GENERAL RELATIVITY
FINAL EXAM

June 6, 2016

Name:

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1. Write down how the following objects transform under a coordinate transformation in four dimensional curved space.
   a) A_μ , b) B^μ , c) $\partial_\mu H$, where H is a scalar function
   d) $T^{\alpha\beta}{}_{\mu\nu}$, e) $A_\mu B^\mu$

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2. a) Write down the squared infinitesimal interval ds^2 describing the spacetime outside a static spherical distribution of mass M.

b) What is the infinitesimal radial distance along a radial line?

c) Calculate the radial distance between two events A and B with coordinates r_A and r_B along such a line in the limit $r_S/r \ll 1$, where $r_S \equiv 2GM/c^2$.

d) What is the proper time $\Delta\tau$ measured by a clock between two events at its location?

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3. Consider two light signals emitted at a radial coordinate r_E. The signals are received at the radial coordinate r_R.

a) What is the ratio $\frac{\Delta\tau_R}{\Delta\tau_E}$, where $\Delta\tau_R$ and $\Delta\tau_E$ are the proper times measured by clocks at rest at the radial coordinates r_R and r_E?

b) What is the ratio $\frac{f_E}{f_R}$ between the frequencies of the emitted and received light signals?

c) Calculate $\frac{f_E}{f_R}$ in the limit $\frac{r_S}{r_R} \ll 1$ and $\frac{r_S}{r_E} \ll 1$.

3d) Express $\frac{f_E}{f_R}$ in terms of the radial difference $h = r_R - r_E$ and the gravitational acceleration g measured at the emission radius r_E.

e) What is the effect used to observe this gravitational redshift?

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4. Consider a fictitious two dimensional line element given by

$$ds^2 = x^2 dx^2 + 2dxdy - dy^2$$

a) Write down g_{ij} in matrix form.

b) Obtain g^{ij}. (Note that ds^2 is not diagonal)

4c) Given $V_i = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $W^i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Find V^i and W_i.

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5. Two black holes have masses $M_1 = 4 M_\odot$ and $M_2 = 6 M_\odot$ where $M_\odot$ is the solar mass.

a) Find the ratio T_1/T_2 of the temperatures of the black holes.

b) A black hole which has a temperature of $T_{BH} = 2.0K$ at $t=0$ is in contact with the cosmic background radiation whose temperature is $2.7 K$. Describe what happens to the temperature of the black hole at a later time.

5c) A black hole which has a temperature of $T_{BH} = 3.3\,K$ at $t=0$ is in contact with the cosmic background radiation. Describe what happens to the temperature of the black hole at a later time.

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6. Assuming that the universe is homogeneous and isotropic its line element is given by the so called Robertson-Walker line element

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2\right],$$

where the scalar function $a(t)$ is called the scale factor.

a) Find the relation between the proper time τ and the coordinate time t.

b) Consider the 3 dimensional spatial metric

$$d\Sigma^2 = \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

What must be the value of the constant k if the 3 dimensional spatial space is flat?

7. The line element squared ds^2 for a two dimensional space is given by

$$ds^2 = \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 d\phi^2$$

a) What is the infinitesimal radial distance Δs ?

b) Is this space flat or curved? Why?

c) Embed this space in a three dimensional space described by cylindrical coordinates (r, ϕ, z).
(Do not calculate the integral. Only give the integral equation for $z(r)$)

d) Is the embedding space flat or curved?

8\. a) Write down the Einstein tensor $G_{\mu\nu}$ including the cosmological constant Λ.

b) Write down the Einstein equation with Λ in the presence of matter and energy.

c) Express the equation in a form which contains only $R_{\mu\nu}$ on the left side of the equation. Show all your steps. (Keep the c's)

d) How can the Λ-term on the right side of the equation be interpreted in terms of a stress-energy tensor?

9. The Kerr metric is given by

$$ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right)c^2 dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$+\left(r^2 + a^2 + \frac{r_s \cdot r a^2 \sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\phi^2 - \frac{2 r_s \cdot r a \sin^2\theta}{\rho^2} d\phi\, c\, dt$$

, where $r_s \equiv 2GM/c^2$, $\rho^2 \equiv r^2 + a^2\cos^2\theta$

$\Delta \equiv r^2 - r_s \cdot r + a^2$, $a \equiv \frac{J}{Mc}$, J = angular momentum

a) Express the metric in the equatorial plane with $\theta = \pi/2$. (use the open expressions for r_s, ρ^2, and Δ)

b) How many "infinite redshift surfaces" does this reduced metric have? What are the coordinate(s) r for these surfaces?

c) How many "event horizons" does this reduced metric have? Find the coordinates r of these horizons.

d) At what value of r does a rotating object become a black hole?

e) What is the value of r for "extreme Kerr black holes?

10. Assume that some creatures live in a world whose geometry is described by the metric

$$ds^2 = -\left(1 - \frac{r^2}{R^2}\right)c^2dt^2 + \left(1 - \frac{r^2}{R^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2,$$

where R = constant and $0 \leq r < \infty$. Suppose the creatures live near $r = 0$. If a creature were to cross the radius $r = R$,

a) could the creature return back? Explain why.

b) A creature observes that its clock registers zero time even though the clock is perfect. Describe where this creature is.

c) Is such a hypothetical world flat or curved?

11. a) Write down Maxwell's equations in Electromagnetism

b) Write down the corresponding equations in Gravitation

c) Write down the Lorentz force equation for a particle of mass m, velocity $\vec{v}$ moving in a superimposed gravitoelectric and gravitomagnetic fields.

d) What is Lense-Thirring Precession? Describe briefly.

12. For a binary star system with masses m_1 and m_2 seperated by a distance D rotating around their center of mass with angular frequency ω, the metric perturbation is given by

$$h_{TT}^{jk} = -\frac{4GM\eta D^2\omega^2}{c^4 R_0}\begin{bmatrix} \cos\left[2\omega\left(t-\frac{R_0}{c}\right)\right], & \sin\left[2\omega\left(t-\frac{R_0}{c}\right)\right], & 0 \\ \sin\left[2\omega\left(t-\frac{R_0}{c}\right)\right], & X, & 0 \\ 0, & 0, & 0 \end{bmatrix}$$

, where $M \equiv m_1 + m_2$, $\eta = \frac{m_1 m_2}{M^2}$, and R_0 is the distance in the z direction from the center of mass of the system.

a) What is X equal to, and why?

b) Write h_{TT}^{jk} as a sum of "plus" and "cross" polarization waves.

c) What is the angular frequency of these gravitational waves?

d) What are the amplitudes A_+ and $A_\otimes$?

e) What type of polarization do these waves have?

13. Explain what is "the cosmic censorship hypothesis" for a rotating black hole. (Hint: The Kerr metric parameters in Q.9 may be helpful.)

14. In the weak-field solution for a rotating spherical object the metric perturbations are given as

$$h^{tt} = h^{xx} = h^{yy} = h^{zz} = \frac{2GM}{c^2 R}$$

$$h^{tx} = -\frac{2GSY}{c^3 R^3}, \quad h^{ty} = \frac{2GSX}{c^3 R^3}, \quad h^{tz} = 0$$

where S is the spin angular momentum.

Obtain ds^2 for such a rotating object in quasi Cartesian coordinates (X, Y, Z) in the weak-field limit.

15. A binary star system with masses m_1 and m_2 rotate around their center of mass in the z direction. Let r_1 and r_2 be distances of the stars to the center of mass.

a) Show that

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) D \,, \quad r_2 = \left(\frac{m_1}{m_1+m_2}\right) D \,,$$

where $D = r_1 + r_2$.

b) Assuming that the stars move slowly enough their velocities are non-relativistic and that they are far enough apart so that Newtonian gravitational theory is adequate to predict their motion, apply Newton's second law to m_1 and show that

$$D^3 = \frac{GM}{\omega^2} \,,$$ where ω is the angular speed of the stars, and $M = m_1 + m_2$.

c) Assuming that Newtonian approximations are adequate, write down the total energy E of a binary star system

d) Use the values of r_1 and r_2 from a) and D^3 from b) and express E in terms of G, m_1, m_2, and D.

16. Consider a plane-wave metric perturbation having the form

$$H^{\mu\nu} = A^{\mu\nu} \sin(k_\sigma x^\sigma),$$

where $A^{\mu\nu}$ is a constant matrix and k_σ is a constant wave-number covector.

a) Show that the Einstein Equation $\Box^2 H^{\mu\nu} = 0$ for empty space implies that $k^2 = 0$.

b) Show that such a wave moves at the speed of light.

c) Find the condition on $A^{\mu\nu}$ such that $H = \eta_{\mu\nu} H^{\mu\nu} = 0$.

17. For gravitational waves moving in the $+x$ direction from the source, the metric perturbation is given by

$$H_{TT}^{jk} = \frac{2GM\Omega D^2\omega^2}{c^4 R_o}\begin{bmatrix} a & 0 & 0 \\ b & 0 & \sin\left[2\omega\left(t-\frac{R_o}{c}\right)\right] \\ d & \sin\left[2\omega\left(t-\frac{R_o}{c}\right)\right] & 0 \end{bmatrix}$$

a) Determine the values of the constants a, b, and d. Explain why?

b) What kind of polarization does this wave have?

c) Determine the amplitudes A_+ and $A_\otimes$.

d) Can this wave be said to be circularly polarized? Why?

18. The Einstein tensor $G_{\mu\nu}$ can be expanded in powers of the metric perturbation $h_{\mu\nu}$ and $H_{\mu\nu}$ (recall $H_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$) as

$$G_{\mu\nu} = G_{\mu\nu}^{(1)} + G_{\mu\nu}^{(2)} + \cdots$$

where $G_{\mu\nu}^{(1)}$ is evaluated to first order in $h_{\mu\nu}$ and $H_{\mu\nu}$ and $G_{\mu\nu}^{(2)}$ is evaluated to second order.

a) Write down the Einstein equation to first order in $h_{\mu\nu}$ and $H_{\mu\nu}$

b) Write down the Einstein equation to second order in $h_{\mu\nu}$ and $H_{\mu\nu}$.

c) Rewrite the equation in part b) by keeping only the $H_{\mu\nu}$ term on the left side of the second order equation

d) Reexpress the equation in part c) by defining a gravitational stres-energy tensor $T_{\mu\nu}^{GW}$

19. For a Kerr black hole, the metric on the event horizon's surface is

$$ds^2 = \rho_+^2 d\theta^2 + \left(\frac{r_s r_+}{\rho_+}\right)^2 \sin^2\theta \, d\phi^2 ,$$

where $\rho_+ \equiv r_+^2 + a^2\cos^2\theta$, $r_+ \equiv \frac{GM}{c^2} + \sqrt{\frac{GM}{c^2} - a^2}$

This surface can be embedded in a three dimensional flat space in cylindrical coordinates whose metric is

$$ds^2 = dR^2 + R^2 d\phi^2 + dz^2$$

a) What is $R(\theta)$ equal to?

b) What is $\frac{dR^2}{d\theta^2} + \frac{dz^2}{d\theta^2}$ equal to?

c) Find the expression for $\frac{dz}{d\theta}$.

20. a) Find the expression the Kerr metric in Q. 9 reduces to in the limit $M \to 0$.

b) Is the geometry around such a zero-mass object flat or curved? Why?

c) What is the stress-energy tensor $T_{\mu\nu}$ equal to for an object (massive and rotating) such that the solution of the Einstein equation is the Kerr metric in Q. 9?

GENERAL RELATIVITY
FINAL EXAM

02/01/2017

Name:

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1. In four dimensional curved spacetime write down how the following objects transform under a coordinate transformation $x^{\mu} \longrightarrow x^{\mu'}$:
a) A^{μ} b) B_{μ}, c) $\partial_{\mu}\Phi$, where Φ is a scalar function d) $T^{\mu\nu}{}_{\alpha\beta}$, e) $A_{\mu}B^{\mu}$

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2. In the hypothetical spacetime described by

$$ds^2 = -c^2dt^2 + \frac{dr^2}{1-\frac{R}{r}} + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2,$$

a covariant vector and a contravariant vector are given by

$$X_{\mu} = \begin{bmatrix} 1 \\ 0 \\ r^2 \\ \sin^2\theta \end{bmatrix}, \quad Y^{\mu} = \begin{bmatrix} 0 \\ 1-\frac{R}{r} \\ 1 \\ 1/\sin^2\theta \end{bmatrix}$$

a) Obtain X^{μ} and Y_{μ}.

b) Calculate $X_N Y^N$:

3. a) Consider the following equations

i) $p^{\mu} = m\, g^{\mu\nu} U_{\nu} U^{\alpha}$, ii) $\eta_{\mu\nu} p^{\mu} p^{\nu} + m^2 c^2 = 0$

iii) $\eta_{\alpha\beta} = \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\gamma}{}_{\beta}$, iv) $\frac{dp^{\alpha}}{d\tau} = q F^{\alpha\beta} U^{\beta}$

v) $\delta^{\mu}{}_{\nu} \delta^{\nu}{}_{x}$. Circle the equations that violate the rules for indices

b) Calculate the following

i) $g_{\alpha 3} g^{\alpha 3} =$

ii) $\delta^{\mu}{}_{1} \delta^{1}{}_{\mu} =$

iii) $\eta_{ij} \eta^{ij} =$

iv) $g^{\alpha\beta} \delta^{\mu}{}_{\beta} \delta^{\nu}{}_{\alpha} g_{\mu\nu} =$

v) $\delta^{\mu}{}_{\alpha} \delta^{\alpha}{}_{\beta} \delta^{\beta}{}_{\mu} =$

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4. a) Write down the Schwarzschild metric inside the event horizon.

b) What is the proper time interval $\Delta\tau$ for an object moving radially inward from $r = r_s = 2GM/c^2$ to $r = 0$?

5. The Reissner – Nördström metric describes the spacetime geometry outside a spherical object with mass M and electric charge Q. It is given by (in GR units):

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2 + \left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

a) Calculate the radii of the infinite red shift surfaces.

b) What are the radii of the event horizons?

c) What is the condition for a naked singularity?

d) For what value of Q the radius of the event horizon is equal to the radius of the Schwarzschild event horizon?

e) What does cosmic censorship require Q^2 to be?

6. a) Write down Maxwell's equations in Electromagnetism.

b) Write down the corresponding equations in Gravitation.

c) Write down the Lorentz force equation for a particle of mass m, velocity $\vec{v}$ moving in a superimposed gravitoelectric and gravitomagnetic fields.

d) What is the gravitoelectric field?

e) If a gyroscope is put in an equatorial orbit around the Earth, what effects is it subject to?

7. Suppose the geometry outside a spherical object with mass M is described by

$$ds^2 = -e^{-r_s/r} c^2 dt^2 + e^{+r_s/r} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2,$$

where $r_s \equiv \frac{2GM}{c^2}$.

a) What is the condition that this metric approximates the Schwarzschild metric?

b) Is there an event horizon? Explain.

c) Are there black holes according to this metric? Explain

d) If an object falling inward toward $r=0$ crosses $r=r_s$, can this object return back? Explain.

e) If an object at $r=r_s$ emits a light signal, is it possible for the signal to be infinitely red shifted?

8. Consider two black holes with masses $M_1 = 4M_\odot$ and $M_2 = 8M_\odot$, where $M_\odot$ is the solar mass.

a) What is the ratio $\frac{T_1}{T_2}$ of the temperatures of the black holes?

b) Suppose the black holes are surrounded by the cosmic background radiation whose temperature is 2.7K

Suppose at $t_0 = 0$ $T_1 = 2.8$ K and $T_2 = 1.4$ K.

Describe what happens to the temperatures of these black holes at a later time $t > t_0$.

c) Can a black hole be in thermal equilibrium with a heat reservoir?

9. a) In the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski metric and the perturbation $h_{\mu\nu}$ satisfies $|h_{\mu\nu}| \ll 1$. The inverse metric $g^{\mu\nu}$ can be written as $g^{\mu\nu} = \eta^{\mu\nu} + b^{\mu\nu}$, where $|b^{\mu\nu}| \ll 1$. Find $b^{\mu\nu}$ in terms of $h^{\mu\nu}$ to first order in the metric perturbation. Show your steps.

b) Consider a metric given by

$$ds^2 = g_{tt} c^2 dt^2 + g_{ii} dx^i dx^i$$

Find the value of g_{tt} such that the proper time and the coordinate time become identical.

10. a) Write down the stress-energy (or energy-momentum) tensor for a perfect fluid in ~~of~~ curved spacetime.

b) Write down the expression for conservation of energy and momentum.

c) Write down the expression for the stress-energy tensor for dust.

d) Write down the Einstein tensor $G^{\mu\nu}$.

e) What is $\nabla_\mu G^{\mu\nu} = ?$. Why?

Fall 2017-18 GENERAL RELATIVITY

FINAL EXAM

Instructor: Murat Özer 9/01/2018

Student's Name:

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1. a) Find the value of $g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$

b) What is the physical meaning of the above expression?

c) Evaluate the t-component of the four-velocity $\underset{\sim}{U}$ for an object at rest in the Schwarzschild spacetime.

d) The geodesic equation for any spacetime is

$$g_{\alpha\beta} \frac{d^2 x^\beta}{d\tau^2} = -\left(\partial_\nu g_{\alpha\mu} - \frac{1}{2} \partial_\alpha g_{\mu\nu}\right) U^\mu U^\nu$$

Obtain the geodesic equation for r in the Schwarzschild spacetime.

2. Consider a hypothetical spherical object with a negative mass $-M$.

a) Write down the Schwarzschild line element (metric) outside this object when there is no matter outside it.

b) Is there an infinite redshift surface for this object? Explain why.

c) Is there an event horizon for this object. Explain.

d) What is the value of r for which there is a singularity?

e) What kind of singularity is this?

3. a) Consider the following equations

(i) $P_\mu - P^\mu = 0$ (ii) $P_\alpha \eta^{\alpha\beta} P_\beta + m^2c^2 = 0$

(iii) $\eta_{\mu\gamma} = \eta_{\mu\alpha} \Lambda^\alpha{}_\beta \Lambda^\beta{}_\gamma$ (iv) $g^{\alpha\beta} A_\beta - g^{\alpha\nu} B_\gamma = 0$

(v) $F^\mu{}_\gamma = \partial^\mu A_\gamma - \partial^\gamma A_\mu$

Circle the equations that violate the rules for indices.

b) In four dimensional curved spacetime write down how the following objects transform under a coordinate transformation.

(i) $\Phi(x^\alpha)$, where Φ is a scalar function

(ii) $A^\alpha B_\alpha$, (iii) $A^\alpha{}_\beta$, (iv) $\partial^\alpha \Phi$

(v) $T^{\alpha\beta}{}_\mu$

4. The infinitesimal spacetime distance $d\underset{\sim}{s}$ between two neighboring points can be written as (in spherical polar coordinates)

$$d\underset{\sim}{s} = c\,dt\,\underset{\sim}{e}_t + dr\,\underset{\sim}{e}_r + d\theta\,\underset{\sim}{e}_\theta + d\phi\,e_\phi ,$$

where the $\underset{\sim}{e}_\mu$'s are the coordinate basis vectors.

a) What are the basis vectors (in column vector form) for the Minkowski metric?

b) What are the dual basis vectors $\underset{\sim}{e}^\mu$'s for the Minkowski metric? (Hint: $g_{\mu\nu} = \underset{\sim}{e}_\mu \cdot \underset{\sim}{e}_\nu = (\vec{e}_\mu)_\alpha (\vec{e}_\nu)^\alpha = (\vec{e}_\mu)_\alpha \eta^{\alpha\beta} (\vec{e}_\nu)_\beta$)

c) Write down the expression for $ds^2 = d\underset{\sim}{s} \cdot d\underset{\sim}{s}$ in spherical coordinates in terms of the basis vectors. (Hint: In spherical polar coordinates $\underset{\sim}{e}_\mu \cdot \underset{\sim}{e}_\nu = 0$ if $\mu \neq \nu$)

d) What are the basis vectors $\underset{\sim}{e}_\mu$ for the Schwarzschild metric?

e) What are the dual basis vectors $\underset{\sim}{e}^\mu$ for the Schwarzschild metric?

5. The electromagnetic field tensor $F^{\mu\nu}$ is defined as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu ,$$

where A^μ is the electromagnetic four potential.

a) Show that $F^{\mu\nu}$ is antisymmetric.

b) Find the value of $\partial_\mu \partial_\nu F^{\mu\nu}$

c) Find the value of $\partial^\alpha F^{\mu\nu} + \partial^\nu F^{\alpha\mu} + \partial^\mu F^{\nu\alpha}$

6, The Kerr-Newman metric for a rotating massive and charged object is given by

$$ds^2 = -\left(1 - \frac{r_s r}{\rho^2}\right)c^2 dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$+\left(r^2 + a^2 + \frac{r_s r a^2 \sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\phi^2 - \frac{2 r_s r a \sin^2\theta}{\rho^2} c\, dt\, d\phi,$$

where $r_s \equiv 2GM/c^2$, $\rho^2 \equiv r^2 + a^2\cos^2\theta$

$a \equiv \frac{J}{Mc}$, $\Delta \equiv r^2 - r_s r + a^2 + r_Q^2$

$r_Q^2 = k_e Q^2/c^4$, J = angular momentum.

a) From where a light signal should be emitted so that the received signal at r_R is infinitely red shifted?

b) Are there event horizons? If yes, find their coordinates.

c) What is the value of r for "extreme Kerr-Newman" black holes?

7. a) Find the conditions for which the Kerr-Newman metric reduces to the Minkowski metric for finite r.

b) Show the "ergoregion" for the Kerr-Newman metric on a figure.

c) What is the condition so that there are no event horizons?

8. a) What is a naked singularity? Explain using the Kerr-Newman metric.

b) What is the cosmic sensorship hypothesis?

9. a) Write down Maxwell's equations in Electromagnetism.

b) What is the "gravito electric field" in Gravitation?

c) Give a reason why there must be a "gravito magnetic field" in Gravitation.

d) Write down Maxwell's corresponding equations in Gravitation.

10. a) Write down the Einstein Equation with a cosmological constant Λ in the presence of matter and energy. Identify each term in the equation.

b) Is it possible to add a term like $g_{\mu\nu}\Phi(x^\alpha)$ to the energy-momentum tensor, where $\partial_\alpha \Phi \neq 0$? Explain.

c) Find the expression for the curvature scalar R in terms of Λ and T, where T is the trace of $T_{\mu\nu}$.

GENERAL RELATIVITY
FINAL EXAM

07/01/2019

Name:

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1. For weak fields the Schwarzschild metric can be approximated by the following

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 + \frac{r_s}{r}\right)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2,$$

where $r_s \equiv \dfrac{2GM}{c^2}$.

a) Write down the metric for $r < r_s$.

b) Is the metric for $r < r_s$ a spacetime metric? Why?

c) Does the source of this metric become a black hole if it collapses to a size $r < r_s$. Explain why?

d) If an object passes to the region $0 < r < r_s$, can it return back to the region $r_s < r < \infty$? Explain why?

2. Two blackholes have temperatures $T_1(0) = 2.5\,K$ and $T_2(0) = 3.4\,K$ at $t_0 = 0$.

The blackholes are in contact with the cosmic background radiation whose temperature is $T_R = 2.7\,K$.

a) At a later time $t_1 > 0$, the blackholes are observed to have temperatures $3.6\,K$ and $2.3\,K$.

(i) The temperature of the first blackhole at t_1 is

$$T_1(1) = .$$

The temperature of the second blackhole at t_1 is

$$T_2(1) =$$

b) What is the ratio $\dfrac{M_1(1)}{M_2(1)}$ of the blackholes at t_1 equal to ?

c) Say True or False. At a time $t_N > 0$, the temperatures of the blackholes may be equal to each other.

3. In an alternative theory, the metric outside a spherical object with mass M and electric charge $Q>0$ is given by

$$ds^2 = -\left(1 - 2\frac{GM}{c^2 r} + 2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right)c^2 dt^2 + \left(1 - 2\frac{GM}{c^2 r} + 2\frac{e}{m_e}\frac{k_e Q}{c^2 r}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 ,$$

where $q=+e$, and m_e are the electric charge and the mass of a positron. which is moving in the viscinity of the spherical object.

a) What is the condition on M and Q which makes the space time flat?

b) Find the "radius" r_{EH} of the "event horizon" for this system.

c) What is the condition on the radius r_0 of the spherical object so that this system may be a black hole?

d) From part b), what is the condition on M so that there is no event horizon?

4. a) Write down the Einstein field equation in the presence of external matter, energy and cosmological constant.

b) For a massive but neutral object, write down the Einstein field equation which has only vacuum outside it (i.e. no mass, no energy, and no cosmological constant)

c) Write down the metric for ds^2 for a spherical object as in part b).

d) If a test particle moves on a geodesic of the space-time described by the metric in part c), what is the condition on the electric charge of this test particle?

e) Write down the geodesic equation for this test charge.

5. Consider a hypothetical two-dimensional spacetime metric given by

$$ds^2 = -\left(1-\frac{R}{r}\right)c^2 dt^2 + 2c\,dt\,dr + \left(1-\frac{R}{r}\right)^{-1} dr^2$$

a) Write down the matrix for the metric g_{ij}; $(i=0,1)$

b) Find the matrix for g^{ij}.

c) Given the contravariant vector $X^i = \begin{bmatrix} 1 \\ r \end{bmatrix}$. Obtain the covariant vector X_i.

6. a) Given an anti-symmetric tensor $F^{\mu\nu}$. Calculate the trace of $F^{\mu\nu}$ given by $F = F^{\mu}{}_{\mu}$

b) Calculate $\partial_\mu \partial_\nu F^{\mu\nu}$.

c) The following equations are given in flat spacetime

(i) $P_\alpha P^\alpha + m^2 c^2 = 0$, (ii) $\frac{dp^\alpha}{dt} = q F^{\alpha\beta} \frac{dx_\beta}{dt}$

(iii) $\eta_{\mu\nu} \eta^{\mu\nu} = 4$

Write down the corresponding equations in a curved spacetime.

7. In a hypothetical multi-dimensional spacetime the following relation is given:

$$g_{\alpha\beta}\, g^{\alpha\beta} = 10$$

a) What are the numerical values that α and β can take?

b) What is the dimension of this spacetime?

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8. Write down how the following objects transform under a coordinate transformation $x^{\alpha} \longrightarrow x^{\alpha'}$ in four dimensional curved spacetime.

a) A^{μ}, b) B_{μ}, c) $\partial_{\mu} C$, where C is a scalar function.

d) $G^{\mu\nu}$, e) $H^{\mu\nu}{}_{\alpha\beta}$

9. The line element squared ds^2 for a two dimensional space is given by

$$ds^2 = \left(1 + \frac{R}{r}\right) dr^2 + r^2 d\phi^2 ,$$

where R is a constant.

a) what is the infinitesimal angular distance ds_ϕ ?

b) what is the total angular distance Δs_ϕ ?

c) Embed this space in a three dimensional space described by cylindrical coordinates (r, ϕ, z). Calculate the integral and find $z(r)$.

d) Is the embedding space flat or curved ?

10. From the roof of a building of height h two signals are sent down to the ground. Let the frequencies of the emitted and received signals be f_E and f_R. Find the ratio $\frac{f_R}{f_E}$ from the Schwarzschild metric.

b) Find $\frac{f_R - f_E}{f_E}$. What kind of a shift is this?

11, The four-velocity of a particle in the Schwarzschild spacetime is given by

$$\underset{\sim}{U} = \begin{bmatrix} c\,dt/d\tau \\ dr/d\tau \\ d\theta/d\tau \\ d\phi/d\tau \end{bmatrix}$$

Consider the vectors $\underset{\sim}{\xi} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\underset{\sim}{\gamma} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Calculate the quantities $e = -\underset{\sim}{\xi} \cdot \underset{\sim}{U}$ and $k = \underset{\sim}{\gamma} \cdot \underset{\sim}{U}$ in the Schwarzschild spacetime.

12. a) Write down the Einstein tensor $G_{\mu\nu}$ including the cosmological constant Λ. (include the Λ-term in $G_{\mu\nu}$)

b) Write down the Einstein Field Equation in the presence of (external) matter and energy.

c) Express the equation in a form which contains only $R_{\mu\nu}$ on the left side of the equation. Show your work.

d) Let the stress-energy tensor for the vacuum $T_{\mu\nu}^{(vac)}$ be proportional to Λ. Write the expression for $T_{\mu\nu}^{(vac)}$.

13. The Yılmaz metric for a spherical object of mass M is given by

$$ds^2 = -\exp(-r_s/r)c^2dt^2 + \exp(r_s/r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2,$$

where $r_s \equiv 2GM/c^2$.

a) Is there an event horizon for this metric? Why?

b) Are there blackholes predicted by this metric? Why?

c) If the spherical object has also a net electric charge Q (as in Problem 3). What would r_s be when there is an electron moving in the viscinity of the spherical object?

14. a) Given the tensor $R_{\mu\nu}$. Raise the index to find $R^{\mu}{}_{\nu}$:

b) Given the tensor $T^{\mu\nu}$. Lower the index to find $T_{\mu}{}^{\nu}$.

c) Say True or False: $A^{\mu\nu}{}_{\alpha\beta} + B^{\alpha\beta}{}_{\mu\nu} = C^{\mu\nu}{}_{\alpha\beta}$:

d) Say True or False: $A^{\mu\nu} B_{\alpha\beta} C^{\gamma}_{\rho} = D^{\mu\nu\gamma}{}_{\alpha\beta\rho}$

e) Say True or False:

$A_{\mu} B^{\mu} + C_{\alpha\beta} D^{\alpha\beta} = F$, where F is a scalar.

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15. Given the equation $\frac{dx^{\mu}}{d\tau} = \frac{q}{m} F^{\mu\nu} \frac{dx_{\nu}}{d\tau}$

a) Write the equation in terms of $\frac{dx^{\nu}}{d\tau}$ in flat spacetime.

b) Write the equation in terms of $\frac{dx^{\nu}}{d\tau}$ in a curved spacetime.

c) Consider a spacetime whose metric is $ds^2 = -e^{-\frac{x}{a}}(c\,dt)^2 + dx^2$ where a is a constant.

(i) What is the dimension of this spacetime?

(ii) Is this spacetime flat or curved?