MAT 5120 - Advanced Algebra - 2023-2024 Spring

Homework Assignment 2

Due March 18th 2024

There are 10 questions each worth 10 points.

(1) Let H be a subgroup of the group G.

(a) Show that $H \leq N_G(H)$. Give an example to show that it is not necessarily true that $H \subseteq N_G(H)$ if H is not a subgroup.

(b) Show that $H \leq C_G(H)$ if and only if H is abelian.

(2) **(a)** Let H be a subgroup of order 2 in G. Show that $N_G(H) = C_G(H)$. Deduce that if $N_G(H) = G$ then $H \leq Z(G)$.

(b) Show that $Z(G) \leq N_G(A)$ for any subset A of G.

(3) Let G be a finite group and let $x, g \in G$.

(a) Show that $|gxg^{-1}| = |x|$.

(b) Show that $g \in N_G(\langle x \rangle)$ if and only if $gxg^{-1} = x^a$ for some $a \in \mathbb{Z}$.

(4) Let $G = D_8$.

(a) Find the centralizer of each element of G.

(b) Find the center of G.

(c) Find the normalizer of each subgroup of G.

(d) Determine all normal subgroups of G.

(e) For each normal subgroup N of G, determine the isomorphism type of G/N.

(5) Let $G = Q_8$.

(a) Find the centralizer of each element of G.

(b) Find the center of G.

(c) Find the normalizer of each subgroup of G.

(d) Determine all normal subgroups of G.

(e) For each normal subgroup N of G, determine the isomorphism type of G/N.

(6) Let $G = \left\langle x, y \mid x^4 = y^4 = 1,\ xy = yx \right\rangle \cong Z_4 \times Z_4$ and let $\overline{G} = G/\left\langle x^2y^2 \right\rangle$.

(a) Show that $|\overline{G}| = 8$.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{x}^a\overline{y}^b$ for some integers a and b.

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).

(d) Show that $\overline{G} \cong Z_4 \times Z_2$.

(7) Let $G = D_{16} = \left\langle x, y \mid x^8 = y^2 = 1,\ xy = yx^{-1} \right\rangle$ and let $\overline{G} = G/\left\langle x^4 \right\rangle$.

(a) Show that $|\overline{G}| = 8$.

(b) Exhibit each element of $\overline{G}$ in the form $\overline{x}^a\overline{y}^b$ for some integers a and b.

(c) Find the order of each of the elements of $\overline{G}$ exhibited in (b).
(d) Show that $\overline{G} \cong D_8$.

(8) Let $N \trianglelefteq G$ and let $M \trianglelefteq H$. Show that $(N \times M) \trianglelefteq (G \times H)$ and $(G \times H)/(N \times M) \cong (G/N) \times (H/M)$.

(9) Let $N \lhd G$ where $p = |G : N|$ is a prime. Show that for all $K \leq G$, either $K \leq N$ or $G = NK$ where $|K : N \cap K| = p$.

(10) A subgroup H of a group G is called a maximal subgroup of G if there is no subgroup K of G with $H < K < G$. Show that if H is a maximal subgroup of G and $H \lhd G$, then $|G : H|$ is a prime.