}_{\triangleq \gamma\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right)} \qquad \forall\ t \in [t_0, \infty) \setminus B \quad (P4)$$

Then by (P3) and (P4)

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup_{\tau \geq t_0} \|u(\tau)\|\right) \quad \forall\ t \geq t_0 \geq 0$$

By causality it follows that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right) \quad \forall\ t \geq t_0 \geq 0$$

//

which concludes the proof □

Example (Khalil, pp 177)

Consider the system

$$\dot{x} = -ax^3 + u \quad , \quad a > 0$$

with the Lyapunov candidate function $V(x) = \frac{1}{2}x^2$.

Note that $V(x)$ is p.d. with

$$\underbrace{\gamma_1(\|x\|)}_{\frac{1}{4}x^2} \leq V(x(t)) \leq \underbrace{\gamma_2(\|x\|)}_{x^2}$$

$\leftarrow$ dikkat

taking the derivative of V we have

$$\dot{V} = x(-ax^3 + u)$$

$$\dot{V} = -ax^4 + xu$$

for $0 < \theta < 1$ we have

$$\dot{V} = -ax^4 - a\theta x^4 + a\theta x^4 + xu$$

$$= -a(1-\theta)x^4 - x(a\theta x^3 - u)$$

$$\leq -\underbrace{(1-\theta)x^4}_{\gamma_3(\|x\|)} \quad \forall \ |x| > \left(\frac{|u|}{a\theta}\right)^{1/3}$$

$\left\{ \leq -(1-\theta)x^4 \right.$ provided that $x(a\theta x^3 - u) > 0$

$\Rightarrow$ which is also valid $|x| > \left(\frac{|u|}{a\theta}\right)^{1/3} \}$

that is the given system is globally input to state stable with $\gamma(u) = \left(\frac{|u|}{a\theta}\right)^{1/3}$.

Theorem

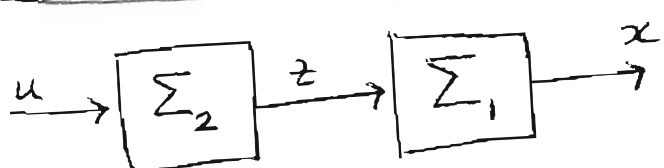
A continuous function $V : D \to \mathbb{R}_+$ is an ISS Lyapunov function on D {for the system $\dot{x} = f(x,u)$} if and only if there exists class $\mathcal{K}$ functions $\gamma_1, \gamma_2, \gamma_3$ and σ such that the following two conditions are satisfied:

$$\gamma_1(\|x\|) \leq V(x(t)) \leq \gamma_2(\|x\|) \qquad \forall x \in D, t > 0 \qquad (i)$$

$$\frac{\partial V}{\partial x} f(x,u) \leq -\gamma_3(\|x\|) + \sigma(\|u\|) \qquad \forall x \in D, u \in D_u$$

V is an ISS Lyapunov function if $D = \mathbb{R}^n$, $D_u = \mathbb{R}^m$ and $\gamma_1, \gamma_2, \gamma_3$ and $\sigma \in \mathcal{K}_\infty$.

Cascade - Connected systems



Consider this system where

$$\Sigma_1 : \dot{x} = f(x,z)$$

$$\Sigma_2 : \dot{z} = g(x,u)$$

If both Σ_1 and Σ_2 are ~~both~~ input-to-state stable then the composite system $\Sigma : u \to \begin{bmatrix} x \\ z \end{bmatrix}$ is input to state stable //

Example

Consider the system

$$\dot{x}_1 = -x_1 + x_2^2$$

$$\dot{x}_2 = -x_2 + u$$

first investigate the stability properties of the unforced system ($u(t) = 0$)

use $V = \frac{1}{2}x_1^2 + \frac{1}{4}a x_2^4$ with $a > 0$

as a Lyapunov function candidate

$\frac{d}{dt}$ of V gives

$$\dot{V} = x_1 \cdot (-x_1 + x_2^2) + a x_2^3 (-x_2)$$

$$= -x_1^2 + x_1 x_2^2 - a x_2^4$$

$$= -(x_1 - \tfrac{1}{2}x_2^2)^2 - \underbrace{(a - \tfrac{1}{4})x_2^4}$$

when a is selected to satisfy $a > \frac{1}{4}$ we obtain g.a.s of $x = 0$

Now for the actual system we use the same Lyapunov candidate function and obtain $\dot{V}$ with $u \neq 0$

$$\dot{V} = x_1 (-x_1 + x_2^2) + x_2^3 (-x_2 + u)$$

$$\dot{V} = \underbrace{-\tfrac{1}{2}(x_1 - x_2^2)^2 - \tfrac{1}{2}(x_1^2 + x_2^4)}_{\text{always negative}} + x_2^3 u$$

$$\dot{V} \leq \underbrace{-\tfrac{1}{2}(x_1^2 + x_2^4)}_{\text{needs to dominant } x_2^3 u} + x_2^3 \cdot u$$

//

rewrite $\dot{V}$ as

$$\dot{V} \leq -\frac{1}{2}(1-Q)(x_1^2+x_2^4) \underbrace{-\frac{1}{2}Q(x_1^2+x_2^4)}_{①} + \underbrace{|x_2|^3|u|}_{②}$$

with $0 < Q < 1$

If ① > ② we will end up with

$$\dot{V} \leq \underbrace{-\frac{1}{2}(1-Q)\cdot(x_1^2+x_2^4)}_{\gamma_3}$$

and ISS will be achieved.

that is we need

$$-\frac{1}{2}Q(x_1^2+x_2^4) + |x_2|^3|u| \leq 0$$

which is satisfied when

$$|x_2| \geq \frac{2|u|}{Q} \quad \text{or} \quad |x_2| \leq \frac{2|u|}{Q} \text{ and } |x_1| \geq \left(\frac{2|u|}{Q}\right)^2$$

which is implied by

$\Rightarrow$

$$\max\{|x_1|, |x_2|\} \geq \max\left\{\frac{2|u|}{Q}, \left(\frac{2|u|}{Q}\right)^2\right\}$$

or more generally

$$\gamma(u) = \max\left\{\frac{2u}{Q}, \left(\frac{2u}{Q}\right)^2\right\}$$

or simply use the theorem to achieve a similar result