GENERAL RELATIVITY
EXAM 2

Name: Take Home Exam 31 December, 2016

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1. An object falls radially inward toward a black hole with mass M, starting at rest at infinity. How much time will a clock on the object register between the times the object passes the Schwarzschild radial coordinate $r = 10\,\frac{GM}{c^2}$ and $r = 2\,\frac{GM}{c^2}$?

Hint: The quantity $\tilde{E}$ given by

$$\tilde{E} = \frac{1}{2c^2}\left(\frac{dr}{d\tau}\right)^2 - \frac{GM}{c^2 r} + \frac{\ell^2}{2r^2c^2} - \frac{GM\ell^2}{c^4 r^3}$$

is conserved. Here $\ell \equiv r^2(d\phi/d\tau)$.

2. The metric for a 2-dimensional space is given by

$$ds^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\phi^2 \ ; \quad R = \text{const.}$$

a) Is this space curved or flat ? Why

b) Embed a surface with this metric in a three dimensional Euclidean space described by cylindrical coordinates r, ϕ, and z. (Find $z(r)$.)

3. a) In the Schwarzschild space, what is the infinitesimal (or differential) distance ds corresponding to a purely radial infinitesimal (differential) displacement dr? (Keep c^2; do not put $c = 1$!)

b) What is the ordinary distance between points with coordinates r_2 and r_1; $(r_2 > r_1)$?

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4. a) Write down the Schwarzschild metric inside the event horizon.

b) What is the proper time $\Delta\tau$ for an object moving radially inward from $r = r_S = 2GM/c^2$ to $r = 0$?

5. The Schwarzschild metric in Kruskal - Szekeres coordinates is given by $(c = 1)$

$$ds^2 = -\frac{32(GM)^3}{r} e^{-r/2GM} (dv^2 - du^2) + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

a) What are the timelike and spacelike coordinates?

b) Find the equation for radial photon worldlines.

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6. a) Write down the expression for the absolute gradient (covariant derivative) of a 4-vector A^μ.

b) Starting from $\nabla_\alpha (A^\mu B_\mu)$ obtain the absolute gradient of B_μ.

7 a) For a second rank tensor $A_{\mu\nu}$ the covariant derivative is defined as $\nabla_\alpha A_{\mu\nu} = \partial_\alpha A_{\mu\nu} - \Gamma^\beta_{\alpha\mu} A_{\beta\nu} - \Gamma^\beta_{\alpha\nu} A_{\mu\beta}$.

Using $g_{\mu\nu} = \underset{\sim}{e}_\mu \cdot \underset{\sim}{e}_\nu$, where $\underset{\sim}{e}_\mu$ are the basis vectors

calculate the covariant derivative $\nabla_\alpha g_{\mu\nu}$.

(Hint: $\partial_\alpha \underset{\sim}{e}_\mu \equiv \Gamma^\beta_{\mu\alpha} \underset{\sim}{e}_\beta$)

b) An object's 4-velocity $\underset{\sim}{u} = u^\mu \underset{\sim}{e}_\mu$ satisfies $\frac{d\underset{\sim}{u}}{d\tau} = 0$. Evaluate this and obtain the geodesic equation for a freely falling object.

8. Suppose that the geometry outside a spherical object with mass M is described by

$$ds^2 = -e^{-r_s/r}c^2dt^2 + e^{+r_s/r}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\, d\phi^2,$$

where $r_s \equiv \dfrac{2GM}{c^2}$.

a) What is the condition that this metric approximates the Schwarzschild metric?

b) Is there an event horizon? Explain your answer.

c) Are there black holes according to this metric? Explain.

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d) If an object falling inward toward $r=0$ crosses $r=r_s$, can this object return back? Explain.

e) If an object at $r=r_s$ emits a light signal, is it possible for the signal to be infinitely red shifted?

9. Since the metric in question 8 does not depend on time, the time component of the geodesic equation reads

$$0 = \frac{d}{d\tau}\left(g_{t\nu} \frac{dx^\nu}{d\tau} \right)$$

Compare this equation to the geodesic equation with the Christoffel symbols and obtain Γ^t_{rt} and Γ^t_{tr} for that metric.

10. a) Maxwell's equations in flat spacetime are

i) $\partial_\alpha F^{\beta\alpha} = 4\pi k J^\beta$

ii) $\partial^\mu F^{\alpha\beta} + \partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\beta\mu} = 0$

Write down these equations in curved space time.

b) Write down the Einstein Equation in the presence of matter, energy. (Keep the c's.), and the cosmological constant.

c) Express the Einstein Equation in a form which contains only $R_{\mu\nu}$ on the left side of the equation. Show your steps.

GENERAL RELATIVITY
MIDTERM EXAM 2

May 14, 2016

Name: ______________________________

Number: ____________________________

ANSWER ALL THE QUESTIONS

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1. a) Two black holes have masses $M_1 = 3M_\odot$ and $M_2 = 5M_\odot$, where $M_\odot$ is the solar mass. What is the ratio $\frac{T_1}{T_2}$ of the temperatures of the black holes?

b) A black hole which has a temperature of $T_{BH} = 2.5\,K$ at $t=0$ is in contact with the cosmic background radiation whose temperature is $2.7\,K$. Describe what happens to the temperature of the black hole at a later time.

c) A black hole which has a temperature of $T_{BH} = 4.1\,K$ at $t=0$ is in contact with the cosmic background radiation. Describe what happens to the temperature of the black hole at a later time.

2. a) A freely-falling frame is at rest in the Schwarzschild coordinate system. Find the expression for U^t in Schwarzschild coordinates. (Hint: Recall the value of $\underset{\sim}{U} \cdot \underset{\sim}{U}$ in any coordinate system. $\underset{\sim}{U}$ is the four-momentum)

b) The energy E of a particle is given by $E = -\underset{\sim}{U} \cdot \underset{\sim}{p}$, where $p^{\mu} \equiv mc\,(dx^{\mu}/d\tau)$ is the 4-momentum of the particle. Find the value of E.

3. a) Find the value of the absolute gradient of the metric $g_{\mu\nu}$; $\nabla_{\alpha}\, g_{\mu\nu} = ?$. Explain your reasoning clearly.

b) Write down the expressions for the absolute gradient (covariant derivative) of a 4-vector A^μ and a covector B_μ.

c) Calculate $\nabla_\alpha (A^\mu B_\mu)$.

4. a) An object's 4-velocity $\underset{\sim}{u} = u^\mu \underset{\sim}{e}_\mu$ satisfies $\frac{d\underset{\sim}{u}}{d\tau} = 0$. Evaluate this and obtain the geodesic equation for a freely falling object. (Hint: Recall that $\frac{\partial \underset{\sim}{e}_\alpha}{\partial x^\mu} \equiv \Gamma^\nu_{\alpha\mu} \underset{\sim}{e}_\nu$)

4b) Since the Schwarzschild metric does not depend on time, the time component of the geodesic equation reads

$$0 = \frac{d}{d\tau}\left(g_{t\nu}\frac{dx^{\nu}}{d\tau}\right)$$

Compare this equation to the geodesic equation with the Christoffel symbols and obtain Γ^{t}_{rt} for the Schwarzschild metric.

5. The square of the infinitesimal interval for a unit 2-sphere is given by

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

The $\theta\phi\theta\phi$ component of the Riemann tensor is found to be $R_{\theta\phi\theta\phi} = \sin^2\theta$

a) Calculate all the components of the Ricci tensor $R_{\beta\nu}$.

b) Calculate the curvature scalar R.

6. a) Write down the stress-energy tensor for a perfect fluid of mass density ρ_0 and pressure p_0 in a "locally inertial frame" whose 4-velocity is U^μ. (Keep the speed of light as c)

6 b) Generalize the above expression to a frame in a curved spacetime.

c) Write down the expression expressing the conservation of the stress-energy tensor in a curved spacetime.

7. a) Write down the Einstein tensor $G_{\mu\nu}$ including the Cosmological Constant Λ.

b) Write down the Einstein Equation in the presence of matter and energy.

c) Express the equation in a form which contains only $R_{\mu\nu}$ on the left side of the equation. Show your steps.

7. d) Let the stress-energy tensor for the vacuum $T_{\mu\nu}^{vac}$ be proportional to Λ. Write the exact expression for $T_{\mu\nu}^{vac}$.

8. In the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $\eta_{\mu\nu}$ is the Minkowski tensor and the perturbation $h_{\mu\nu}$ satisfies $|h_{\mu\nu}| \ll 1$. The inverse metric $g^{\mu\nu}$ can be written as $g^{\mu\nu} = \eta^{\mu\nu} + b^{\mu\nu}$, where $|b^{\mu\nu}| \ll 1$. Find $b^{\mu\nu}$ in terms of $h^{\mu\nu}$ to first order in the metric perturbation. Show your steps.

9. The metric in the empty space outside a static infinite planar slab of mass M might be

$$ds^2 = -c^2dt^2 + f(x)dx^2 + dy^2 + dz^2$$

The only nonzero Christoffel symbol for this metric is

$$\Gamma^x_{xx} = \frac{1}{2f}\frac{df}{dx}$$

a) Calculate the Riemann tensor $R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\nu\gamma}\Gamma^\gamma_{\beta\mu}$

b) Is this spacetime flat or curved? State your reasoning.

c) Could this metric describe a real gravitational field in spacetime? State your reasoning.

10. a) Write down the weak-field Einstein Equation in the presence of matter and energy outside the source.

b) Write down a plane-wave solution in empty space in terms of a constant matrix $A^{\mu\nu}$ and the constant covector $k_\sigma = [-\frac{\omega}{c}, \vec{k}]$.

c) Find the relation between k_μ and $A^{\mu\nu}$ implied by the Lorentz gauge condition on $H^{\mu\nu}$.

d) Show that such a wave propagates at the speed of light c. (Hint: $|\vec{k}| = \frac{\omega}{v}$)

e) What are the transverse-traceless gauge conditions on $A^{\mu\nu}$?

GENERAL RELATIVITY
EXAM 2

Instructor: Murat Özer 02/01/2018

Student's Name: ____________________

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1. Suppose that we have a three dimensional spacetime with an interval

$$ds^2 = -c^2dt^2 + dz^2 + R(t)^2 d\phi^2,$$

where $R(t)$ is some increasing function of time.

a) Write down $g_{\mu\nu}$ and $g^{\mu\nu}$ in matrix form

$$[g_{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(t)^2 \end{bmatrix}, \quad [g^{\mu\nu}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{R(t)^2} \end{bmatrix}$$

b) The Christoffel symbols are defined as

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\mu\beta} - \partial_\beta g_{\mu\nu}\right)$$

Find the non-zero Christoffel symbols

Noting that the derivatives can only be with respect to t:

$\mu = t: \quad g_{\nu\beta} = g_{\phi\phi} = R^2; \quad g^{\alpha\beta} = g^{\phi\phi} \Rightarrow \Gamma^{\phi}_{t\phi} = \frac{1}{2}\frac{1}{R^2}\partial_t R^2 = \frac{\dot{R}}{R}$

$\cancel{\mu} = \cancel{\nu}$ ($\nu = t$)$: \quad g_{\mu\beta} = g_{\phi\phi} = R^2; \quad g^{\alpha\beta} = g^{\phi\phi} \Rightarrow \Gamma^{\phi}_{\phi t} = \frac{1}{2}\frac{1}{R^2}\partial_t R^2 = \frac{\dot{R}}{R}$

$\beta = t: \quad g_{\mu\nu} = g_{\phi\phi} = R^2; \quad g^{\alpha\beta} = g^{tt} \Rightarrow \Gamma^{t}_{\phi\phi} = \frac{1}{2}\frac{1}{R^2}\partial_t R^2 = \frac{\dot{R}}{R}$

c) Calculate the non-zero components of the Riemann tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\beta\nu} - \partial_\nu \Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\nu\gamma}\Gamma^{\gamma}_{\beta\mu} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\gamma}_{\beta\nu},$$

where $\alpha, \beta, \mu, \nu, \gamma = 0, 1, 2$ (or t, z, ϕ)

$\partial_\mu \Gamma^{\alpha}_{\beta\nu}$: $\mu = t,\ \alpha = \phi,\ \beta = t,\ \nu = \phi$

$$R^{\phi}{}_{tt\phi} = \partial_t(\Gamma^{\phi}_{t\phi}) - \cancelto{0}{\partial_\phi(\ )} + \Gamma^{\phi}_{\phi t}\cancelto{0}{\Gamma^{t}_{tt}} + \Gamma^{\phi}_{t\phi}\Gamma^{\phi}_{t\phi}$$

$$= \frac{\ddot{R}R - \dot{R}\dot{R}}{R^2} + \left(\frac{\dot{R}}{R}\right)^2 = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} + \frac{\dot{R}^2}{R^2} = \frac{\ddot{R}}{R}$$

$\mu = t,\ \alpha = t,\ \beta = \nu = \phi$

$$R^{t}{}_{\phi t\phi} = \partial_t(\Gamma^{t}_{\phi\phi}) - \cancelto{0}{\partial_\phi(\ )} + \boxed{\Gamma^{t}_{\phi\phi}\Gamma^{\phi}_{\phi t}} \rightarrow = \frac{\dot{R}^2}{R^2}$$

$$= \frac{\ddot{R}}{R}$$

$\partial_\nu \Gamma^{\alpha}_{\beta\mu}$: $\nu = t,\ \alpha = \phi,\ \beta = t,\ \mu = \phi$

$$R^{\phi}{}_{t\phi t} = \cancelto{0}{\partial_\phi(\ )} - \partial_t(\Gamma^{\phi}_{t\phi}) + \Gamma^{\phi}_{t\phi}\Gamma^{\phi}_{t\phi} + \Gamma^{\phi}_{\phi t}\cancelto{0}{\Gamma^{t}_{tt}}$$

$$=$$

d) What is the condition on $R(t)$ so that the spacetime is flat?

For the spacetime to be flat the Riemann tensor must vanish. This means that $\dot{R} = 0 \Rightarrow R = \text{constant}$.

2) The Schwarzschild solution in the presence of a cosmological constant Λ is given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2$$

$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 .$$

a) Is this metric asymptotically flat? Explain why.

Since $\lim_{r \to \infty} \left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right) \to \infty$, this metric is not asymptotically flat.

b) Find the equation satisfied by the infinite redshift surface.

$$g_{tt} = -\left(1 - \frac{r_s}{r} - \frac{\Lambda r^2}{3}\right) = 0$$

$$\Rightarrow 3r - 3r_s - \Lambda r^3 = 0$$

c) Is there an event horizon? Explain why.

There is no event horizon because $\left(1 - \frac{2GM}{c^2 r} - \frac{\Lambda r^2}{3}\right)$ does not change sign for all values of r.

3. a) Write down the line element (the metric) in the empty space around a massive spherical object in the weak-field limit.

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 + \frac{2GM}{c^2 r}\right)\left(dx^2 + dy^2 + dz^2\right)$$

b) In the weak-field limit $g_{\mu\nu}$ can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Write down the matrix for $h_{\mu\nu}$

$$[h_{\mu\nu}] = \begin{bmatrix} r_s/r & 0 & 0 & 0 \\ 0 & r_s/r & 0 & 0 \\ 0 & 0 & r_s/r & 0 \\ 0 & 0 & 0 & r_s/r \end{bmatrix}, \text{ where } r_s \equiv \frac{2GM}{c^2}$$

c) $g^{\mu\nu}$ can be written as $g^{\mu\nu} = \eta^{\mu\nu} + b^{\mu\nu}$. Find $b^{\mu\nu}$

$$g_{\mu\nu} g^{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\mu\nu} + b^{\mu\nu}) = 4$$

$$= 4 + \eta_{\mu\nu} b^{\mu\nu} + h_{\mu\nu}\eta^{\mu\nu} + \underbrace{h_{\mu\nu} b^{\mu\nu}}_{\text{negligible}} = 4$$

$$\Rightarrow \eta_{\mu\nu} b^{\mu\nu} + h^{\mu\nu}\eta_{\mu\nu} = 0$$

$$\Rightarrow b^{\mu\nu} = -h^{\mu\nu}$$

d) Is there an event horizon in the weak-field limit? Explain why.

There is no event horizon because all the space coordinates remain as space coordinates for all values of r.

4) a) Write down the expression for the absolute gradient (covariant derivative) of a covector A_μ.

$$\nabla_\alpha A_\mu = \partial_\alpha A_\mu - \Gamma^\nu_{\alpha\mu} A_\nu$$

b) Write down the expression for the absolute gradient of a scalar $\Phi(x^\mu)$.

$$\nabla_\alpha \Phi = \partial_\alpha \Phi$$

c) Starting from $\nabla_{\alpha}(A^\mu B_\mu)$ obtain the absolute gradient of A^μ.

$$\nabla_\alpha (A^\mu B_\mu) = \partial_\alpha (A^\mu B_\mu)$$
$$= (\partial_\alpha A^\mu) B_\mu + A^\mu (\partial_\alpha B_\mu)$$

$$(\nabla_\alpha A^\mu) B_\mu + A^\mu (\nabla_\alpha B_\mu) =$$
$$(\nabla_\alpha A^\mu) B_\mu + \underbrace{A^\mu [\cancel{\partial_\alpha B_\mu} - \Gamma^\nu_{\alpha\mu} B_\nu]}_{\text{rename } \mu \to \nu \to \mu} = (\partial_\alpha A^\mu) B_\mu + \cancel{A^\mu (\partial_\alpha B_\mu)}$$

$$(\nabla_\alpha A^\mu) B_\mu = (\partial_\alpha A^\mu) B_\mu + (\Gamma^\mu_{\alpha\nu} A^\nu) B_\mu$$

$$\Rightarrow \nabla_\alpha A^\mu = \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\nu} A^\nu$$

d) Write down the expression for the double absolute gradient $\nabla_\mu \nabla_\nu \Phi$ of a scalar function $\Phi(x^\alpha)$.

$$\nabla_\mu \nabla_\nu \Phi = \partial_\mu \partial_\nu \Phi$$

5. Consider an object moving freely in a gravitational field.

a) Write down the equation satisfied by the object's 4-velocity $\underset{\sim}{U} = U^{\alpha} \underset{\sim}{e}_{\alpha}$, where $\underset{\sim}{e}_{\alpha}$ are the coordinate basis vectors.

$$\frac{d\underset{\sim}{U}}{d\tau} = \frac{d(U^{\alpha}\underset{\sim}{e}_{\alpha})}{d\tau} = 0$$

b) Evaluate the expression in a) and obtain the geodesic equation for an object moving freely in a gravitational field. (Hint: $\partial_{\mu} \underset{\sim}{e}_{\alpha} \equiv \Gamma^{\beta}_{\mu\alpha} \underset{\sim}{e}_{\beta}$)

$$\frac{d}{d\tau}(U^{\alpha}\underset{\sim}{e}_{\alpha}) = \frac{dU^{\alpha}}{d\tau} \cdot \underset{\sim}{e}_{\alpha} + U^{\alpha}\frac{d\underset{\sim}{e}_{\alpha}}{d\tau}$$

$$= \frac{dU^{\alpha}}{d\tau}\underset{\sim}{e}_{\alpha} + U^{\alpha}\frac{\partial \underset{\sim}{e}_{\alpha}}{\partial x^{\mu}}\frac{\partial x^{\mu}}{\partial \tau} = U^{\alpha}\Gamma$$

$$\frac{dU^{\alpha}}{d\tau}\underset{\sim}{e}_{\alpha} + \underbrace{U^{\alpha}\Gamma^{\beta}_{\mu\alpha} e_{\beta}\frac{dx^{\mu}}{d\tau}}_{\text{rename } \beta\to\alpha\to\beta} = 0$$

$$\left(\frac{dU^{\alpha}}{d\tau} + \Gamma^{\alpha}_{\mu\beta} U^{\beta}U^{\mu}\right)\underset{\sim}{e}_{\alpha} = 0$$

$$\Rightarrow \frac{dU^{\alpha}}{d\tau} + \Gamma^{\alpha}_{\mu\beta} U^{\mu}U^{\beta} = 0$$

6. a) Evaluate $\nabla_\alpha g_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor.

In a locally inertial frame $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$

$$\nabla_\alpha g_{\mu\nu} = \nabla_\alpha \eta_{\mu\nu} = 0$$, because $\eta_{\mu\nu}$ are const.

b) Maxwell's equations in Minkowski space are given as

$$\partial_\nu F^{\mu\nu} = 4\pi k_e J^\mu$$

$$\partial^\mu F^{\alpha\beta} + \partial^\beta F^{\mu\alpha} + \partial^\alpha F^{\beta\mu} = 0 .$$

Write down these equations in any arbitrary coordinate system in any arbitrary spacetime.

$$\nabla_\nu F^{\mu\nu} = 4\pi k_e J^\mu$$

$$\nabla^\mu F^{\alpha\beta} + \nabla^\beta F^{\mu\alpha} + \nabla^\alpha F^{\beta\mu} = 0$$

c) Find the value of $\nabla_\alpha (g_{\mu\nu} T^{\mu\nu})$, where $T^{\mu\nu}$ is a stress-energy tensor.

$$\nabla_\alpha (g_{\mu\nu} T^{\mu\nu}) = \underbrace{(\nabla_\alpha g_{\mu\nu})}_{=0} T^{\mu\nu} + g_{\mu\nu} \underbrace{(\nabla_\alpha T^{\mu\nu})}_{=0}$$

$$= 0$$

7. In isotropic coordinates (ct, ρ, θ, ϕ) the Schwarzschild line element is given as

$$ds^2 = -\left(1-\frac{GM}{2c^2\rho}\right)^2\left(1+\frac{GM}{2c^2\rho}\right)^{-2}c^2dt^2$$

$$+\left(1+\frac{GM}{2c^2\rho}\right)^4\left(d\rho^2+\rho^2d\theta^2+\rho^2\sin^2\theta\, d\phi^2\right)$$

a) Write down the expression for the proper time $d\tau$,

$$d\tau = \left(1-\frac{GM}{2c^2\rho}\right)\left(1+\frac{GM}{2c^2\rho}\right)^{-1}dt$$

b) Is there an infinite redshift surface? Explain why.

Yes there is because $d\tau = 0$ for $\rho = \dfrac{GM}{2c^2}$

$$\frac{\lambda_R}{\lambda_E} = \frac{d\tau_R}{d\tau_E} = \frac{\left(1-\frac{GM}{2c^2\rho_R}\right)\left(1+\frac{GM}{2c^2\rho_R}\right)^{-1}}{\left(1-\frac{GM}{2c^2\rho_E}\right)\left(1+\frac{GM}{2c^2\rho_E}\right)^{-1}} \to \infty \text{ for } \rho_E = \frac{GM}{2c^2}$$

c) Is there an event horizon? Explain why.

There is no event horizon because the time and space coordinates always remain as they are.

d) Are there black holes according to this metric? Explain why.

There are no black holes because there is no event horizon.

8. An observer at $r = r_1$ in a Schwarzschild field sends a light signal in the radial direction toward $r = r_2$, where $r_2 > r_1$.

a) What is the instantaneous coordinate velocity $\frac{dr}{dt}$ of the signal?

For light signals $ds^2 = 0 = -\left(1 - \frac{r_s}{r}\right)c^2 dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2$ ← radially

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{r_s}{r}\right)^2 c^2 \Rightarrow \frac{dr}{dt} = +\left(1 - \frac{r_s}{r}\right)c \quad \text{(for increasing } r\text{)}$$

b) How long does it take the signal to travel from r_1 to r_2 in t time? (You have to evaluate a simple integral.)

$$c\int dt = c\Delta t = \int_{r_1}^{r_2} \frac{dr}{(1 - r_s/r)} = \int_{r_1}^{r_2} \frac{r\,dr}{r - r_s} \quad ; \quad \frac{r}{r - r_s} = 1 + \frac{r_s}{r - r_s}$$

$$c\Delta t = \int_{r_1}^{r_2} \left(1 + \frac{r_s}{r - r_s}\right) dr = \left[r + r_s \ln(r - r_s)\right]\Big|_{r_1}^{r_2}$$

$$c\Delta t = r_2 - r_1 + r_s \ln\left(\frac{r_2 - r_s}{r_1 - r_s}\right)$$

$$\Delta t = \frac{1}{c}\left[r_2 - r_1 + r_s \ln\left(\frac{r_2 - r_s}{r_1 - r_s}\right)\right].$$

c) How long does it take the signal to travel from r_1 to r_2 in proper time? Why?

For light $ds^2 = c^2 d\tau^2 = 0 \Rightarrow \Delta\tau = 0.$

9. The Robertson-Walker metric is defined as

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right].$$

where $a(t)$ is the scale factor.

a) What is the relation between proper time τ and coordinate time t?

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 \Rightarrow d\tau = dt \Rightarrow \tau = t.$$

b) Find the circumference C of an equatorial circle.

$\theta = \pi/2$

$$C = \int_0^{2\pi} a(t) r \, d\phi = a(t) r \, 2\pi.$$

10. Consider two black holes with masses $M_1 = 2M_\odot$ and $M_2 = 4M_\odot$.

a) If their temperatures are T_1 and T_2, find the ratio $\frac{T_2}{T_1}$.

Since $T \propto \frac{1}{M}$

$$\frac{T_2}{T_1} = \frac{2M_\odot}{4M_\odot} = \frac{1}{2}$$

b) Which black hole is hotter?

Since $T \propto \frac{1}{M}$, the black hole with mass M_1 is hotter.

GENERAL RELATIVITY

EXAM 2

Name: 28/12/2018

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1. Consider the following spacetime metric

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

a) Write down the metric for $r < \frac{2GM}{c^2}$.

b) Is the metric for $r < \frac{2GM}{c^2}$ a spacetime metric? Explain why?

c) Does the source of this metric become a blackhole if it collapses to a size $r < \frac{2GM}{c^2}$? Explain why?

2. a) Write down the expressions for the absolute gradient (covariant derivative) of a four-vector A^{μ} and a covector B_{μ}.

b) Calculate $\nabla_{\alpha}(A_{\mu}B^{\mu})$

c) Explain why $\nabla_{\alpha} g^{\mu\nu} = 0$.

3. a) An object's four-velocity $\underset{\sim}{U} = U^{\mu} \underset{\sim}{e}_{\mu}$ satisfies locally $\frac{d\underset{\sim}{U}}{d\tau} = 0$. Evaluate this and obtain the geodesic equation for a freely falling object
(Hint: Recall that $\frac{\partial \underset{\sim}{e}_{\alpha}}{\partial x^{\mu}} = \Gamma^{\nu}{}_{\alpha\mu} \underset{\sim}{e}_{\nu}$)

b) A tensor $T^{\alpha\beta}$ has the 4x4 matrix form

$$[T^{\alpha\beta}] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$ in flat spacetime, (a, b, c, d are constants)

What is the value of $\nabla_{\mu} T^{\alpha\beta}$ in the Schwarzschild spacetime, where ∇_{μ} is the covariant derivative?

4. The square of the infinitesimal interval for a two-sphere of radius R_0 is given by

$$ds^2 = R_0^2 d\theta^2 + R_0^2 \sin^2\theta \, d\phi^2 .$$

The $\theta\phi\theta\phi$ component of the Riemann tensor is found to be $R_{\theta\phi\theta\phi} = R_{\phi\theta\phi\theta} = R_0^2 \sin^2\theta$

a) Calculate all the components of the Ricci tensor $R_{\beta\gamma}$ given by $R_{\beta\gamma} = R^{\alpha}{}_{\beta\alpha\gamma} = g^{\mu\alpha} R_{\mu\beta\alpha\gamma}$.

b) Calculate the curvature scalar $R \equiv g^{\mu\nu} R_{\mu\nu}$.

5. Two black holes have masses $M_1 = 3 M_\odot$ and $M_2 = 9 M_\odot$, where $M_\odot$ is the solar mass.

a) Find the ratio T_2/T_1 of the temperatures of the black holes.

b) A black hole which has a temperature of $T_{BH} = 3.0\,K$ at $t=0$ is in contact with the cosmic background radiation. Describe what happens to the temperature of the black hole at a later time.

c) A black hole which has a temperature of $T_{BH} = 2.0\,K$ at $t=0$ is in contact with the cosmic background radiation. Describe what happens to the temperature of the black hole at a later time.

6. Assume that some creatures live in a hypothetical world whose spacetime geometry is described by the line element

$$ds^2 = -\left(1 - \frac{R_0^2}{r^2}\right)c^2 dt^2 + \left(1 - \frac{R_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

where R_0 is a constant and $0 \leq r < \infty$. Suppose the creatures live near on the right of R_0 such that $r > R_0$. If a creature were to pass to the left of R_0, namely to the region $r < R_0$,

a) could the creature return back to the region $r > R_0$? Explain why?

b) A creature observes that a light signal emitted by it undergoes an infinite redshift. What is the value of r for this creature? Why?

7. A massive particle moves near a gravitating source of mass M which curves the spacetime around it.

The energy of the particle is given by

$$E = \frac{mc^2 (-g_{00})^{1/2}}{\sqrt{1-\frac{v^2}{c^2}}},$$

where g_{00} is the 00-component of $g_{\mu\nu}$.

a) What is the energy E of the particle in flat spacetime? Explain.

b) The particle obeys the geodesic equation in curved spacetime. Write down this equation.

c) What does the geodesic equation reduce to in flat spacetime?

e) What is the condition on M for the spacetime to be flat?

8. The spacetime geometry outside a spherical object of mass M and electric charge Q is described by the Reissner-Nordström metric given by

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r} + \frac{Gk_e Q^2}{c^4 r^2}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r} + \frac{Gk_e Q^2}{c^4 r^2}\right)^{-1} dr^2$$
$$+ r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

a) Write down the Einstein field equation to which this ds^2 is a solution.

b) Does a neutral particle move on a geodesic of this spacetime?

c) The g_{00} component of the metric can be written as

$$g_{00} = -\left(1 - \frac{2GM_{eff}}{c^2 r}\right)$$

Find M_{eff}.

d) What is the value of Q which makes the spacetime flat?

9. a) Write down the Einstein tensor $G_{\mu\nu}$ including the cosmological constant Λ.

b) Write down the Einstein field equation in the absence of matter and energy.

c) Write down the Einstein field equation in the presence of matter and energy.

d) Express the equation in a form which contains only $R_{\mu\nu}$ on the left side of the equation. Show your work.

e) For a neutral object having no mass, no charge and no energy in the space outside it, what is the Einstein field equation?

10. The energy-momentum of a perfect fluid is given by

$$T^{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) U^\mu U^\nu + p g^{\mu\nu},$$

where ρ and p are the mass density and the pressure. $U^\mu = \dfrac{dx^\mu}{d\tau}$ is the four velocity.

a) What is the expression for $T^{\mu\nu}$ in a locally inertial frame?

b) Write $T^{\mu\nu}$ in matrix form in a locally inertial frame.

c) What is $\nabla_\mu T^{\mu\nu} = ?$

d) What are the dimensions of $T^{\mu\nu}$?