MAT 5120 - Advanced Algebra - 2023-2024 Spring

Homework Assignment 1

Due March 4th 2024

There are 10 questions each worth 10 points.

(1) Prove that $(\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ is a (nonabelian) group under the operation $\star$ defined as $(a, b) \star (c, d) = (ac, bc + d)$ for all $(a, b), (c, d) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$.

(2) Let G be a group and $a, b \in G$. Suppose that $a^2 = 1$ and $ab^7a = b^{13}$. Show that $b^{120} = 1$.

(3) Let $D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle = \{s^i r^j \mid i, j \in \mathbb{Z},\ 0 \leq i \leq 1,\ 0 \leq j \leq n - 1\}$.
(a) Show that for any element $x = sr^j \in D_{2n}$, where $0 \leq j \leq n-1$, we have $rx = xr^{-1}$.
(b) Show that any element $x = sr^j \in D_{2n}$, where $0 \leq j \leq n-1$, has order 2.
(c) Let $n \geq 3$ and suppose that $s^i r^j$ is a nonidentity element of D_{2n} which commutes with all elements of D_{2n}. Show that $i = 0$, n is even, and $j = n/2$.

(4) **(a)** Find the order of $(1\ 8)(2\ 11\ 3\ 7\ 10)(4\ 9\ 15)(5\ 17\ 16\ 13)(6\ 12\ 14) \in S_{17}$.
(b) Let $\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12) \in S_{12}$. For which positive integers i is σ^i also a 12-cycle?
(c) Let $\tau = (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 10)(11\ 12)(13\ 14) \in S_{14}$. Is there a cycle $\sigma \in S_{14}$ such that $\tau = \sigma^k$ for some integer k.
(d) Let $\tau = (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)(10\ 11\ 12)(13\ 14\ 15) \in S_{15}$. Is there a cycle $\sigma \in S_{15}$ such that $\tau = \sigma^k$ for some integer k.

(5) **(a)** Find all numbers n such that S_7 contains an element of order n. Explain your answer.
(b) Find the number of elements of order 4 in S_7. Explain your answer.
(c) Find the smallest value of n such that S_n contains an element of order 35. Explain your answer.

(6) **(a)** Write out the group tables for S_3, D_6, D_8, and Q_8.
(b) Compute the order of each of the elements in S_3, D_6, D_8, and Q_8.

(7) **(a)** Show that S_3 and D_6 are isomorphic.
(b) Show that D_8 and Q_8 are nonisomorphic.

(8) Let $G = GL_2(\mathbb{F}_2)$.
(a) Write out all elements of G and compute their orders.
(b) Show that G is nonabelian.
(c) Show that $G \cong S_3$.

(9) Let G be a group.
(a) Show that for any $x \in G$, $|x| = |x^{-1}|$.
(b) Let $\phi : G \to G$ be defined as $\phi(x) = x^{-1}$. Show that ϕ is a homomorphism if and only if G is abelian.

(10) Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$, let $k \in \mathbb{Z}$ where $k > 1$, and define $\phi : G \to G$ as $\phi(z) = z^k$. Show that ϕ is a surjective homomorphism but is not an isomorphism.