Ch. 30: Gauge Freedom

Gauge Freedom in Electromagnetism:

The electric potential Φ_E satisfies the equation $\vec{E} = -\vec{\nabla}\Phi_E$

But, if a function of time $g(t)$ is added to Φ_E

$$\Phi_E' = \Phi_E + g(t)$$

then

$$\vec{E}' = -\vec{\nabla}\Phi_E' = -\vec{\nabla}\left[\Phi_E + g(t)\right] = \underbrace{-\vec{\nabla}\Phi_E}_{=\vec{E}} \underbrace{-\vec{\nabla}g(t)}_{=0}$$

$$\vec{E}' = \vec{E}$$

Similarly, the vector potential $\vec{A}$ satisfying

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

can be added a function of the form $\vec{\nabla}f$, where $f = f(\vec{r},t)$, can be added to $\vec{A}$:

$$\vec{A}' = \vec{A} + \vec{\nabla}f$$

$$\vec{B}' = \vec{\nabla}\times\vec{A}' = \nabla\times\left[\vec{A} + \vec{\nabla}f\right] = \underbrace{\vec{\nabla}\times\vec{A}}_{=B} + \underbrace{\vec{\nabla}\times\vec{\nabla}f}_{=0}$$

$$\Rightarrow \quad \vec{B}' = \vec{B}$$

The ability to add functions to the potentials describing the field without changing any physically observable aspects of the field "gauge freedom".

Review of the Weak-Field Limit:

The gravitational waves produced will be, far from the source, only tiny perturbations of flat spacetime. Therefore we can use the weak field approximation to express the Einstein equation in linear form.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad h_{\mu\nu} = h_{\nu\mu} \text{ and } |h_{\mu\nu}| \ll 1$$

$h_{\mu\nu}$ is called the metric perturbation

To first order in $h_{\mu\nu}$ $\quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$, where $h^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} h_{\alpha\beta}$

The Rieman tensor is $R_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\partial_\beta \partial_\mu h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\alpha \partial_\mu h_{\beta\nu} - \partial_\beta \partial_\nu h_{\alpha\mu}\right]$

The indices of $h_{\mu\nu}$ are raised and lowered using the flat spacetime metric $\eta_{\mu\nu}$, for example $h^{\mu}{}_{\nu} = \eta^{\mu\alpha} h_{\alpha\nu}$

The Trace-Reversed Perturbation:

When working with gravitational waves it turns out to be more convenient to use the equation

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T^{\mu\nu}$$

$$R^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} R_{\alpha\beta} = g^{\mu\alpha} g^{\nu\beta} R^{\sigma}{}_{\alpha\sigma\beta}$$
$$= g^{\mu\alpha} g^{\nu\beta} g^{\delta\sigma} R_{\delta\alpha\sigma\beta} \approx \eta^{\mu\alpha}\eta^{\nu\beta}\eta^{\delta\sigma} R_{\delta\alpha\sigma\beta}$$

$$R = g^{\delta\sigma} g^{\alpha\beta} R_{\delta\alpha\sigma\beta} \approx \eta^{\delta\sigma}\eta^{\alpha\beta} R_{\delta\alpha\sigma\beta}$$

To first order in the metric perturbation the EE becomes

$$\frac{1}{2}\left(\partial^\gamma \partial_\mu h^{\mu\sigma} + \partial^\sigma \partial_\mu h^{\mu\gamma} - \partial^\gamma \partial^\sigma h - \partial^\mu \partial_\mu h^{\gamma\sigma} - \eta^{\gamma\sigma} \partial_\beta \partial_\mu h^{\mu\beta} + \eta^{\gamma\sigma} \partial^\mu \partial_\mu h\right) = \frac{8\pi G}{c^4} T^{\gamma\sigma} \quad (\star)$$

where $h \equiv \eta^{\mu\nu} h_{\mu\nu}$

We define the trace-reversed metric perturbation

$$H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$H \equiv \eta^{\mu\nu} H_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu} h$$
$$= h - \frac{1}{2}(4) h = h - 2h = -h$$

Substituting $h_{\mu\nu} = H_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} h$

$= H_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} H$ into equation $(\star)$

the weak-field EE becomes

$$\Box^2 H^{\gamma\sigma} - \partial^\gamma \partial_\mu H^{\mu\sigma} - \partial^\sigma \partial_\mu H^{\mu\gamma} + \eta^{\gamma\sigma} \partial_\beta \partial_\mu H^{\mu\beta} = -\frac{16\pi G}{c^4} T^{\gamma\sigma} \quad (**)$$

where $\Box^2 = \eta^{\alpha\beta} \partial_\alpha \partial_\beta = -\frac{\partial^2}{\partial t^2} + \nabla^2$

Gauge Transformation :

Imagine that we make a coordinate transformation of the nearly cartesian coordinates x^α to new nearly cartesian coordinates x'^α given by

$$x'^\alpha = x^\alpha + \xi^\alpha \text{, where}$$

$\xi^\alpha = \xi^\alpha(t, x, y, z)$ and $|\xi^\alpha| \ll 1$

The partial derivatives are

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \frac{\partial}{\partial x^\beta}\left[x^\alpha + \xi^\alpha\right] = \frac{\partial x^\alpha}{\partial x^\beta} + \partial_\beta \xi^\alpha = \delta^\alpha_\beta + \partial_\beta \xi^\alpha$$

$$\frac{\partial x^\beta}{\partial x'^\alpha} = \frac{\partial}{\partial x'^\alpha}\left[x'^\beta - \xi^\beta\right] = \frac{\partial x'^\beta}{\partial x'^\alpha} - \frac{\partial \xi^\beta}{\partial x'^\alpha}$$

$$= \delta^\beta_\alpha - \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial \xi^\beta}{\partial x^\mu} = \delta^\beta_\alpha - \frac{\partial}{\partial x'^\alpha}\left(x'^\mu - \xi^\mu\right)\partial_\mu \xi^\beta$$

$$= \delta^\beta_\alpha - \left(\delta^\mu_\alpha - \partial'_\alpha \xi^\mu\right)\partial_\mu \xi^\beta = \delta^\beta_\alpha - \delta^\mu_\alpha \partial_\mu \xi^\beta + \underbrace{\partial'_\alpha \xi^\mu \partial_\mu \xi^\beta}_{\text{2nd order in } \xi \text{ (negligible)}}$$

$$\approx \delta^\beta_\alpha - \partial_\alpha \xi^\beta$$

It is shown in Box 30.4 that

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$$H'_{\mu\nu} = H_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial_\alpha \xi^\alpha$$

Such a transformation does not effect the Rieman tensor (to our level of approximation) :

$$R'_{\alpha\beta\mu\nu} = \frac{1}{2}\left[\partial_\beta \partial_\mu h'_{\alpha\nu} + \partial_\alpha \partial_\nu h'_{\beta\mu} - \partial_\alpha \partial_\mu h'_{\beta\nu} - \partial_\beta \partial_\nu h'_{\alpha\mu}\right]$$

$$\approx \frac{1}{2}\left[\partial_\beta \partial_\mu h_{\alpha\nu} + \partial_\alpha \partial_\nu h_{\beta\mu} - \partial_\alpha \partial_\mu h_{\beta\nu} - \partial_\beta \partial_\nu h_{\alpha\mu}\right]$$
$$= R_{\alpha\beta\mu\nu}$$

Therefore, such a transformation will have no effect on the Einsten Equation in the weak-field limit.

The Lorentz Gauge:

We can always find a Gauge Transformation $x'^\mu = x^\mu + \xi^\mu$ that converts $H_{\mu\nu}$ to $H'_{\mu\nu}$ such that

$$\partial_\mu H'^{\mu\nu} = 0$$

(This condition is called the Lorentz gauge)

Dropping the prime we have $\partial_\mu H^{\mu\nu} = 0$. Using this condition in $(**)$ yields

$$\Box^2 H^{\mu\nu} = \frac{-16\pi G}{c^4} T^{\mu\nu}$$

provided $\partial_\mu H^{\mu\nu} = 0$.

Note: In the literature $H_{\mu\nu}$ is denoted by $\bar{h}_{\mu\nu}$.

Ch. 31 : Detecting Gravitational Waves

Gravitational waves are ripples in the curvature of spacetime

In empty space, the relevant equations for exploring gravitational wave solutions are

$$\underbrace{\Box^2 H^{\mu\nu}}_{\partial^\alpha \partial_\alpha H^{\mu\nu} = 0} = 0 \quad \text{subject to the restriction } \partial_\nu H^{\mu\nu} = 0 \quad (*)$$

Note: If $H^{\mu\nu}$ solves equations $(*)$, then

$$H'^{\mu\nu} = H^{\mu\nu} - \partial^\mu \xi^\nu - \partial^\nu \xi^\mu + \eta^{\mu\nu} \partial_\alpha \xi^\alpha$$

with $\Box^2 \xi^\mu = \partial^\alpha \partial_\alpha \xi^\mu = 0$ does as well.

A Plane Wave Solution:

$$\Box^2 H^{\mu\nu} = \partial^\alpha \partial_\alpha H^{\mu\nu} = \left[-\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \right] H^{\mu\nu} = 0$$

we expect to find plane wave solutions of the form

$$H^{\mu\nu}(t, x, y, z) = A^{\mu\nu} \cos(k_\sigma x^\sigma)$$

$$= A^{\mu\nu} \cos(k_t x^t + \vec{k} \cdot \vec{r})$$

$$= A^{\mu\nu} \cos\left(-\frac{\omega}{c} ct + \vec{k} \cdot \vec{r}\right)$$

, where $k_\sigma = \left(-\frac{\omega}{c}, k_x, k_y, k_z\right)$ is a constant covector.

Such a wave is a plane wave perpendicular to the $\vec{k}$ direction that moves in the $\vec{k}$ direction with speed $v = \omega/k$

The EE $\Box^2 H^{\mu\nu} = 0$ implies:

$$\partial^\alpha \left[-A^{\mu\nu} \sin(k_\sigma x^\sigma) k_\sigma \partial_\alpha x^\sigma \right] = 0$$

$$A^{\mu\nu} \cos(k_\sigma x^\sigma) k_\gamma \partial^\alpha x^\gamma (k_\sigma \partial_\alpha x^\sigma) - A^{\mu\nu} \sin(\quad) k_\sigma \overbrace{\partial^\alpha \partial_\alpha x^\sigma}^{\partial^\alpha \delta^\sigma_\alpha = 0} = 0$$

$$A^{\mu\nu} \cos(\quad) k_\gamma \eta^{\alpha\beta} \partial_\beta x^\gamma \cdot k_\sigma \delta^\sigma_\alpha = 0$$

$$A^{\mu\nu} \cos(\quad) k_\gamma \eta^{\alpha\beta} \delta^\gamma_\beta k_\alpha = 0$$

$$A^{\mu\nu} \cos(\quad) k_\gamma \eta^{\alpha\gamma} k_\alpha = A^{\mu\nu} \cos(\quad) k_\gamma k^\gamma = 0 \Rightarrow \boxed{k^\alpha k_\alpha = 0}$$

Lorentz gauge $\partial_\mu H^{\mu\nu} = 0$ implies:

$$\partial_\mu \left[A^{\mu\nu} \cos(k_\sigma x^\sigma) \right] = 0$$

$$- A^{\mu\nu} \sin(k_\sigma x^\sigma) \underbrace{\underbrace{k_\gamma \partial_\mu x^\gamma}_{k_\gamma \delta_\mu^\gamma}}_{k_\mu} = 0$$

$$- A^{\mu\nu} k_\mu \sin(k_\sigma x^\sigma) = 0 \Rightarrow \boxed{k_\mu A^{\mu\nu} = 0}$$

Symmetry $H^{\mu\nu} = H^{\nu\mu}$ implies $A^{\mu\nu} = A^{\nu\mu}$

$$k^\alpha k_\alpha = -\left(\frac{\omega}{c}\right)^2 + \vec{k}^2 = -\left(\frac{\omega}{c}\right)^2 + \left(\frac{\omega}{v}\right)^2 = 0$$

$\Rightarrow \boxed{v = c}$! The wave moves with speed c.

$\therefore$ Gravitational waves move at the speed of light.

Transverse-Traceless Gauge:

Let us choose ξ^μ as $\xi^\mu = \overset{\text{constant.}}{B^\mu} \sin(k_\sigma x^\sigma)$

Note that this satisfies $\Box^2 \xi^\mu = 0$

$$\partial^\alpha \partial_\alpha \xi^\mu = \partial^\alpha \left[+B^\mu \cos(k_\sigma x^\sigma)\, k_\sigma \partial_\alpha x^\sigma \right]$$

$$= -B^\mu \sin(\quad)\, k_\sigma \partial^\alpha x^\sigma \cdot (k_\sigma \delta_\alpha^\sigma)$$

$$= -B^\mu \sin(\quad)\, k_\sigma \eta^{\alpha\beta} \underbrace{\partial_\beta x^\sigma}_{} \cdot k_\alpha$$

$$= -B^\mu \sin(\quad)\, k_\sigma \eta^{\alpha\sigma} k_\alpha \delta_\beta^\sigma$$

$$= -B^\mu \sin(\quad)\, \underbrace{k^\alpha k_\alpha}_{=0} = 0$$

It is shown in Box 31.2 that choosing the components of B^μ appropriately, we can convert any original choice of A^μ to a new $A^{\mu\nu}$ such that $A^{\mu t} = A^{t\mu} = 0$

and $A^\mu{}_\mu = 0$

We say that a gravitational plane wave solution satisfying these conditions is in "transverse-traceless gauge" and denote it as $H^{\mu\nu}_{TT}$.

Note that $h_{\mu\nu}^{TT} = H_{\mu\nu}^{TT} - \frac{1}{2}\eta_{\mu\nu} H_{TT}$

$$= H_{\mu\nu}^{TT} - \frac{1}{2}\eta_{\mu\nu}\Big[\underbrace{A^\mu{}_\mu}_{=0} \cos(k_\sigma x^\sigma)\Big] = H_{\mu\nu}^{TT}$$

$\therefore$ In transverse-traceless gauge there is no distinction between $h_{\mu\nu}$ and $H_{\mu\nu}$.

A wave propagating in the +z direction:

For a wave propagating in the +z direction

$$k_\sigma = \left(-\frac{\omega}{c}, 0, 0, \frac{\omega}{c}\right) \text{; } \omega = \text{angular frequency of the wave}$$

The Lorentz condition $k_\mu A^{\mu\nu} = 0$

becomes $-\frac{\omega}{c} A^{t\nu} + \frac{\omega}{c} A^{z\nu} = 0 \Rightarrow A^{t\nu} = A^{z\nu}$

We also have:

Symmetry and TT: $A^{t\nu} = A^{\nu t} = 0 \Rightarrow A^{00} = A^{tt} = 0$

$A^{z\nu} = A^{\nu z}$

$$A^{tz} = A^{zz} = 0$$

TT Gauge: $A^\mu{}_\mu = \underset{0}{A^{tt}} + A^{xx} + A^{yy} + \underset{0}{A^{zz}} = 0$

$$\Rightarrow A^{xx} + A^{yy} = 0$$

non-zero elements

$$A^{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= A_+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{= e_+^{\mu\nu}} + A_\times \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{= e_\times^{\mu\nu}}$$

Polarization Matrices

$e_+^{\mu\nu}$ = upright or pluss polarization

$e_\times^{\mu\nu}$ = diagonal or cross polarization

The physical effects of such a wave:

With lowered indices $h_{\mu\nu}^{TT} = H_{\mu\nu}^{TT} = A_{\mu\nu} \cos(k_\sigma x^\sigma)$

$$A_{\mu\nu} = \eta_{\mu\alpha}\, \eta_{\nu\beta}\, A^{\alpha\beta}$$

$$A_{tt} = \eta_{t\alpha}\, \eta_{t\beta}\, A^{\alpha\beta} = \underset{=(-1)}{\eta_{tt}}\, \underset{=(-1)}{\eta_{tt}}\, A^{tt} = A^{tt}$$

$$A_{ij} = \eta_{i\alpha}\, \eta_{j\beta}\, A^{\alpha\beta} = \underset{=+1}{\eta_{ii}}\, \underset{=+1}{\eta_{jj}}\, A^{ij} = A^{ij}$$

$$\Rightarrow \quad A_{\mu\nu} = A^{\mu\nu}$$

Now, consider a particle initially at rest, Its four-velocity to leading order is $U^\alpha = \begin{bmatrix} c\,dt/d\tau \\ U^x \\ U^y \\ U^z \end{bmatrix} \approx \begin{bmatrix} c(-g_{tt})^{-1/2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \approx \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix}$

The geodesic equation is

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu} U^\mu U^\nu$$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu} \right]$$

$$= \frac{1}{2} \eta^{\alpha\beta} \left[\partial_\mu h^{TOT}_{\nu\beta} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu} \right]$$

For $\alpha = x$: $\Gamma^x_{\mu\nu} U^\mu U^\nu = \Gamma^x_{tt} U^t U^t = c^2 \Gamma^x_{tt}$

$$\Gamma^x_{tt} = \frac{1}{2} \eta^{x\beta} \left[\partial_t h_{t\beta} + \partial_t h_{\beta t} - \partial_\beta h_{tt} \right]$$

$$= \frac{1}{2} \eta^{xx} \left[\partial_t h_{tx} + \partial_t h_{xt} - \partial_x h_{tt} \right]$$

But in the TT gauge $h_{tx} = h_{xt} = h_{tt} = 0$

$$\Rightarrow \Gamma^x_{tt} = 0$$

$$\Gamma^y_{tt} = \frac{1}{2} \eta^{yy} \left[\partial_t \overset{=0}{h_{ty}} + \partial_t \overset{=0}{h_{yt}} - \partial_y \overset{=0}{h_{tt}} \right]$$

$$\Gamma^z_{tt} = \frac{1}{2} \eta^{zz} \left[\partial_t \underset{0}{\underset{\parallel}{h_{tz}}} + \partial_t \underset{0}{\underset{\parallel}{h_{zt}}} - \partial_z \underset{0}{\underset{\parallel}{h_{tt}}} \right]$$

$$\Rightarrow \frac{d^2 x^\alpha}{d\tau^2} = 0$$

The coordinate system that we have adopted by using the transverse - traceless gauge is a comoving coordinate system where the coordinates of a free particle remain fixed.

Ch. 32 : Gravitational Wave Energy

Our goal in this chapter is to find out how much energy a gravitational wave of a given amplitude carries.

The EE
$$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu},$$
where the stress-energy tensor $T^{\mu\nu}$ describes the density of matter and energy excluding the energy associated with the gravitational field.

- Everything having to do with the gravitational field is embedded in the Einstein tensor $G^{\mu\nu}$; the way that the gravitational field energy acts as a source for the field is expressed through the nonlinear aspects of $G^{\mu\nu}$.

The equation $\nabla_\mu T^{\mu\nu} = 0$ was obtained from the equation $\partial_\mu T^{\mu\nu} = 0$, which is correct in a locally inertial frame (LIF) in which we can use cartesian coordinates.

- It turns out that in a general curved space-time, the equation $\nabla_\mu T^{\mu\nu} = 0$ cannot be integrated over a finite region of space to yield conserved quantities the way we can in a flat spacetime.

Conservation of non-gravitational energy and momentum only makes sense in a flat spacetime.

In Nearly Flat Spacetime:

The EE to first order in the metric perturbation $h_{\mu\nu}$ is
$$\Box^2 H_{\mu\nu} = \partial_\alpha \partial^\alpha H_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

or $$-2G^{(1)}_{\mu\nu} = \partial_\alpha \partial^\alpha H_{\mu\nu} = \frac{-16\pi G}{c^4} T_{\mu\nu} \quad (*)$$

$\uparrow$ to first order in $h_{\mu\nu}$

$(H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h)$

$T_{\mu\nu}$: stress-energy tensor of matter and non-gravitational energy

Equation $(*)$ does not include any contribution from gravitational energy. In order to look at the gravitational energy we have to expand the field equations to second order in $h_{\mu\nu}$:

$$-2G^{(1)}_{\mu\nu} - 2G^{(2)}_{\mu\nu} = \partial_\alpha \partial^\alpha H_{\mu\nu} - 2G^{(2)}_{\mu\nu} = \frac{-16\pi G}{c^4} T_{\mu\nu}$$

$$\partial_\alpha \partial^\alpha H_{\mu\nu} = -16\pi G T_{\mu\nu} + 2G^{(2)}_{\mu\nu}$$

$$= -\frac{16\pi G}{c^4}\left(T_{\mu\nu} + T_{\mu\nu}^{GW}\right) \quad (**)$$

, where $T_{\mu\nu}^{GW} \equiv -\frac{c^4 G^{(2)}_{\mu\nu}}{8\pi G}$

Equation $(**)$ expresses how the energy of the gravitational field feeds back on itself to create a stronger gravitational field. This issue is expressed by the non-linearities in $G_{\mu\nu}$.

Since $\partial_\mu H^{\mu\nu} = 0 \Rightarrow \partial_\mu (T^{\mu\nu} + T^{\mu\nu}_{GW}) = 0$

$*$ $T_{\mu\nu}^{GW}$ has no meaning at a point but only becomes meaningful after it has been averaged over several wavelenghts of the gravitational wave:

$$T_{\mu\nu}^{GW} = -\frac{c^4 \langle G^{(2)}_{\mu\nu} \rangle}{8\pi G}$$

Evaluating the Stress-Energy for a Gravitational Wave:

Consider an uprightly polarized gravitational ~~field~~ wave moving in the $+z$ direction.

Let us define

$$h_+(ct,z) = A_+ \cos\left(\frac{\omega ct}{c} - \frac{\omega}{c}z\right) = h_{xx}^{TT} = -h_{yy}^{TT}$$

$$\dot{h}_+(ct,z) = \partial_{(ct)} h_+ = -A_+\frac{\omega}{c}\sin\left(\frac{\omega ct}{c} - \frac{\omega}{c}z\right)$$

$$\partial_z h_+ = +\frac{\omega}{c}A_+ \sin\left(\frac{\omega ct}{c} - \frac{\omega}{c}z\right) = -\partial_{ct} h_+$$

$$\ddot{h}_+ = \partial_{(ct)}\partial_{(ct)} h_+ = -\frac{\omega^2}{c^2}A_+\cos\left(\frac{\omega ct}{c} - \frac{\omega}{c}z\right) = -\dot{h}_+$$

$$= \partial_z\partial_z h_+ = -\partial_z\partial_{(ct)} h_+ = -\frac{\omega^2}{c^2}h_+$$

The diagonal elements of the Ricci tensor are

$$R_{tt} = R_{zz} = h_+\ddot{h}_+ + \frac{1}{2}\dot{h}_+\dot{h}_+ = \left(\dot{h}_+\dot{h}_+ + h_+\ddot{h}\right) - \frac{1}{2}\dot{h}_+\dot{h}_+$$

$$R_{xx} = R_{yy} = 0 \quad , \langle R_{tt}^{(2)}\rangle = \langle R_{zz}^{(2)}\rangle = \langle \dot{h}_+\dot{h}_+ + h_+\ddot{h}_+\rangle - \frac{1}{2}\langle \dot{h}_+\dot{h}_+\rangle$$

If we average over several gravitational wave lengths

$$\langle \dot{h}_+\dot{h}_+ + h_+\ddot{h}_+\rangle = \Big\langle \left[-\frac{\omega}{c}A_+\sin\theta\right]^2 + A_+\cos\theta\left[-\frac{\omega^2}{c^2}A_+\cos\theta\right]\Big\rangle$$

$$= A_+^2\frac{\omega^2}{c^2}\langle \sin^2\theta - \cos^2\theta\rangle = -A_+^2\frac{\omega^2}{c^2}\underbrace{\langle\cos 2\theta\rangle}_{=0} = 0$$

, where $\theta \equiv \frac{\omega (c)t}{c} - \frac{\omega z}{c}$

So $\langle R_{tt}^{(2)}\rangle = \langle R_{zz}^{(2)}\rangle = -\frac{1}{2}\langle \dot{h}_+\dot{h}_+\rangle$

It can be shown that

$$\langle R^{(2)}\rangle = 0$$

second order portion of the curvature scalar

Since $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

$$G_{\mu\nu}^{(2)} = R_{\mu\nu}^{(2)} - \frac{1}{2} g_{\mu\nu} R^{(2)} = T_{\mu\nu}^{GW}$$

$$\langle G_{\mu\nu}^{(2)} \rangle = \langle R_{\mu\nu}^{(2)} \rangle - \frac{1}{2} g_{\mu\nu} \langle R^{(2)} \rangle$$

$$= \langle T_{\mu\nu}^{GW} \rangle$$

$$\langle T_{\mu\nu}^{GW} \rangle = \langle R_{\mu\nu}^{(2)} \rangle$$

$$\langle T_{tt}^{GW} \rangle = \langle R_{tt}^{(2)} \rangle = -\frac{c^4}{8\pi G} \langle G_{\mu\nu}^{(2)} \rangle$$

$$= -\frac{c^4}{8\pi G} \left(-\frac{1}{2} \langle \dot{h}_+ \dot{h}_+ \rangle \right) = \frac{c^4}{16\pi G} \langle \dot{h}_+ \dot{h}_+ \rangle$$

This is the effective energy density of an uprightly-polarized wave

Note that an uprightly polarized wave can be converted to a diagonally polarized wave simply by rotating the coordinate system by $45°$ around the z axis.

Therefore the total energy density of an arbitrary gravitational wave must be

$$\langle T_{tt}^{GW} \rangle = \frac{c^4}{16\pi G} \langle \dot{h}_+ \dot{h}_+ + \dot{h}_\otimes \dot{h}_\otimes \rangle$$

where $h_\otimes = h_{xy}^{TT} = h_{yx}^{TT}$, $h_+ = h_{xx}^{TT} = -h_{yy}^{TT}$

Since $\dot{h}_{jk} \dot{h}^{jk} = 2(\dot{h}_+ \dot{h}_+ + \dot{h}_\otimes \dot{h}_\otimes) \Rightarrow$

$$\langle T_{tt}^{GW} \rangle = \frac{c^4}{32\pi G} \langle \dot{h}_{jk}^{TT} \dot{h}^{jk}_{TT} \rangle$$

For a gravitational wave moving in the $+z$ direction the flux (energy transported per unit time per unit area by the wave in the direction of its motion) is given by

$$\langle T_{tz}^{GW} \rangle = \frac{c^4}{}$$

$$\langle T_{tz}^{GW} \rangle = \langle R_{tz}^{(2)} \rangle = -\frac{c^4}{8\pi G} \langle G_{tz}^{(2)} \rangle$$

It turns out that $\langle T_{tz}^{GW} \rangle = \langle T_{tt}^{GW} \rangle$

$$= \frac{c^4}{32\pi G} \langle \dot{h}_{jk}^{TT} \dot{h}^{jk}_{TT} \rangle$$

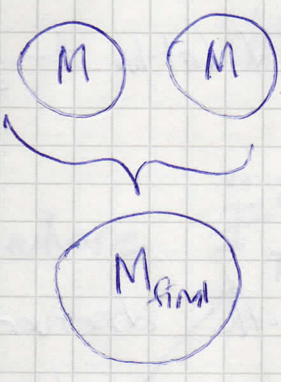
Ch. 33 : Generating Gravitational Waves

In this chapter, we will explore how we can calculate the gravitational waves emitted by a dynamic source.

Crude Estimates:

At a place where $|h_{\mu\nu}| \approx 1$ the strength of the gravitational wave is maximum. Such a maximum-strength wave is realized near the event horizon of a black hole where $g_{tt} = -\left(1 - \frac{2GM}{c^2}\right) \approx -(1-1) = 0$.

A wave of maximum amplitude might be produced near the event horizon of two merging black holes



If $M \approx 1.5 M_\odot$, then $\frac{2GM_{final}}{c^2} = \frac{4GM}{c^2}$

$M_{final} \approx 2M$

$$= \frac{6GM_\odot}{c^2}$$

$$\cong 6(1.5 \text{ km}) \cong 10 \text{ km}$$

$$= 10^4 \text{ m}$$

$\left(\text{Recall } \frac{GM_\odot}{c^2} = 1.485 \text{ km} \approx 1.5 \text{ km}\right)$

If $h = [A_+^2 + A_\otimes^2]^{1/2} \approx 1$ at a radius of $\frac{2GM_{final}}{c^2}$

then $h \approx \frac{1\text{m}}{r} = \frac{1\text{m}}{10^{24}\text{ m}} = \left(\frac{1}{10^4}\right)\frac{1}{10^{20}} = 1 \times 10^{-20}$

at a radius of $r = 10^{24}$ m.

(We have assumed that, like EM waves, gravitational waves fall off as $\frac{1}{r}$)

A gravitational wave detector will need to have a sensitivity of at least $h \approx 10^{-21}$ to have a good chance of seeing such events. LIGO has $h \approx 10^{-22}$.

Overview of the Small-Source Approximation

We assume that

1. The source is small compared to both a wavelength of the wave and the distance to the observer
2. The source is weak because $|h_{\mu\nu}| \ll 1$ even very near the source
3. The source is slow (parts of the source move at speeds $\ll c$)

In this approximation, conservation of energy and momentum in the source implies that

$$H^{jk} \text{ (at time } t) = \frac{2G}{c^4 R} \frac{d^2}{dt^2} \int \rho\, x^j x^k \, dV \text{ (at time } t - \tfrac{R}{c})$$

$$= \frac{2G}{c^4 R} \ddot{I}^{jk} \text{ (at time } t - \tfrac{R}{c}),$$

where R is the distance from the source to the observer and $I^{jk} = \int_{\text{source}} \rho\, x^i x^j \, dV$ is the quadropole moment 3-tensor of the source

We define the reduced quadropole moment 3-tensor $\mathcal{I}^{jk}$

$$\mathcal{I}^{jk} = \int_{\text{source}} \rho \left(x^j x^k - \frac{1}{3} \eta^{jk} r^2 \right) dV$$

Note that $\underbrace{\mathrm{Tr}}_{\text{trace}} \mathcal{I}^{jk} = \eta_{jk} \mathcal{I}^{jk} = \int \rho \left(\eta_{jk} x^j x^k - \frac{1}{3} \eta_{jk} \eta^{jk} r^2 \right) dV$

$$= \int \rho \left(\underbrace{x_1^2 + x_2^2 + x_3^2}_{= r^2} - \frac{1}{3}(3) r^2 \right) dV$$

$$= \int \rho \cdot (0)\, dV = 0$$

If we expand the Newtonian gravitational potential Φ at some large distance R from a compact and static but asymmetrical source whose center of mass is at the origin, we get

$$\Phi = -\frac{GM}{R} + \overbrace{\frac{3\mathcal{I}_{jk}}{2R^3}\left(\frac{X^j}{R}\right)\left(\frac{X^k}{R}\right)}^{\text{leading deviation from sphericity.}} + \text{higher-order terms}$$

, where X^j is a component of the radius vector $\vec{R}$ from the source to the observer.

We will find that, for waves moving in the direction specified by a unit vector $\hat{n}$ ($\vec{k} = k\hat{n}$), we have

$$h^{jk}_{TT} = H^{jk}_{TT} = \frac{2G}{c^4R}\ddot{I}^{jk}_{TT} = \frac{2G}{c^4R}\ddot{\mathcal{I}}^{jk}$$

reduced

, where $\ddot{\mathcal{I}}^{jk}_{TT} \equiv \left(P_m^{\ j}P_n^{\ k} - \frac{1}{2}P^{jk}P_{mn}\right)$

with $P^j_{\ n} \equiv \delta^j_{\ n} - n^j n_m$

evaluated at time $t - \frac{R}{c}$

The flux of gravitational wave energy flowing in direction $\hat{n}$ is

$$\text{Flux} = \frac{c^4}{32\pi G}\left\langle \dot{h}^{jk}_{TT}\,\dot{h}^{TT}_{jk}\right\rangle =$$

$$= \frac{c^4}{32\pi G}\cdot\left(\frac{2G}{c^4R}\right)^2\left\langle \dddot{\mathcal{I}}^{jk}_{TT}\,\dddot{I}^{TT}_{jk}\right\rangle$$

$$= \frac{G}{8\pi c^4R^2}\left\langle \qquad \right\rangle$$

done on p. 343

Integrating the flux over all possible directions

the total gravitational wave energy radiated / unit time =

$$\frac{dE}{dt} = -\frac{G}{c^4 5} \langle \dddot{I}_{jk} \dddot{I}^{jk} \rangle$$

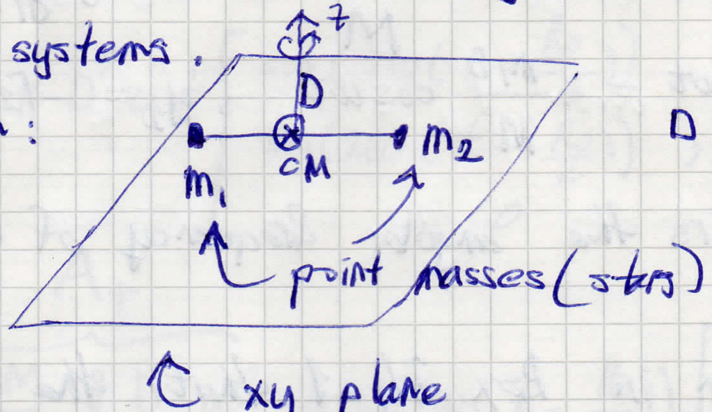
Derivation of these results:

Ch. 34 ; Gravitational Wave Astronomy

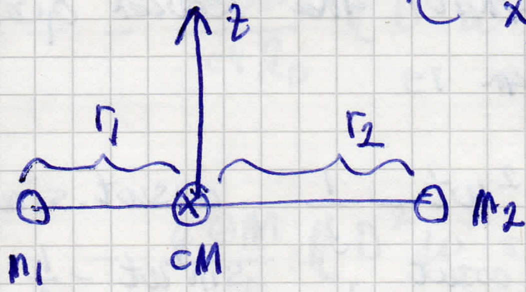
Gravitational radiation from a rotating dumbell :

The most common sources of gravitational waves are binary star systems.

Approximation:



D : distance betw. centers of stars



$$D = r_1 + r_2$$

$$\vec{r}_{CM} = 0 = \frac{-m_1 r_1 \hat{e} + m_2 r_2 \hat{e}}{(m_1 + m_2)} \quad , \quad r_1 = D - r_2$$

$$\Rightarrow \quad -m_1 (D - r_2) + m_2 r_2 = 0$$

$$-m_1 D + r_2 (m_1 + m_2) = 0$$

$$\Rightarrow \quad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) D$$

$$\Rightarrow \quad -m_1 r_1 + \left(\frac{m_1 m_2}{m_1 + m_2}\right) D = 0$$

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) D$$

Note that $\dfrac{r_1}{r_2} = \dfrac{m_2}{m_1}$

Let us define $t=0$ to be the instant when mass 1 crosses the $+x$ axes. The coordinates are then

$$x_1 = r_1 \cos \omega t = \frac{m_2 D}{M} \cos \omega t \; ; \; y_1 = r_1 \sin \omega t = \frac{m_2 D}{M} \sin \omega t$$

$$x_2 = -r_2 \cos \omega t = -\frac{m_1 D}{M} \cos \omega t \; , \; y_2 = -r_2 \sin \omega t = -\frac{m_1 D}{M} \sin \omega t$$

, where ω is the angular frequency of the orbit.

It is shown in Box 34.1 that the reduced quadrupole moment tensor for this system is

$$I^{jk} = \frac{m_1 m_2 D^2}{M} \begin{bmatrix} \cos^2 \omega t - \frac{1}{3} \; , & \cos\omega t . \sin\omega t \; , & 0 \\ \sin\omega t . \cos\omega t \; , & \sin^2 \omega t - \frac{1}{3} \; , & 0 \\ 0 \; , & 0 & 0 - \frac{1}{3} \end{bmatrix}$$

Define $\eta \equiv \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{m_1 m_2}{M^2}$

using $\cos\theta . \sin\theta = \frac{1}{2} \sin 2\theta$, $\cos^2\theta = \frac{1}{2} + \frac{1}{2} \cos 2\theta$

$\sin^2\theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta$

$$\Rightarrow \quad I^{jk} = \frac{1}{2} M \eta D^2 \begin{bmatrix} \frac{1}{3} + \cos 2\omega t \; , & \sin 2\omega t \; , & 0 \\ \sin 2\omega t \; , & \frac{1}{3} - \cos 2\omega t \; , & 0 \\ 0 \; , & 0 \; , & -\frac{1}{3} \end{bmatrix}$$

The double time derivative is

$$\ddot{I}^{jk} = 2 M \eta D^2 \omega^2 \begin{bmatrix} -\cos 2\omega t & -\sin 2\omega t & 0 \\ -\sin 2\omega t & \cos 2\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(Note Trace $\ddot{I}^{jk} = 0$)

For an observer in the +z direction and a distance R_0 from the system's center of mass, the metric perturbation is $h^{jk}_{TT} = H^{jk}_{TT} = \frac{2G}{c^4 R_0} \ddot{I}^{jk}_{TT}$

$$= -\frac{4GM}{c^4 R_0} \eta D^2 \omega^2 \begin{bmatrix} \cos(2\omega(t-\frac{R_0}{c})) , & \sin(2\omega(t-\frac{R_0}{c})), & 0 \\ \sin(2\omega(t-\frac{R_0}{c})) & ,-\cos(\quad) & ,0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$H^{jk}_{TT} = -\overbrace{\frac{4GM}{c^4 R_0} \eta D^2 \omega^2}^{A_+} \cos(2\omega(t-\frac{R_0}{c})) \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$- \underbrace{\frac{4GM}{c^4 R_0} \eta D^2 \omega^2}_{A_\otimes} \sin 2\omega(t-\frac{R_0}{c}) \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_+ = A_\otimes = \frac{4GM\eta D^2 \omega^2}{c^4 R_0}$$

$\Rightarrow$ i) The wave has equal amounts of plus and cross polarization.

ii) The plus and cross polarizations are 90° out of phase, implying the wave is circularly polarized

iii) The wave has an angular frequency equal to twice the rotational frequency of the system.

To find the gravitational waves radiated in another direction, we can calculate $\ddot{I}^{jk}$ using the equation

$$\ddot{I}^{jk}_{TT} = \left(P^j_m P^k_n - \frac{1}{2} P^{jk} P_{mn} \right) \ddot{I}^{mn}$$

$$\text{with } P^j_m = \delta^j_m - n^j n_m$$

For waves moving in the $+x$ direction, it can be shown that

$$h^{jk}_{TT} = H^{jk}_{TT} = \frac{2GM\eta D^2 \omega^2}{c^4 R_0} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos(2\omega(t-\frac{R_0}{c})) & 0 \\ 0 & 0 & -\cos(2\omega(t-\frac{R_0}{c})) \end{bmatrix}$$

$$= A_+ \cos(2\omega(t-\frac{R_0}{c})) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$+ A_\otimes \cdot \underset{\text{zero}}{0}$$

It can be shown that the total power carried away in all directions by the gravitational waves generated by this dumbell system is

$$-\frac{dE}{dt} = \frac{32 G M^2 \eta^2 D^4 \omega^6}{5 c^5}$$

Application to a Binary Star System:

Assume that the stars move slowly enough that their velocities are non-relativistic and that they are far enough so that Newtonian gravitational theory is adequate to predict their motion.

Newton's 2nd law applied to m_1 yields

$$\frac{G m_1 m_2}{D^2} = \frac{m_1 v_1^2}{r_1} \Rightarrow \frac{G m_2}{D^2} = r_1 \underbrace{\left(\frac{v_1}{r_1}\right)^2}_{= \omega^2}$$

$$= \underbrace{\left(\frac{m_2}{m_1+m_2}\right) D}_{= r_1} \omega^2$$

$$\Rightarrow \frac{G}{D^2} = \frac{D \omega^2}{M} \Rightarrow D^3 = \frac{GM}{\omega^2}$$

Plugging this into $A_+ = A_{⊗} = \frac{4GM \eta D^2 \omega^2}{c^4 R_0}$

$$= \frac{4GM}{c^4 R_0} \eta \left(\frac{GM}{\omega^2}\right)^{2/3} \cdot \omega^2$$

$$= \frac{4GM}{c^4 R_0} \eta (GM\omega)^{2/3}$$

The radiated power is

$$-\frac{dE}{dt} = \frac{32 \eta^2}{5 G c^5} (GM\omega)^{10/3}$$

It is shown in Box 34.3 that the total energy of the binary system is $E = -\frac{G m_1 m_2}{2D} = -\frac{1}{2} M (GM\omega)^{2/3} \eta$

$-\frac{dE}{dt}$ shows that as ω increases $\frac{dE}{dt}$ decreases and $D = \left(\frac{GM}{\omega^2}\right)^{1/3}$ decreases.

A Realistic Binary System:

The ι Boötis System: $M_1 = 1 M_\odot$, $M_2 = 0.5 M_\odot$

T (period) = 23,300 s, distance for this system

$R_0 = 4 \times 10^{17}$ m $\quad A_+ = 7.7 \times 10^{-21}$

$$-\frac{dE}{dt} = 1.1 \times 10^{23}\ W$$

Ch. 35 Gravitomagnetism

Review of the weak-field limit:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \quad h_{\mu\nu} = h_{\nu\mu} \,, \quad |h_{\mu\nu}| \ll 1$$

$$H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

Lorentz gauge : $\partial_\mu H^{\mu\nu} = 0$

Note: This gauge condition is equivalent to

$$0 = \eta^{\alpha\mu}\left(\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\mu}\right)$$

We saw in Ch. 22 that the EE becomes

$$(*) \qquad \partial^\alpha \partial_\alpha h^{\mu\nu} = -\frac{16\pi G}{c^4}\left(T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} T\right)$$

Solving the Einstein Equation:

The solution to Eq. (*) is

$$h^{\mu\nu}_{(t,\vec{R})} = \frac{4G}{c^4}\int_{\text{source}} \frac{1}{s}\left[T^{\mu\nu}\left(t-\frac{s}{c},\vec{r}\right) - \frac{1}{2}\eta^{\mu\nu} T\left(t-\frac{s}{c},\vec{r}\right)\right] \times dV$$

where $\vec{s} = \vec{R} - \vec{r}$, $s = |\vec{R} - \vec{r}|$

The Slow-Source Approximation:

Assume source = nonrelativistic perfect fluid. Then $\frac{P_0}{c^2}$ will be negligible compared to mass density ρ_0.

Also, $u^t \approx c$, $u^i \approx v^i \ll c$

$$T^{\mu\nu} = \left(\rho_0 + \frac{P_0}{c^2}\right) u^\mu u^\nu + P_0 g^{\mu\nu}$$

$$T^{ij} \approx \rho_0 u^i u^j \approx \rho_0 v_i v_j \approx 0$$

$$T^{tt} \approx \rho_0 u^t u^t \approx \rho_0 c^2$$

$$T^{ti} \approx \rho_0 u^t u^i \approx \rho_0 c v^i \,, \quad T^{ii} \approx \rho_0 u^i u^i \approx \rho_0 (v^i)^2 \approx 0 \quad \text{because } (v^i)^2 \ll c^2$$

This means that $T = \eta_{\alpha\beta} T^{\alpha\beta} = T^{\alpha}{}_{\alpha}$

$$\approx \eta_{tt} T^{tt} + 0 + 0 + 0$$

$$\approx - T^{tt} \approx - \rho_0 c^2$$

$$T^{tt} - \frac{1}{2}\eta^{tt} T = \rho_0 c^2 - \frac{1}{2}(-1)(-\rho_0 c^2) = \frac{1}{2}\rho_0 c^2$$

$$T^{ti} - \frac{1}{2}\eta^{ti} T = \rho_0 u^t u^i - 0 \approx \rho_0 c v^i$$

$$T^{ij} - \frac{1}{2}\eta^{ij} T \approx 0 - \frac{1}{2}\eta^{ij}(-\rho_0 c^2) \approx \frac{1}{2}\eta^{ij}\rho_0 c^2$$

Therefore

$$h^{tt}(t, \vec{R}) = \frac{4G}{c^4}\int \frac{1}{s}\underbrace{\Big[\rho_0 c^2 - \frac{1}{2}\overset{(-1)}{\overset{\shortparallel}{\eta^{tt}}}(-\rho_0 c^2)\Big]}_{\frac{1}{2}\rho_0 c^2} dV$$

$$= \frac{2}{c^2}\int \frac{G\rho_0(t - \frac{s}{c}, \vec{r})}{s} dV$$

$$h^{xx}(t, \vec{R}) = \frac{4G}{c^4}\int \frac{1}{s}\left[T^{xx} - \frac{1}{2}\eta^{xx} T\right] dV$$

$$\underbrace{\left[\frac{1}{2}\eta^{xx}\rho_0 c^2 - \frac{1}{2}\eta^{xx}(-\rho_0 c^2)\right]}_{\rho_0 c^2}$$

$$= \frac{4}{c^2}\int \frac{G\rho_0}{s} dV$$

$$h^{ti}(t, \vec{R}) = h^{it}(t, \vec{R}) = \frac{4G}{c^4}\int \frac{\Big[\rho_0 c v^i - \frac{1}{2}\overset{0}{\overset{\shortparallel}{\eta^{ti}}}(\rho_0 c^2)\Big]}{s} dV$$

$$= \frac{4}{c^3}\int \frac{G J^i(t - \frac{s}{c}, \vec{r})}{s} dV$$

where $\vec{J} = \rho_0 \vec{v}$ is the mass current density.

Let us define

the gravitational scalar potential $\Phi_G = -\frac{1}{2} h^{tt} =$

$$= -\frac{1}{c^2}\int_{\text{source}} \frac{G\rho_0}{s}\, dV$$

the gravitational vector potential

$$A^i_G = -\frac{1}{4} h^{ti} = -\frac{1}{4} h^{it} = -\frac{1}{c^3}\int \frac{G J^i}{s}\, dV$$

It is shown in Box 35.1 that

$$0 = \eta^{\alpha\mu}\left(\partial_\mu h_{\alpha\nu} - \frac{1}{2}\partial_\nu h_{\alpha\mu}\right)$$

requires that $\quad \vec{\nabla}\cdot\vec{A}_G = -\dfrac{\partial \Phi_G}{c\,\partial t}$

The Gravitational Maxwell Equations:

sign change

$\vec{\nabla}\cdot\vec{E} = 4\pi k_e \rho_Q$	$\vec{\nabla}\cdot\vec{E}_G = -4\pi G \rho_M$
$\vec{\nabla}\cdot\vec{B} = 0$	$\vec{\nabla}\cdot\vec{B}_G = 0$
$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	$\vec{\nabla}\times\vec{E}_G = -\dfrac{\partial\vec{B}_G}{\partial t}$
$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0 \dfrac{\partial\vec{E}}{\partial t}\right)$	$\vec{\nabla}\times\vec{B}_G = -\dfrac{4\pi G}{c^2}\vec{J} + \dfrac{1}{c^2}\dfrac{\partial\vec{E}_G}{\partial t}$
$\vec{E} = -\nabla\Phi - \dfrac{\partial\vec{A}}{\partial t}$; $\vec{B} = \vec{\nabla}\times\vec{A}$	$\vec{E}_G = -\nabla\Phi_G - \dfrac{\partial\vec{A}_G}{\partial t}$
$k_e = \dfrac{1}{4\pi\varepsilon_0}$, $\varepsilon_0\mu_0 = \dfrac{1}{c^2}$	$\vec{B}_G = \vec{\nabla}\times\vec{A}_G$
$\mu_0 = \dfrac{1}{c^2\varepsilon_0} = \dfrac{4\pi k_e}{c^2}$	$\vec{\nabla}\cdot\vec{A}_G + \dfrac{\partial\Phi_G}{c\,\partial t} = 0$

$\underbrace{\hphantom{\mu_0 = \frac{1}{c^2\varepsilon_0} = \frac{4\pi k_e}{c^2}}}_{\text{Electromagnetism}}$

$\vec{\nabla}\cdot\vec{A} + \dfrac{\partial\Phi}{c\,\partial t} = 0$

(*) The sign change is because the gravitational force between positive masses is attractive, but the electromagnetic force between positive charges is repulsive.

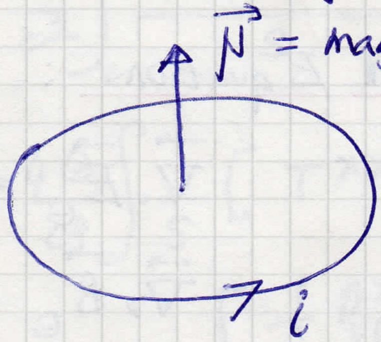
$\vec{E}_G = \vec{g}$; $\vec{B}_G$: Gravitomagnetic field
(Gravimagnetic)

If the field is static, for non-relativistic velocities, $v \ll c$,
$$\frac{d^2 x^i}{dt^2} \approx \frac{1}{2}\eta^{ik}\partial_k h + \eta^{jk}(\partial_k h_{tj} - \partial_j h_{tk}) \cdot v^j$$
(Eq. (22.14))

It is shown in box 35.3 that the gravitational force acting on the particle is
$$\vec{F}_G = m\frac{d^2\vec{x}}{dt^2} = m(\vec{E}_G + \vec{v} \times 4\vec{B}_G)$$
(Check this out for Electricity)

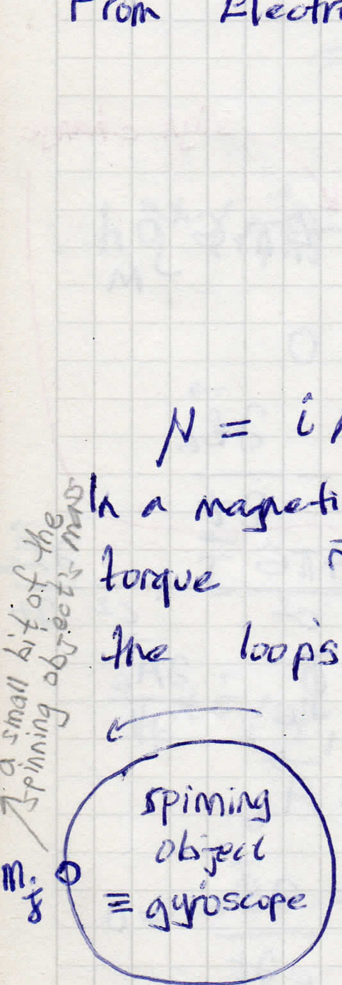
Gravitomagnetic Effects on a Gyroscope:

From Electromagnetic theory:



$\mu = iA$

In a magnetic field $\vec{B}$, such a current loop experiences a torque $\vec{\tau} = \vec{\mu} \times \vec{B}$. This torque tries to align the loops magnetic moment $\vec{\mu}$ with the field $\vec{B}$



$\vec{S}$ = the gyroscope's total spin angular momentum
$\vec{\mu}_G$ = the gyroscope's gravitomagnetic moment (due to spinning mass bits)

mass current $(i_m)_j = \frac{m_j}{T_j} = \frac{m_j}{(2\pi r / v_j)}$
(see Box 35.4)

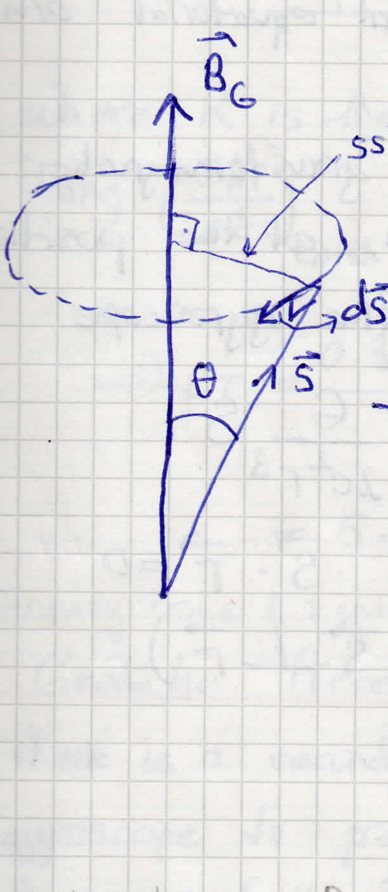
$$(\mu_G)_j = (i_m \cdot A)_j = \frac{m_j \cdot v_j}{2\pi r_j} \cdot (\pi r_j^2) = \frac{1}{2} m_j v_j r_j = \frac{1}{2} s_j$$

$$\mu_G = \sum_j (\mu_G)_j = \frac{1}{2} \sum s_j = \frac{1}{2} s \quad ; \quad \vec{\mu}_G = \frac{1}{2} \vec{s}$$

Such a gyroscope should experience a torque

$$\vec{\tau} = \vec{\mu}_G \times 4\vec{B}_G = \vec{s} \times 2\vec{B}_G$$

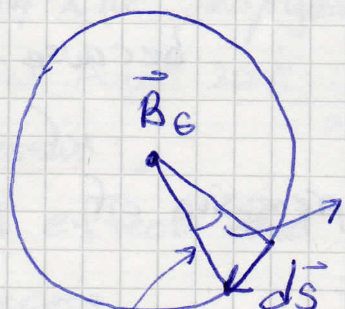
But also $\vec{\tau} = \frac{d\vec{s}}{dt}$ (compare with $\vec{F} = \frac{d\vec{p}}{dt}$)



$$|\vec{\tau}| = 2 s B_G \sin\theta = \frac{ds}{dt}$$

$$\Omega_{LT} = \frac{d\phi}{dt} = \frac{ds/s\sin\theta}{dt} = \frac{ds}{dt}\left(\frac{1}{s\sin\theta}\right)$$

$$= \frac{2 s B_G \sin\theta}{s\sin\theta} = 2 B_G$$



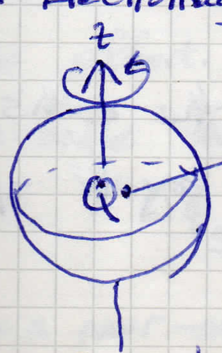
$$d\phi = \frac{ds}{s\sin\theta}$$

in vector form: $\vec{\Omega}_{LT} = -2\vec{B}_G$

Rotational Frame Dragging Effect

Lense-Thirring Precession Near a Spinning Object

From Electromagnetic Theory: magnetic field produced at $\vec{r}$ by the spinning charged object

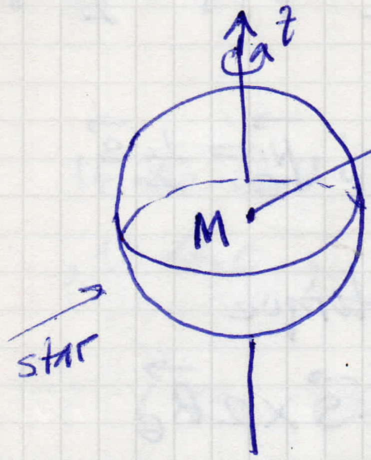


$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi c^2 r^3}\left[(3\vec{\mu}\cdot\vec{r})\hat{r} + \vec{\mu}\right]$$

$$\mu_0 = \frac{4\pi k_e}{c^2} = \frac{1}{c^2 \varepsilon_0}$$

a spinning charged object with spherical symmetry

By analogy ;



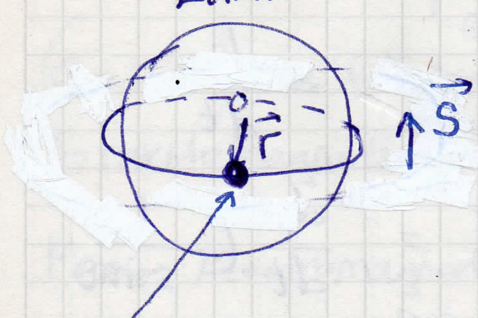
gravitomagnetic field produced by the spherical spinning star

$$\vec{B}_G(\vec{r}) = -\frac{G}{c^2 r^3}\left[3(\vec{\mu}_G \cdot \vec{r})\hat{r} - \vec{\mu}_G\right]$$

using $\vec{\mu}_G = \frac{1}{2}\vec{S}$

$$\vec{B}_G(\vec{r}) = \frac{G}{2c^2 r^3}\left[\vec{S} - 3(\vec{S}\cdot\hat{r})\hat{r}\right]$$

For example, consider a gyroscope in an equatorial orbit around the earth



The gravitomagnetic field at the position of the gyroscope

$$B_G(r) = \frac{G}{2c^2 r^3} S$$

because $\vec{S}\cdot\vec{r} = 0$
$(\vec{S} \perp \vec{r})$

The angular speed of precession of the gyroscope will be

$$\Omega_{LT} = 2B_G = \frac{GS}{c^2 r^3} \quad ; \quad S = I\omega$$

(Remark: $\vec{p} = m\vec{v}$
$\vec{S} = I\vec{\omega}$)

moment of inertia

$$= \frac{GI\omega}{c^2 r^3},$$

where I and ω are the earth's moment of inertia and angular speed.

$$\omega_{Earth} = \frac{2\pi}{day}$$

$$I = \alpha M R^2$$

For a perfect sphere $\alpha = \frac{2}{5} = 0.4$

But since the earth is denser near its center, we expect α for the earth to be smaller than 0.4.

Estimates imply that $\alpha = 0.33$

Therefore, a good estimate of the Lense-Thirring precession rate for the orbiting gyroscope is

$$\Omega_{LT} = \frac{G(\alpha M R^2)\omega}{c^2 r^3} = 0.33\left(\frac{GM/c^2}{R}\right)\left(\frac{R}{r}\right)^3 \omega$$

, where R is the earth's radius $\approx$ 6380 km.

Using $\left(\frac{GM}{c^2}\right)_{earth} = 4.45$ mm , $r \approx R$

$$\Omega_{LT} = \frac{0.33(4.5 \times 10^{-3} m)}{6,380,000\ m}\left(\frac{2\pi\ rad}{day}\right)\frac{365.25\ day}{year}$$

$$= 5.4 \times 10^{-7}\ rad/year$$, a very small number to measure!

Gravity Probe B (ended August 2005): good agreement with theory

Geodetic Precession:

There is a second effect that will also cause an orbiting gyroscope to precess.

Because of the curvature of spacetime, a gyroscope orbiting even a non-spinning object will precess: this phenomenon is called geodetic precession. This is not a gravitomagnetic effect.

The angle through which the gyroscope precesses is

$$\Delta\phi_{gd} \approx \frac{3GM}{c^2 R} \text{ per orbit}$$

Since a near-earth orbit takes about 85 minutes, the precession rate is $\Omega_{gd} = \dfrac{\Delta\phi_{gd}}{T}$

$$= \frac{3\pi(4.45 \times 10^{-3}\,m)}{(85 \times 60\,sec)} \frac{1}{(6\,380\,000\,m)} \frac{(3.16 \times 10^{7}\,sec)}{year}$$

$$= 4.1 \times 10^{-5}\ rad/year$$

Note that this is about two orders of magnitude larger than the Lense-Thirring effect for a gyroscope in low earth orbit.

Ch. 36: The Kerr Metric

Weak-Field Solution for a Rotating Sphere:

Consider a slowly rotating spherically symmetric object. The object is said to be stationary if the stress-energy tensor is time-independent inside the object (indicating that the object's mass and fluid velocity distributions are unchanging despite its rotation).

Then the metric perturbations will be time-independent and the retardation due to light travel time can be ignored.

The metric perturbations will be

$$h^{tt}(\vec{R}) = h^{xx}(\vec{R}) = h^{yy}(\vec{R}) = h^{zz}(\vec{R})$$

$$= \frac{2G}{c^2}\int_{\text{source}} \frac{\rho_0(r)\,dV}{|\vec{R}-\vec{r}|}$$

$$h^{ti}(\vec{R}) = h^{it}(\vec{R}) = \frac{4G}{c^3}\int \frac{\rho_0(r)\,v^i(\vec{r})}{|\vec{R}-\vec{r}|}\,dV.$$

, other $h^{\mu\nu} \simeq 0$.

Here $\vec{R}$ is the position where the field is being evaluated.

We have $h^{tt} = h^{xx} = h^{yy} = h^{zz} = \frac{2GM}{c^2 R}$

Assume that the angular velocity of the object is $\vec{\Omega} = \Omega\hat{k}$ (object rotating around the z-axis)

$$\vec{v}(\vec{r}) = \vec{\Omega}\times\vec{r} = \Omega\,\hat{k}\times\vec{r}$$

$$\vec{v} = \Omega(\hat{k}\times\vec{r}) = \Omega[\hat{k}\times(x\hat{i}+y\hat{j}+z\hat{k})]$$

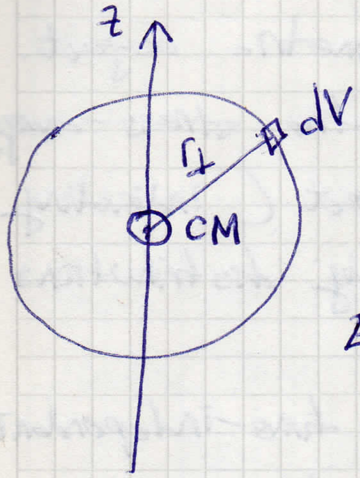
$$= \Omega[x\hat{j} - y\hat{i}] \Rightarrow v^x = -\Omega y,\; v^y = \Omega x$$

$$v^z = 0$$

ijk

The object's total spin angular momentum $\vec{S}$ is

$$\vec{S} = I\vec{\Omega} = \vec{\Omega}\int \rho_0 (x^2+y^2)\, dV$$



$$r_\perp = x^2+y^2$$

Expansion of $\frac{1}{|\vec{R}-\vec{r}|}$:

$$\frac{1}{|\vec{R}-\vec{r}|} = \frac{1}{R} + \frac{\vec{R}\cdot\vec{r}}{R^3} + \frac{3(\vec{R}\cdot\vec{r})^2 - R^2 r^2}{R^5} + \cdots$$

Then $$h^{tx}(\vec{R}) = \frac{4G}{c^3}\frac{1}{R}\int \rho_0 (-\Omega y)\, dV$$

$$+ \frac{4G}{c^3}\frac{1}{R^3}\int \rho_0 (-\Omega y)\, \vec{R}\cdot\vec{r}\, dV + \cdots$$

$$= -\frac{4G\Omega}{c^3 R}\underbrace{\int \rho_0(r)\, y\, dV}_{\substack{\text{The } y \text{ position}\\ \text{of the object's}\\ \text{center of mass}\\ =0}} - \underbrace{\frac{4G\Omega}{c^3R^3}\int \rho_0(r)\,(\vec{R}\cdot\vec{r})\, y\, dV}_{= -\frac{2GSY}{c^3R^3}\ \text{(Box 36.2)}} + \cdots$$

$$\Rightarrow h^{tx}(\vec{R}) = h^{xt}(\vec{R}) = -\frac{2GSY}{c^3R^3}$$

$$h^{ty}(\vec{R}) = h^{yt}(\vec{R}) = +\frac{2GSX}{c^3R^3}$$

$$\vec{R} = (X, Y, Z)$$

since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and $h_{tx} = \eta_{t\alpha}\eta_{x\beta}h^{\alpha\beta}$

$$= \underset{-1}{\eta_{tt}}\ \underset{+1}{\eta_{xx}}\, h^{tx} = -h^{tx}$$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$= \left(\eta_{\mu\nu} + h_{\mu\nu}\right) dx^\mu dx^\nu$$

$$= \left(\eta_{tt} + h_{tt}\right)(c\,dt)^2 + \left(\eta_{ii} + h_{ii}\right)(dx^i)^2$$

$$+ \left(\eta_{tx} + h_{tx}\right) c\,dt\, dX + \left(\eta_{xt} + h_{xt}\right) c\,dt\, dX$$

$$+ \left(\eta_{ty} + h_{ty}\right) c\,dt\, dY + \left(\eta_{yt} + h_{yt}\right) c\,dt\, dY$$

$$= -\left(1 - \frac{2GM}{c^2 R}\right)(c\,dt)^2 + \left(1 + \frac{2GM}{c^2 R}\right)\left(dX^2 + dY^2 + dZ^2\right)$$

$$\underbrace{+2\left(0 + \frac{2GS}{c^3 R^3} Y\right) dX\, dt + 2\left(0 - \frac{2GS}{c^3 R^3} X\right) dY\, dt}_{= -4GS/c^3R^3\,(X\,dY - Y\,dX)\,dt}$$

Defining $a \equiv \dfrac{S}{Mc}$, the angular momentum per Mc

The metric in polar coordinates becomes

$$ds^2 = -\left(1 - \frac{2GM}{c^2 R}\right) c^2 dt^2 + \left(1 + \frac{2GM}{c^2 R}\right)\left(dR^2 + R^2 d\theta^2 + R^2 \sin^2\theta\, d\phi^2\right)$$

$$- \frac{4GMa}{c^2 R} \sin^2\theta\, d\phi\, (c\,dt)$$

off-diagonal term

- This is an approximate solution to the EE.

This metric is useful for calculations near rotating planets and ordinary stars, but does not apply to objects with strong fields such as black holes or neutron stars.

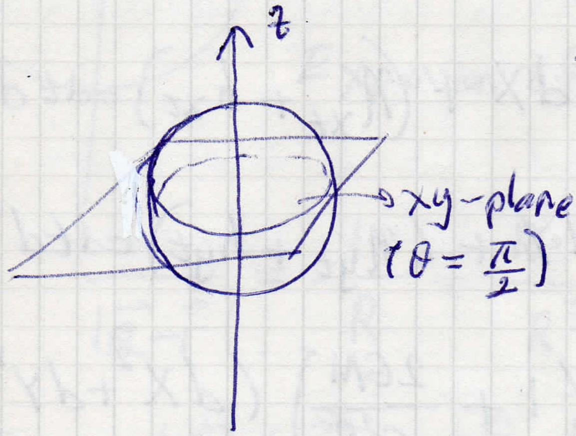
- For black holes and neutron stars we need an exact solution.

Introduction to the Kerr Solution:

We look for a vacuum solution with the following properties

1. As $r \to \infty$, the metric should reduce to that for flat space spherical coordinates
2. The metric should be time-independent
3. The metric should be axially symmetric

$\Rightarrow$ it will be independent of ϕ.



The metric should be of the form

$$ds^2 = g_{tt}\, c^2 dt^2 + g_{rr}\, dr^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}\, d\phi^2 + 2\, g_{t\phi}\, c\,dt\, d\phi$$

Define

$$\rho^2 = r^2 + a^2 \cos^2\theta$$

$$\Delta = r^2 - \underbrace{2\frac{GM}{c^2}}_{= r_s} r + a^2$$

$$ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right) c^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{r_s \cdot r a^2 \sin^2\theta}{\rho^2}\right) \sin^2\theta\, d\phi^2 - \frac{2 r_s \cdot r\, a \sin^2\theta}{\rho^2} d\phi\, c\,dt$$

Ch. 38 Ergoregion and Horizon

For a Schwarzschild Black Hole:

* The surface with radius $r_S = \frac{2GM}{c^2}$ is an infinite-redshift surface where $g_{tt} = 0$, which means that clocks at rest on that surface measure zero proper time relative to clocks at infinity

$$d\tau = \sqrt{1 - \frac{2GM}{c^2 r}}\, dt \rightarrow 0 \text{ at } r = r_S.$$

* The surface with $r = r_S$ is also an event horizon, that is, a surface beyond which no ingoing particle can return. For $r < r_S$, g_{tt} and g_{rr} switch signs. Decreasing r becomes the future.

For a Kerr Black Hole:

$$g_{tt} = -\left(1 - \frac{\frac{2GM}{c^2} \cdot r}{r^2 + a^2\cos^2\theta}\right) = -\left(\frac{r^2 + a^2\cos^2\theta - \frac{2GM}{c^2} \cdot r}{r^2 + a^2\cos^2\theta}\right)$$

$$g_{tt} = 0 \quad \Rightarrow \quad r_{1,2} = \frac{1}{2}\left(\frac{2GM}{c^2} \pm \sqrt{\left(\frac{2GM}{c^2}\right)^2 - 4a^2\cos^2\theta}\right)$$

$$r_{1,2} = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2\cos^2\theta}$$

Thus, the Kerr spacetime has two infinite redshift surfaces at r_1 and r_2. as long as $a < \frac{GM}{c^2}$.

$r_1 = r_e$: radius of the outer surface

$$r_e = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2\cos^2\theta}$$

At the poles ($\theta = 0$ and π) $r_e = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right) - a^2}$

At the equator ($\theta = \frac{\pi}{2}$) $r_e = \frac{2GM}{c^2}$

The Event Horizon:

see ds^2 for Kerr solution in Ch. 37.

The quantity Δ vanishes at the radii

$$\Delta = r^2 - \frac{2GM}{c^2} r + a^2$$

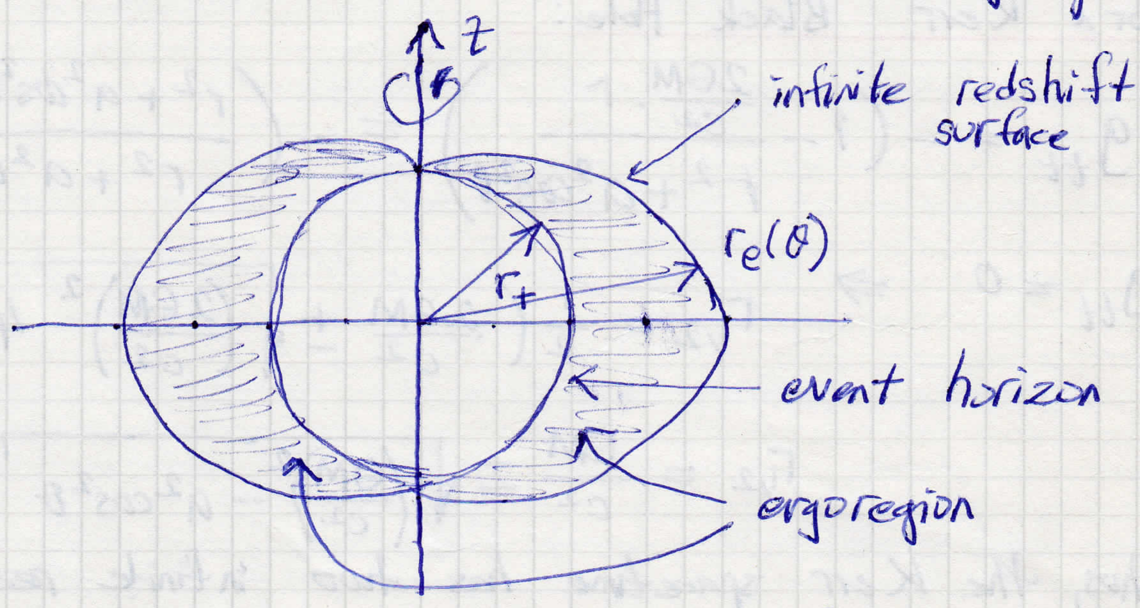
$$r_\pm = \frac{1}{2}\left(\frac{2GM}{c^2} \pm \sqrt{\left(\frac{2GM}{c^2}\right)^2 - 4a^2} \right)$$

$$r_\pm = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}, \quad a \leq \frac{GM}{c^2}$$

At $r = r_+ = \frac{GM}{c^2} + \sqrt{\left(\frac{GM}{c^2}\right)^2 - a^2}$ the massive rotating object becomes a black hole.

Black holes with the limiting value $a = \frac{GM}{c^2}$ are called extreme "Kerr black holes".

The region between the infinite redshift surface and the event horizon is called the "ergoregion".



(For this figure $a = \frac{GM}{c^2}$)

Putting $r = r_+$ in the Kerr line element and taking a $t = \text{const.}$ slice yields a 2 dimensional surface with the line element $(c\,dt = dr = 0)$

$$d\Sigma_+^2 = \rho_+^2 \, d\theta^2 + \left(\frac{\frac{2GM}{c^2} r_+}{\rho_+} \right)^2 \sin^2\theta \, d\phi^2 ,$$

where $\rho_+^2 \equiv r_+^2 + a^2 \cos^2\theta$

and $\left(r_+ - \frac{GM}{c^2}\right)^2 - \left[\left(\frac{GM}{c^2}\right)^2 - a^2\right] = 0$

$$r_+^2 - 2\frac{GM}{c^2} \cdot r_+ + \cancel{\left(\frac{GM}{c^2}\right)^2} - \cancel{\left(\frac{GM}{c^2}\right)^2} + \cancel{\left(\frac{GM}{c^2}\right)} \cdot a^2 = 0$$

$$r_+^2 - 2\left(\frac{GM}{c^2}\right) \cdot r_+ + \cancel{\left(\frac{GM}{c^2}\right)}^2 \cdot a^2 = 0$$

The surface $\overline{d\Sigma}$ has a constant coordinate value r_+ its intrinsic geometry is not spherically symmetric.

The area of the Event horizon is

$$A = 8\pi \left(\frac{GM}{c^2}\right) \cdot r_+$$

Cosmic Censorship:

If $a > \frac{GM}{c^2}$ there are no event horizons

The curvature of spacetime becomes infinite where

$r^2 + a^2\cos^2\theta = 0$. (true geometric singularity.)

If $a > \frac{GM}{c^2}$ there would be no event horizons surrounding this singularity. Such a singularity is called a naked singularity. Observer could visit this singularity and send information back.

The cosmic censorship hypothesis asserts that the gravitational collapse of a physically reasonable mass distribution can never produce such a naked singularity unclothed by an event horizon.

It is assumed that always $a < \frac{GM}{c^2}$ and the singularity is clothed by event horizons.