MAT 5120 - Advanced Algebra - 2023-2024 Spring

Homework Assignment 3

Due April 22nd 2024

There are 10 questions each worth 10 points.

(1) Determine the conjugacy classes and the number of elements in each conjugacy class for the groups D_8 and Q_8. Show your work.

(2) Determine the conjugacy classes and the number of elements in each conjugacy class for the groups S_6 and A_6. Show your work.

(3) Show that A_5 has no subgroup of order 15, 20 or 30. (Hint: Consider the action of A_5 on the set of left cosets of a subgroup.)

(4) Let G be a group.
(a) Show that $Z(G)$ is characteristic in G.
(b) Show that $Inn(G)$ is a normal subgroup of $Aut(G)$.

(5) Let $H \leq K \leq G$ be groups.
(a) Show that if H is characteristic in K and K is normal in G, then H is normal in G.
(b) Show that if H is characteristic in K and K is characteristic in G, then H is characteristic in G.
(c) Give an example to show that if H is normal in K and K is characteristic in G, then H need not be normal in G.

(6) Let $H \leq K \leq G$ be groups. Show that if H is characteristic in G and K/H is characteristic in G/H, then K is characteristic in G.

(7) Show that there is no simple group of order 28, 56, or 132.

(8) Show that a group of order $231 = 3 \cdot 7 \cdot 11$ must have
(a) a normal Sylow 7-subgroup,
(b) a normal Sylow 11-subgroup in the center.

(9) For $p \in \{2, 3, 5\}$, find $n_p(S_5)$.

(10) Let G be a simple group of order 168.
(a) Find the number of subgroups and the number of elements of order 7 in G.
(b) Show that G is isomorphic to a subgroup of the symmetric group S_8. [Hint: Consider the action of G on the set of Sylow 7-subgroups of G by conjugation.]
(c) Find the number of conjugacy classes of elements of order 7 in G.